On the Cremona dimension of a p-group

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Let X be a smooth complex projective variety. We denote by $\operatorname{Bir} X$ the group of birational transformations $X \dashrightarrow X$.

A very classical object of study is the *plane Cremona group* $\operatorname{Bir} \mathbb{P}^2$, about which many results are known, by a long series of authors starting with Max Noether and Guido Castelnuovo. Igor Dolgachev and Vasily Iskovskikh, in 2009, completed the classification of conjugacy classes of finite subgroups of $\operatorname{Bir} \mathbb{P}^2$. In their paper they say "Very little is known about the Cremona groups in higher-dimensional spaces".

Around the same time Jean Pierre Serre asked the question "Does there exist a finite group which is not embeddable in $\operatorname{Bir} \mathbb{P}^3$?", adding "This looks very likely".

Since then the progress has been considerable.

One can ask in general: what can we say about finite subgroups of $\mathrm{Bir}\,\mathbb{P}^n$?

First of all, by equivariant resolution of singularities, if X is a smooth projective variety, the finite subgroups of $\mathrm{Bir}\,X$ are the finite subgroups of $\mathrm{Aut}\,X'$, where X' is a smooth projective variety that is birational to X. Thus, one way to phrase the question is: given a positive integer n, what can we say about the finite subgroups of $\mathrm{Aut}\,X$, where X is an n-dimensional smooth rational variety? But it has become clear that a better behaved replacement for the class of rational varieties is that of *rationally connected varieties* (those with lots of rational curves, which include unirational varieties).

The *Cremona dimension* $\operatorname{CrD} G$ of a finite group G is the least n such that G appears as a subgroup of a group $\operatorname{Bir} X$, or, equivalently, $\operatorname{Aut} X$, where X is an n-dimensional smooth projective rationally connected variety. Thus, a stronger version of Serre's question is: are there finite groups G with $\operatorname{CrD} G > 3$?

Again in 2009, Yuri Prokhorov gave a complete classification of the simple finite groups G with $\operatorname{CrD} G \leq 3$, showing that there is a finite number of them, and positively answering Serre's question. But of course we want results in arbitrarily high dimension.

A benchmark I want to use is the following: what can we say about the Cremona dimension of the symmetric group S_n for higher values of n? For $n \geq 4$ we have an upper bound $\operatorname{CrD} \mathrm{S}_n \leq n - 3$, and it also known that $\lim_{n\to\infty}(n - \operatorname{CrD} \mathrm{S}_n) = \infty$; but what about lower bounds?

If j is a positive integer, we say that a group G has the *Jordan property of index* j if every finite subgroup $H \subseteq G$ has a normal abelian subgroup $A \subseteq H$ of index at most j. Camille Jordan proved that for each n there exists a positive integer $j(n)$ such that $\mathrm{GL}_n(\mathbb{C})$ has the Jordan property of index $j(n)$. Serre also asked the following question: does the analogous property holds for the groups $\mathrm{Bir}\,\mathbb{P}^n$?

In 2012 Yuri Prokhorov and Constantin Shramov proved the following result. The Borisov–Alexeev–Borisov (BAB) conjecture was an important conjecture on boundedness of singular Fano varieties.

Theorem. For each integer $n \geq 0$, assume that the BAB conjecture holds in dimension n. Then there exists an integer $j(n)$ such that if X is an n-dimensional smooth projective rationally connected variety, then $\mathrm{Bir}\,X$ has the Jordan property of index $j(n)$.

The BAB conjecture was proved by Caucher Birkar in 2016. This answers Serre's question positively. It also shows that

$$\lim_{n\to\infty} \mathrm{CrD}\,\mathrm{S}_n = \infty$$

and that the number of finite simple groups of bounded Cremona dimension is finite.

This was a breakthrough.

Unfortunately no explicit bounds are known for $j(n)$, except for $n = 2$ and $n = 3$: Prokhorov and Shramov showed that $j(3) \leq 107{,}495{,}424$.

