Fundamental gerbes

Angelo Vistoli

Scuola Normale Superiore, Pisa

Joint work with Niels Borne, Université de Lille

Kioloa, March 14, 2016

Let X be a geometrically reduced connected scheme of finite type over a field k, and let $x_0 \in X(k)$ be a rational point. Nori defined the *fundamental group scheme* $\pi_1^{\mathsf{N}}(X, x_0)$; it is a profinite group scheme with the property that, given a finite group scheme G on k, the homomorphisms $\pi_1^{\mathsf{N}}(X, x_0) \to G$ correspond to isomorphism classes of G-torsors $Y \to X$ with a fixed rational point $y_0 \in Y(k)$ lying over x_0.

Nori also produced another kind of fundamental group scheme, the *unipotent fundamental group scheme* $\pi_1^{\mathsf{U}}(X, x_0)$. This is pro-unipotent, and if G is a unipotent group scheme, morphisms $\pi_1^{\mathsf{U}}(X, x_0)$ correspond to isomorphism classes of G-torsors $Y \to X$ with a fixed rational point $y_0 \in Y(k)$ lying over x_0.

Borne and I remove the dependence on a base point, using *gerbes*.

We also explore the issue of what fundamental group schemes, or fundamental gerbes, do exist.

We are interested in *affine gerbes* over k. These are fpqc gerbes $\Gamma \to (\mathrm{Aff}/k)$ whose diagonal $\Gamma \to \Gamma \times \Gamma$ is affine.

If G is an affine group scheme over k, then $\mathscr{B}_k G$ is an affine gerbe, with a distinguished object in $\mathscr{B}_k G(k)$ given by the trivial torsor $G \to \operatorname{Spec} k$.

Conversely, if Γ is an affine gerbe on k, ξ is an object in $\Gamma(k)$, and $\underline{\mathrm{Aut}}_k \, \xi$ is the group scheme of automorphisms of ξ, descent theory gives an isomorphism $\mathscr{B}_k G \simeq \Gamma$.

So, $\{$affine group schemes$\} = \{$affine gerbes with sections$\}$.

Let $\mathscr{C}$ be a class of affine group schemes of finite type over extensions ℓ of k; for each ℓ we denote by $\mathscr{C}(\ell)$ the class of group schemes over ℓ that are in $\mathscr{C}$. We say that $\mathscr{C}$ is *stable* if the following conditions are satisfied.

(1) Each $\mathscr{C}(\ell)$ is closed under isomorphisms.

(2) If ℓ is an extension of k, ℓ' is an extension of ℓ, and G is a group scheme in $\mathscr{C}(\ell)$, then $G_{\ell'}$ is in $\mathscr{C}(\ell')$.

(3) Each $\mathscr{C}(\ell)$ is closed under products.

(4) Subgroups and quotients of groups in $\mathscr{C}(\ell)$ are in $\mathscr{C}(\ell)$.

(5) If G in $\mathscr{C}(\ell)$, every inner form (i.e., each H over ℓ such that $\mathscr{B}_\ell G \simeq \mathscr{B}_\ell H$) of G is in $\mathscr{C}(\ell)$.

Examples are given by commutative group schemes, unipotent group schemes, finite group schemes, all affine group schemes of finite type.

Given a stable class $\mathscr{C}$, a pro-$\mathscr{C}$-group is a group scheme over some extension ℓ of k which is a projective limit of groups in $\mathscr{C}$.

If Γ is an affine gerbe over an extension of k, we say that Γ is a $\mathscr{C}$-gerbe if for all objects ξ of Γ defined over some extension ℓ of k, the group scheme $\underline{\mathrm{Aut}}_\ell\, \xi$ is in $\mathscr{C}$.

A pro-$\mathscr{C}$-gerbe is a projective limit of $\mathscr{C}$-gerbes.

An affine group scheme G is in $\mathscr{C}$ (resp. is a pro-$\mathscr{C}$-group) if and only if $\mathscr{B}_k G$ is a $\mathscr{C}$-gerbe (resp. is a pro-$\mathscr{C}$-gerbe).

Let $X \to (\mathrm{Aff}/k)$ be a fibered category. A $\mathscr{C}$-fundamental gerbe $\Pi^{\mathscr{C}}_{X/k}$ is a pro-$\mathscr{C}$-gerbe over (Aff/k) with a morphism $X \to \Pi^{\mathscr{C}}_{X/k}$ such that for any other pro-$\mathscr{C}$-gerbe $\Gamma \to (\mathrm{Aff}/k)$ every morphism $X \to \Gamma$ factor through $\Pi^{\mathscr{C}}_{X/k}$, uniquely up to a unique isomorphism.

The exact condition is that for every pro-$\mathscr{C}$-gerbe Γ over k, the induced functor $\mathrm{Hom}_k(\Pi^{\mathscr{C}}_{X/k}, \Gamma) \to \mathrm{Hom}_k(X, \Gamma)$ induced by $X \to \Pi^{\mathscr{C}}_{X/k}$ is an equivalence of categories.

