Neutral representations,
the arithmetic of quotient singularities,
and fields of moduli

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Part 1: Residual gerbes and fields of moduli

All fields in this talk will have characteristic 0, purely for ease of exposition; everything goes through in positive characteristic, by adding enough adjectives, like "tame" and "reduced".

Let k be a field, and consider a class $\mathscr{M}$ of objects of type (X, ξ), where X is a proper variety over an extension K of k, and ξ is an additional structure, for example, a polarization on X, or an effective Cartier divisor $D \subseteq X$, or a finite set of marked rational points. We will usually omit the additional structure from the notation, and denote (X, ξ) simply with X.

We will require some conditions on $\mathscr{M}$, which will be satisfied in all the obvious examples: there should be a good notion of family of objects of $\mathscr{M}$, satisfying existence of pullbacks, a finite presentation condition, étale descent, and representability of isomorphism group schemes.

Recall that a *finite étale gerbe* over a field F is a Deligne–Mumford stack $\mathscr{G}$ of finite type over F, such that

(1) any two objects of $\mathscr{G}$ over a scheme S are étale-locally isomorphic, and

(2) $\mathscr{G}(F') \neq \emptyset$ for some finite extension F' of F.

If G is a finite étale group scheme over a field F, the classifying stacks $\mathscr{B}_F G$, whose objects over an F-scheme S are étale G-covers $E \to S$, is a finite étale gerbe. If $\mathscr{G}$ is a finite étale gerbe over F, ξ is an object of $\mathscr{G}(F)$, and $G \stackrel{\text{def}}{=} \underline{\mathrm{Aut}}_F\, \xi$ is its automorphism group scheme, then $\mathscr{G}$ is equivalent to $\mathscr{B}_F G$. Such a gerbe is called *neutral*.

Let X be an object of $\mathscr{M}$ over the algebraic closure $\overline{k}$; this X is defined over a finite extension k' of k. We will always assume that $\mathrm{Aut}(X) = \mathrm{Aut}_{\overline{k}}(X)$ is finite.

One can define a moduli problem of twisted forms of X, giving rise to a Deligne–Mumford stack $\mathscr{G}_X$. If S is a scheme over $\operatorname{Spec} k$, an object of $\mathscr{G}_X(S)$ is given by a family $Y \to S$ that is étale locally isomorphic to a trivial family $X' \times_{\operatorname{Spec} k'} T \to T$, where X' is a form of X over a finite extension k' of k.

This stack $\mathscr{G}_X$ is a finite étale gerbe over a finite extension $M(X)$ of k, which is called the *field of moduli* of X. It is contained in every extension $k \subseteq k' \subseteq \overline{k}$ such that X is defined over k'.

The gerbe $\mathscr{G}_X$ is the *residual gerbe* of X.

When $\mathscr{M}$ is a Deligne–Mumford stack with moduli space $M \to \operatorname{Spec} k$ (for example, smooth projective curves of genus at least 2, or polarized abelian varieties), an object X over $\overline{k}$ gives a morphism $\operatorname{Spec} \overline{k} \to M$; then if p is the image of $\operatorname{Spec} \overline{k} \to M$, then $\mathrm{M}(X)$ is the residue field $k(p)$. The residual gerbe $\mathscr{G}_X$ is the reduced inverse image of $\operatorname{Spec} k(p)$ in $\mathscr{M}$.

Here is the basic question: is X defined over its field of moduli $\mathrm{M}(X)$? In other words, is the residual gerbe $\mathscr{G}_X$ neutral?

Is terms of moduli spaces: if M is the moduli space of $\mathscr{M}$, given a point $p \in M$, is there an object of $\mathscr{M}\big(k(p)\big)$ over $\operatorname{Spec} k(p)$?

We are looking for positive answers that depend on geometric conditions for X over $\overline{k}$, not on arithmetic conditions.

The field of moduli was first defined for polarized varieties by T. Matsusaka in 1958; the definition was clarified and extended by G. Shimura and S. Koizumi. It has been studied since then, particularly for curves and abelian varieties.

Some known results.

(1) It is classical that every elliptic curve is defined over its field of moduli.

(2) In 1971 Shimura gave examples of hyperelliptic curves of any even genus not defined over their fields of moduli.

(3) Recently M. Lieblich and D. Bragg proved *Murphy's law for algebraic stacks:* various natural algebro-geometric moduli stacks, including the stack of smooth curves, have the property that every Deligne-Mumford gerbe over a field appears as the residual gerbe of one of their points.