So the problem of giving explicit examples of finite groups of Cremona dimension at least 5, and explicit lower bounds for $\mathrm{CrD}\,\mathrm{S}_n$, remained open.

A second breakthrough happened in 2018, when Olivier Haution proved the following remarkable fixed point theorem.

Theorem. Let p be a prime, G a finite p-group acting on a smooth projective variety X of dimension less than $p-1$. Assume that p does not divide $\chi(X, \mathscr{O}_X)$. Then $X^G \neq \emptyset$.

If X is rationally connected then $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ for all $i > 0$, hence $\chi(X, \mathscr{O}_X) = 1$, and the second hypothesis is always satisfied.

Corollary (Jinsong Xu). Let p be a prime, X a smooth projective variety such that p does not divide $\chi(X, \mathscr{O}_X)$, G a finite p-subgroup of $\operatorname{Bir} X$. If $\dim X < p-1$, then G is abelian of rank at most $\dim X$.

We can make a birational modification of X and assume $G \subseteq \operatorname{Aut} X$ (recall that $\chi(X, \mathscr{O}_X)$ is a birational invariant). Then by Haution's theorem G must have a fixed point on X; the representation on the tangent space is faithful, and a p-group with a faithful representation V with $\dim V \leq p-1$ is abelian of rank at most $\dim V$.

This implies the following:

(1) A non-abelian p-group has Cremona dimension at least $p-1$.

(2) An abelian p-group of rank r has Cremona dimension at least $\min(r, p-1)$.

Previously Prokhorov and Shramov had shown that a non-abelian p-group has Cremona dimension at least 4 if $p \geq 17$.

The symmetric group S_n contains a non-abelian p-group if and only if $n \geq p^2$; hence if p is a prime with $p \leq \sqrt{n}$ we have $\operatorname{CrD} \mathrm{S}_n \geq p-1$. Since by Bertrand's postulate we have a prime p such that $\lfloor \sqrt{n}/2 \rfloor < p < \sqrt{n}$ we get the estimate

$$\lfloor \sqrt{n}/2 \rfloor \leq \operatorname{CrD} \mathrm{S}_n \leq n-3\,.$$

A limitation of Haution's theorem is in the hypothesis that $\dim X < p-1$; for example, it says nothing about 2-subgroups of $\operatorname{Bir} X$. Our approach, which is based on a new result of relative Brauer groups, does not have this problem, but requires a stronger hypothesis on X, and does not apply to abelian subgroups.

If G is a finite p-group, we are going to define a natural number $\gamma(G)$, the *non-abelian size* of G, which has the following properties.

(1) $\gamma(G)$ is always a multiple of $p-1$.

(2) $\gamma(G) = 0$ if an only if G is abelian.

(3) If $G_1, \ldots, G_r$ are finite p-groups, then
$$\gamma(G_1 \times \cdots \times G_r) = \gamma(G_1) + \cdots + \gamma(G_r)\,.$$

(4) If G is a non-abelian group with center $\mathbb{Z}/p$, then $\gamma(G) = p-1$.

Thus, for example, if G is a non-abelian p-group of order p^3 we have $\gamma(G) = p-1$, and $\gamma(G^r) = r(p-1)$ for any $r \geq 0$.

Our main result is the following.

Theorem (Bresciani–Reichstein–V.). Let p be a prime, X be a smooth projective variety such that $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ when i is odd, and $\chi(X, \mathscr{O}_X) < p$. Let $G \subseteq \operatorname{Bir} X$ be a p-subgroup. Then $\dim X \geq \gamma(G)$.

Corollary. The Cremona dimension of a finite p-group G is greater than or equal to $\gamma(G)$.

Since $\gamma(G) \geq p - 1$ as soon as G is not abelian, and can be much larger, this strengthens considerably the first part of the corollary to Haution's theorem.