Sometimes one can conclude that X is defined over its field of moduli just by looking at $\mathrm{Aut}(X)$.

A finite group G is *of outer splitting type* if the surjection $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ from its group of automorphism to the group of outer automorphism is split.

The Grothendieck–Giraud classification of gerbes (non-abelian cohomology) has the following consequence.

Theorem. If $\mathrm{Aut}(X)$ is finite with trivial center, and of outer splitting type, then X is defined over its field of moduli.

Such groups are fairly common: examples include the symmetric and alternating groups S_n for $n \geq 5$ and A_n for $n = 5$ or $n \geq 7$. Simple groups of outer splitting type have been classified by A. Lucchini, F. Menegazzo, and M. Morigi, and include all sporadic groups.

There are many positive answers in various situations in which the result above does not apply. The following result of Shimura is particularly important for us.

Let A_g be the moduli space of principally polarized abelian varieties over $\mathbb{C}$, k its field of rational functions, X the corresponding abelian variety over $\overline{k}$. Then the field of moduli of X is k. Shimura proved that X is defined over k if and only if g is odd.

The residual gerbe $\mathscr{G}_X$ has a universal family $\mathscr{X} \to \mathscr{G}_X$; the objects of $\mathscr{X}$ over a scheme S over k are objects $Y \to S$ of $\mathscr{G}_X$, together with a section $S \to Y$. This stack $\mathscr{X}$ has a moduli space $\mathbf{X}$, which is a scheme over $\mathrm{M}(X)$; we call this the *compression* of X. The base change $\mathbf{X}_{\overline{k}}$ is isomorphic to $X/\operatorname{Aut}(X)$. In other words, even if X does not descend to $\mathrm{M}(X)$, the quotient $X/\operatorname{Aut}(X)$ does.

Theorem [P. Dèbes – M. Emsalem (1998), Bresciani–V.]. Let X be a smooth connected curve, possibly with additional structure, and assume that $\operatorname{Aut}(X)$ is finite, and acts faithfully on X. If the compression $\mathbf{X}$ has a rational point over $\mathrm{M}(X)$, then X is defined over its field of moduli. $\square$

One case in which $\mathbf{X}$ has a rational point is when $\mathscr{X} \to \mathscr{G}_X$ has a section; this will automatically happen if ξ include a rational point. Thus we get that n-pointed smooth curves of genus g, with $n \geq 1$ and $g \geq 1$, are defined over their field of moduli. This generalizes the classical case $(g, n) = (1, 1)$.

Almost all the results on curves being defined over their fields of moduli known up to 2022 can be recovered easily using this theorem.

But how about the case $\dim X > 1$?

Smooth pointed surfaces are not necessarily defined over their fields of moduli (for example, Shimura showed that the generic principally polarized abelian surface over $\mathbb{C}$ is not defined over its field of moduli).

Part 2: R-singularities

A *singularity* over a field F is a pair (U, p), where U is a scheme of finite type over F, and $p \in U(F)$. Two singularities (U, p) and (U', p') over F are *equivalent* if $\widehat{\mathscr{O}}_{U,p} \simeq \widehat{\mathscr{O}}_{U',p'}$. Two singularities (U, p) and (U', p') over two different fields F and F' are *stably equivalent* if there exists a common extension F'' such that $(U, p)_{F''}$ is stably equivalent to $(U, p)_{F''}$.

Let (U, p) be a singularity over F, and $\pi\colon \widetilde{U} \to U$ a resolution. We say that (U, p) is *liftable* if $\widetilde{U}$ has an F-rational point over p. This condition is independent of the resolution; furthermore, a singularity that is equivalent to a liftable singularity is itself liftable.

When $k = \mathbb{R}$, liftable singularities are called *central points*, and play a role in real algebraic geometry.

Definition. A quotient singularity (U, p) over an algebraically closed field is an *R-singularity* if every singularity that is stably equivalent to (U, p) is liftable.

A smooth point is an R-singularity. A 2-dimensional A_n-singularity is an R-singularity if and only if n is even.

Bresciani has a complete classification of 2-dimensional R-singularities: all non-cyclic quotient singularities, all cyclic singularities of type $\frac{1}{n}(1, d)$ with n odd, and most of the singularities of type $\frac{1}{n}(1, d)$ with n even are R-singularities.

Theorem [Bresciani – V.]. Let X be an object of $\mathscr{M}$ over $\overline{k}$. Assume the following conditions.

(1) $\mathrm{Aut}(X)$ is finite, and acts faithfully on X.