What does this say about S_n? If $m \stackrel{\text{def}}{=} \lfloor n/4 \rfloor$, and D_4 is the dihedral group of order 8, then $\mathrm{D}_4^m \subseteq \mathrm{S}_n$; since $\gamma(\mathrm{D}_4^m) = m$, we get the bounds

$$\lfloor n/4 \rfloor \leq \operatorname{CrD} \mathrm{S}_n \leq n - 3\,.$$

On the hand, this says nothing about abelian groups. Consider the group $(\mathbb{Z}/p)^r$. Since $(\mathbb{Z}/p)^r \subseteq \mathsf{GL}_r$, we have

$$\mathsf{CrD}(\mathbb{Z}/p)^r \leq r\,,$$

while from Haution's theorem we get

$$\mathsf{CrD}(\mathbb{Z}/p)^r \geq \min(r, p-1)\,.$$

One can show that in fact that $\lim_{r\to\infty}\bigl(r - \mathsf{CrD}(\mathbb{Z}/p)^r\bigr) = \infty$.

This leaves open the (extremely unlikely) possibility that $\mathsf{CrD}(\mathbb{Z}/p)^r$, with p fixed, is bounded.

There are two non-abelian groups of order 8, the dihedral group D_4 and the quaternion group Q_8. We have $\gamma(\mathrm{D}_4) = \gamma(\mathrm{Q}_8) = 1$; on the other hand $\mathsf{CrD}\,\mathrm{D}_4 = 1$, while $\mathsf{CrD}\,\mathrm{Q}_8 = 2$.

If p is an odd prime, there are two non-abelian p-groups of order p^3, the Heisenberg group $\mathrm{H}_p = (\mathbb{Z}/p)^2 \rtimes (\mathbb{Z}/p)$, with exponent p, and the semidirect product $(\mathbb{Z}/p^2) \rtimes (\mathbb{Z}/p)$ with exponent p^2. One can show that $\operatorname{CrD} \mathrm{H}_p = p - 1$, while $(\mathbb{Z}/p^2) \rtimes (\mathbb{Z}/p)$ has Cremona dimension either p or $p - 1$.

Let p be an odd prime, r a positive integer, G_r the extra-special group of order p^{1+2r} and exponent p. Then $\mathrm{Z}(G_r) = \mathbb{Z}/p$, while $G_r/\mathrm{Z}(G_r) = (\mathbb{Z}/p)^{2r}$. Hence $\gamma(G_r) = p - 1$. All we can show is that

$$p - 1 \leq \operatorname{CrD} G_r \leq r(p - 1)\,;$$

on the other hand G_r contains a copy of $(\mathbb{Z}/p)^r$, so $\operatorname{CrD} G_r$ is presumably unbounded.

The proof of our theorem is based on a new connection between Cremona dimension and relative Brauer group, and does not use the results of Prokhorov, Shramov and Haution.

Let us define the non-abelian size $\gamma(G)$ of a p-group.

Definition. If A is an abelian p-group, then $A \simeq \prod_{i=1}^{r} \mathbb{Z}/p^{n_i}$ for certain natural numbers $n_1, \ldots, n_r$. We define the *Brauer dimension* of A as

$$\beta(A) = \sum_{i=1}^{r} (p^{n_i} - 1)\,.$$

Clearly $\beta(A)$ is always a multiple of $p - 1$, and $\beta(A) = 0$ if and only if $A = 0$.

So, for example, that Brauer dimension of $(\mathbb{Z}/p)^2$ is $2(p-1)$, and that of $\mathbb{Z}/p^2$ is $p^2 - 1$.

If G is a p-group with center $\mathrm{Z}(G)$, we define its *non-abelian center* $\mathrm{K}(G)$ as the kernel of the composite $\mathrm{Z}(G) \subseteq G \twoheadrightarrow G^{\mathrm{ab}}$. It is easy to see that $\mathrm{K}(G)$ is trivial if and only if G is abelian.