(2) X is integral.

(3) The compression $\mathbf{X}$ has a rational point p, and $(\mathbf{X}_{\overline{k}}, p_{\overline{k}})$ is an R-singularity.

Then X is defined over its field of moduli. □

For integral pointed varieties (X, x_0), this gives a criterion that only depends of the singularity of $\big(X/\mathrm{Aut}(X), x_0\big)$ at the image of p.

This theorem, and more generally our machinery, has been applied by Bresciani to give many new results, for example on plane curves (by considering a plane curve as a structure on $\mathbb{P}^2$), and prove a conjecture of Doyle and Silverman on fields of definition of dynamical systems.

One can show that if n is odd, the cone $\mathbb{A}^n/\{\pm 1\}$ is has an R-singularity at its vertex. From this we obtain the following.

Theorem. Let (X, x_0) be an integral pointed variety over $\overline{k}$. If $\mathrm{Aut}(X, x_0)$ has order 2, $\dim X$ is odd, and x_0 is a smooth isolated fixed point, then (X, x_0) is defined over its field of moduli.

This is a vast generalization of Shimura's result that a generic principally polarized abelian variety of odd dimension is defined over its field of moduli.

Part 3: Neutral representations

The technique above only applies when the compression $\mathbf{X}$ has a rational point over the field of moduli, so mostly to pointed varieties. I am going to explain a new way of showing that X is defined over its field of moduli, by analyzing the action of $\mathrm{Aut}(X)$ on certainly intrinsically defined cohomology groups associated with X, such as $\mathrm{H}^i(X, \mathscr{O}_X)$, that does not have this limitation.

If $\mathscr{G}$ is a finite étale gerbe over a field F, a *vector bundle* $\mathscr{V}$ on $\mathscr{G}$ consists of a functorial assignment of a vector bundle $\mathscr{V}_\xi$ over S for each object ξ of $\mathscr{G}$ over a scheme S. If ξ is an object of $\mathscr{G}$ over $\overline{F}$, by functoriality the group $\mathrm{Aut}(\xi)$ acts on the $\overline{F}$-vector space $\mathscr{V}_\xi$.

Two finite-dimensional representations of G over two fields F and F' are *equivalent* if they become isomorphic by extending scalars to a common extension of F and F'.

Let V be a finite dimensional representation of a finite group G over an algebraically closed field K (for example, $K = \mathbb{C}$). If $\mathscr{G}$ is a gerbe over a field F, we say that V and $\mathscr{G}$ are *compatible* if there exist a vector bundle $\mathscr{V}$ on $\mathscr{G}$, an object ξ of $\mathscr{G}$ over $\overline{F}$, an isomorphism $G \simeq \mathrm{Aut}(\xi)$, and a vector bundle on $\mathscr{G}$, the resulting representation of G on $\overline{F}$ is equivalent to K.

Definition. A finite representation V over an algebraically closed field is *neutral* if every finite étale gerbe that is compatible with V is neutral.

Let $V = \mathrm{H}^i(X, F)$ be a cohomology group of a sheaf F on X that is obtained from $\Omega_{X,\bar{k}}$ by linear algebra operations (for example, $\mathscr{O}_X$, $\Omega_{X,\bar{k}}$ and all of its symmetric, tensor and exterior powers). Then the representation V of $\mathrm{Aut}(X)$ is compatible with the residual gerbe $\mathscr{G}_X$. For example, the representation $\mathrm{H}^i(X, \mathscr{O}_X)$ comes from the vector bundle that associates to each object $\pi\colon Y \to S$ the locally free sheaf $\mathrm{R}^i\pi_*\mathscr{O}_Y$.

If (X, x_0) is a pointed variety, the tangent space $\mathrm{T}_{x_0}X$ with its natural action of $\mathrm{Aut}(X, x_0)$ is also compatible with $\mathscr{G}_X$.

So, if the action of $\mathrm{Aut}(X)$ on one of these intrinsically defined vector spaces is neutral, then X is defined over its field of moduli.

Here is the basic example of a neutral representation.

Theorem. A faithful 1-dimensional representation of a finite group is neutral.

By looking at the action of $\mathrm{Aut}(X)$ on $\mathrm{T}_p X$, this reproves the result of Dèbes and Emsalem, that a smooth marked curve (X, p) is defined over its field of moduli.