We define the non-abelian size $\gamma(G)$ as the Brauer dimension $\beta\bigl(\mathrm{K}(G)\bigr)$. The following properties, already mentioned, are easy to check.

(1) $\gamma(G)$ is always a multiple of $p-1$.

(2) $\gamma(G) = 0$ if an only if G is abelian.

(3) If G_1, …, G_r are finite p-groups, then

$$\gamma(G_1 \times \cdots \times G_r) = \gamma(G_1) + \cdots + \gamma(G_r)\,.$$

(4) If G is a non-abelian group with center $\mathbb{Z}/p$, then $\gamma(G) = p-1$.

Let me describe the ingredients of the proof.

Theorem (Bresciani–Reichstein–V.). Let p be a prime, X be a smooth projective variety such that $\mathrm{H}^i(X, \mathscr{O}_X) = 0$ when i is odd, and $\chi(X, \mathscr{O}_X) < p$. Let $G \subseteq \operatorname{Bir} X$ be a p-subgroup. Then $\dim X \geq \gamma(G)$.

As usual, we assume that $G \subseteq \operatorname{Aut} X$. Call Z the center of G, and set $\overline{G} \stackrel{\mathrm{def}}{=} G/Z$; then $\overline{G}$ acts on X/Z. Let Y be a smooth projective $\overline{G}$-equivariant model of X/Z; the meaning of the condition on X is that it ensures that $\chi(Y, \mathscr{O}_Y)$ is not divisible by p.

Our proof uses a result of Nikita Karpenko and Alexander Merkurjev to reduce it to a theorem on relative Brauer groups.

Recall that if K is a field, its Brauer group $\operatorname{Br} K = \mathrm{H}^2(K, \mathbb{G}_\mathrm{m})$ is the second étale cohomology group with coefficients in the sheaf $\mathbb{G}_\mathrm{m}$ of invertible elements.

If A is a finite abelian group, call $A^\vee \stackrel{\text{def}}{=} \mathrm{Hom}(A, \mathbb{C}^*)$ its dual, which is non-canonically isomorphic to A. Set $\Gamma \stackrel{\text{def}}{=} \mathsf{K}(G)^\vee$; since $\mathsf{K}(G)$ is by definition the kernel of the composite $Z \subseteq G \twoheadrightarrow G^{\mathsf{ab}}$, we see that Γ is the cokernel of the corresponding homomorphism $(G^{\mathsf{ab}})^\vee \to Z^\vee$. Since $\Gamma \simeq \mathsf{K}(G)$ we have $\gamma(G) \stackrel{\text{def}}{=} \beta\big(\mathsf{K}(G)\big) = \beta(\Gamma)$.

Let E/K be a Galois extension with Galois group $\overline{G}$, where K is an extension of $\mathbb{C}$; we associate with this a homomorphism $\Gamma \to \mathrm{Br}(K)$, via a construction due Mathieu Florence.

The extension E/K gives a class $[E/K]$ in the non-abelian cohomology set $\mathrm{H}^1(K, \overline{G})$. The exact sequence

$$1 \longrightarrow Z \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1$$

induces a non-abelian coboundary map $\partial\colon \mathrm{H}^1(K, \overline{G}) \to \mathrm{H}^2(K, Z)$, hence a class $\alpha \stackrel{\text{def}}{=} \partial[E/K] \in \mathrm{H}^2(K, Z)$.

We obtain a homomorphism $Z^\vee \to \mathrm{Br}(K)$ by sending a character $\lambda\colon Z \to \mathbb{G}_\mathrm{m}$ into $\lambda_*\alpha \in \mathrm{H}^2(K, \mathbb{G}_\mathrm{m})$. The characters coming from G^{ab} go to 0, hence we get a homomorphism $\Gamma \to \mathrm{Br}\, K$.

It follows from a result of Karpenko and Merkurjev that if E/K is *versal* (or universal, or generic), this homomorphism is injective.

The action of $\overline{G}$ on Y, which is a smooth birational model of X/Z, gives a twisted version $\widetilde{Y} \stackrel{\text{def}}{=} (Y \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec} E)/\overline{G}$ of Y over K. If $K(\widetilde{Y})$ is the function field of $\widetilde{Y}$, we prove that $\Gamma \subseteq \operatorname{Br}(K)$ is in the kernel of the pullback $\operatorname{Br}(K) \to \operatorname{Br}\bigl(K(\widetilde{Y})\bigr)$. The theorem follows from the following result, of independent interest.

Theorem (Bresciani–Reichstein–V.). Let K be a field, $M \to \operatorname{Spec} K$ a smooth projective variety, p a prime that does not divide $\chi(M, \mathscr{O}_M)$. If Γ is a p-subgroup of the relative Brauer group $\ker\bigl(\operatorname{Br}(K) \to \operatorname{Br}\bigl(K(M)\bigr)\bigr)$, then $\dim M \geq \beta(\Gamma)$.

Let us sketch the proof of this in the simplest case, when $\Gamma = \mathbb{Z}/p$. Thus, we have a a smooth projective variety $M \to \operatorname{Spec} K$ such that $p \nmid \chi(M, \mathscr{O}_M)$ and $\ker\bigl(\operatorname{Br}(K) \to \operatorname{Br}\bigl(K(M)\bigr)\bigr)$ contains an element of order p; we want to show that $\dim M \geq p - 1$.

Recall that classes in the Brauer group $\mathrm{Br}(K)$ are equivalence classes of Brauer–Severi varieties over K, that is, varieties $Q \to \operatorname{Spec} K$ such that $Q_{\overline{K}} \simeq \mathbb{P}^n_{\overline{K}}$ for some n. If L is an extension of K, then $[Q]$ is in the relative Brauer group $\ker\bigl(\mathrm{Br}(K) \to \mathrm{Br}(L)\bigr)$ if and only if $Q(L) \neq \emptyset$. So, if $M \to \operatorname{Spec} K$ is a variety, $[Q] \in \ker\bigl(\mathrm{Br}(K) \to \mathrm{Br}\bigl(K(M)\bigr)\bigr)$ if and only if there is a rational map $M \dashrightarrow Q$.

So, we need the following: if $M \to \operatorname{Spec} K$ is a a smooth projective variety, $Q \to \operatorname{Spec} K$ a Brauer–Severi variety such that $[Q] \in \mathrm{Br}(K)$ has order p, and $f\colon M \dashrightarrow Q$ a rational map, then $\dim M \geq p - 1$. We will assume $\dim M < p - 1$, and derive a contradiction.

The proof is based on the following lemma. Consider the base change $f_{\overline{K}}\colon M_{\overline{K}} \dashrightarrow Q_{\overline{K}} \simeq \mathbb{P}^n_{\overline{K}}$, and consider the pullback $L \stackrel{\text{def}}{=} f^*\mathscr{O}(1)$ (this is well defined because M is smooth).

Lemma. Let $t \in \mathbb{Z}$, $i \in \mathbb{N}$, and set $d_i \stackrel{\text{def}}{=} \dim_{\overline{K}} \mathrm{H}^i(M_{\overline{K}}, L^{\otimes t})$. Then for any $t \geq 0$ we have $d_i t[Q] = 0 \in \mathrm{Br}(K)$.

Since the order of $[Q]$ is p, we get $p \mid d_i$ for all i whenever $p \nmid t$. Now set $\Phi(t) \stackrel{\text{def}}{=} \chi(M_{\overline{K}}, L^{\otimes t})$; then Φ is an integer valued polynomial of degree at most $\dim M < p - 1$, which vanishes for $t = 1, \ldots, p-1$. This implies that Φ must be 0 modulo p, so that $p \mid \Phi(0) = \chi(M_{\overline{K}}, \mathscr{O}_{M_{\overline{K}}}) = \chi(M, \mathscr{O}_M)$.