Let me give a very short proof of this theorem, meant for experts. Let $\mathscr{G}$ be a finite étale gerbe over a field F, with a line bundle $\mathscr{V} \to \mathscr{G}$, such that if ξ is an object of $\mathscr{G}$ over $\overline{F}$, the corresponding 1-dimensional representation of $\mathrm{Aut}(\xi)$ is faithful. This corresponds to a representable morphism $\mathscr{G} \to \mathscr{B}_F\mathbb{G}_{\mathrm{m}}$. If Γ is the band of $\mathscr{G}$, there is an embedding $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}$, hence $\Gamma = \boldsymbol{\mu}_n$ for some n. The class of $\mathscr{G}$ in $\mathrm{H}^2(F, \boldsymbol{\mu}_n)$ maps to 0 in $\mathrm{H}^2(F, \mathbb{G}_{\mathrm{m}})$; but by Hilbert's theorem 90, the homomorphism $\mathrm{H}^2(F, \boldsymbol{\mu}_n) \to \mathrm{H}^2(F, \mathbb{G}_{\mathrm{m}})$ is injective; hence the class of $\mathscr{G}$ is 0, and $\mathscr{G}$ is neutral.

Building on this, one can construct many examples of neutral representations. For example, if V is a representation of a cyclic group C_2 of order 2, that is compatible with a gerbe $\mathscr{G}$ over F. Then $\det(V)$ is also compatible; hence, if $\dim V - \dim V^{\mathrm{C}_2}$ is odd, so that $\det(V)$ is faithful, then V is neutral.

With a some more work one can prove the following.

Theorem. Let C_p be a cyclic group of prime order p, V representation of C_p. Consider the eigenspace decomposition $V = \oplus_{\lambda \in \widehat{G}} V_\lambda$. Then V is neutral if and only if $\dim V_\lambda$ is not divisible by p for some $\lambda \neq 1$.

Corollary. In the situation above, if $\dim V - \dim V^{\mathrm{C}_p}$ is not divisible by p, then V is neutral.

In general we have sufficient conditions for a representation to be neutral, but we do not have a complete classification of neutral representations, even for cyclic groups.

This theorem has several geometric applications.

Let X be a smooth projective curve, such that $G \stackrel{\text{def}}{=} \operatorname{Aut} X$ is finite of prime order p. Look at the action of G on $\mathrm{H}^0(X, \omega_X)$; the dimension of $\mathrm{H}^0(X, \omega_X)$ is the genus $\mathrm{g}(X)$ of X, while the dimension of $\mathrm{H}^0(X, \omega_X)^G$ is $\mathrm{g}(X/G)$. Hence we get the following

Theorem. Let X be a smooth projective curve, such that $G \stackrel{\text{def}}{=} \operatorname{Aut} X$ is finite of prime order p. If $\mathrm{g}(X) - \mathrm{g}(X/G)$ is not divisible by p, then X is defined over its field of moduli.

Another application is to pointed varieties; this was our original motivation for the development of the theory.

Theorem. Let (X, x_0) be an integral pointed variety over $\overline{k}$. If $\operatorname{Aut}(X, x_0)$ has prime order p, $\dim X$ is not divisible by p, and x_0 is a smooth isolated fixed point, then (X, x_0) is defined over its field of moduli.

One can also use this theorem to give examples of varieties that are *not* defined over their field of moduli. Suppose that $X \to \operatorname{Spec} \overline{k}$ is a smooth projective curve of genus $g \leq 2$, such that $X/\operatorname{Aut} X = \mathbb{P}^1$. The the compression $\mathbf{X} \to \operatorname{Spec} k$ is a conic over k, which may or may not have rational points over k.

Theorem. Assume the following conditions:

(1) $d \equiv g \pmod 2$, and

(2) $X/\operatorname{Aut} X = \mathbb{P}^1$.

Then the compression $\mathbf{X}$, which is a conic over the field of moduli of X, has a rational point if and only if X is defined over its field of moduli.

Of course the "if" part follows from the theorem of Dèbes and Emsalem.

There is an interesting connection between neutral representations and R-singularities.

Theorem. Let K be an algebraically closed field, V a faithful representation of G over K.

(1) If V/G has an R-singularity at the origin, then V is neutral.

(2) If $G \subseteq \mathrm{GL}(V)$ does not contain any pseudoreflexion, then the converse also holds.

For example, if $G \subseteq \mathrm{GL}(V)$ is generated by pseudoreflexions, so that V/G is smooth, then V is neutral.

Thus, the problems of classifying neutral representations and classifying R-singularities are strictly related, and very hard. Bresciani's student Tianzhi Yang is classifying 3-dimensional abelian quotient singularities, and this is already highly nontrivial.