Suppose that $X \to \Pi^{\mathscr{C}}_{X/k}$ is a universal pro-$\mathscr{C}$-gerbe, and $x_0 \in X(k)$. We call $\pi_1^{\mathscr{C}}(X, x_0)$ the group scheme of automorphisms of the image of x_0 in $\Pi^{\mathscr{C}}_{X/k}(k)$; this is the *$\mathscr{C}$-fundamental group scheme*. It has the property that for each $G \in \mathscr{C}(k)$ there is a functorial bijection between homomorphism $\pi_1^{\mathscr{C}}(X, x_0) \to G$ and isomorphism classes of G-torsors $Y \to X$ with a rational point $y_0 \in Y(k)$ over x_0.

$\mathscr{C}$-fundamental gerbes don't exist in any kind of reasonable generality for all stable classes $\mathscr{C}$.

For example, suppose that $\mathscr{C}$ contains a semisimple non-trivial group G; let P be a parabolic subgroup. Assume that there exists a non-constant map $X \to G/P$ mapping a rational point x_0 into the image of the identity in G (for example, $X = \mathbb{P}^1$): then the fibered product $X \times_{G/P} G$ is a non-trivial P-torsor on X that becomes trivial when one extends the group to G. This should correspond to a non-trivial homomorphism $\pi_1^{\mathscr{C}}(X, x_0) \to P$ which becomes trivial when composed with the injection $P \to G$, and this is absurd.

One can also show that if $\mathscr{C}$ contains the semidirect product $\mathbb{G}_{\mathrm{m}} \ltimes \mathbb{G}_{\mathrm{a}}$, the fundamental gerbe $\Pi^{\mathscr{C}}_{\mathbb{P}^1/k}$ can not exist.

To define fundamental group gerbes, for wide classes of group schemes one needs to consider objects with additional structures, such as flat connections, stratifications, crystals, F-divisible structures in the sense of Esnault and Hogadi.

There is very interesting work in progress in this direction by Fabio Tonini and Lei Zhang.

Our work goes in a different direction: we find a necessary and sufficient condition for the existence of $\mathscr{C}$-fundamental gerbes without any additional structure.

A stack Z of finite type over an algebraically closed field k is *well-founded* if every reduced closed substack $V \to Z$ with $\mathrm{H}^0(V, \mathscr{O}) = k$ is a gerbe, (or, equivalently, the groupoid $V(k)$ is connected). So, for example, an affine scheme of finite type over k is well-founded. If G is a reductive group scheme acting on an affine scheme X of finite type over k, then the quotient stack $[X/G]$ is well-founded if and only if the action has closed orbits.

An affine group scheme G over k is *well-founded* if for any algebraically closed extension ℓ of k and any two subgroup schemes H and K of G_ℓ, the quotient stack $[G_\ell/H \ \times K]$ is well-founded. Here the action of $H \times K$ on G_ℓ is defined by $g \cdot (h, k) = h^{-1}gk$.

A stable class is *well-founded* if it consists of well-founded groups.

A non-trivial semisimple group G is not well-founded: take $H = \{1\}$, and as K a parabolic subgroup P of G. In this case $H^0([G/P], \mathscr{O}) = k$, but the action of P on G has more than one orbit.

The semidirect product $\mathbb{G}_{\mathrm{m}} \ltimes \mathbb{G}_{\mathrm{a}}$ is also not well-founded (take $H = K = \mathbb{G}_{\mathrm{m}}$).

Here are two large classes of well-founded groups. An affine group scheme G over k is *virtually unipotent*, or *virtually abelian*, if $G_{\overline{k}}$ contains a unipotent, or abelian, subgroup scheme of finite index.

Theorem. A virtually unipotent or virtually abelian group scheme is well-founded.

Assume that $X \to (\mathrm{Aff}/k)$ is a fibered category. We say that X is *reduced* if every morphism from X to an algebraic stack Y over k factors through Y_{red}. We say that X is *geometrically reduced* if for every extension k'/k, the fibered product $X_{k'} \to (\mathrm{Aff}/k')$ is reduced.

Theorem (Borne–V.). Let $\mathscr{C}$ be a well-founded stable class. Let $X \to (\mathrm{Aff}/k)$ be a fibered category satisfying the following conditions.

(a) There exists an fpqc cover $U \to X$, where U is an affine scheme.

(b) X is geometrically reduced.

(c) $\mathrm{H}^0(X, \mathscr{O}) = k$.

Then the fundamental gerbe $X \to \Pi^{\mathscr{C}}_{X/k}$ exists.

Examples of such fibered categories are quasi-compact, quasi-separated geometrically reduced schemes X with $\mathrm{H}^0(X, \mathscr{O}) = k$, and affine gerbes.

It is easy to see how the condition that $\mathscr{C}$ be well-founded comes into play. The point is that if G is an affine group scheme over k, while H and K are subgroup schemes, the fibered product $\mathscr{B}_k H \times_{\mathscr{B}_k G} \mathscr{B}_k K$ is the quotient stack $[G/H \times K]$, where the action is that define earlier by the formula $g \cdot (h, k) = h^{-1} g k$.

So, if the stack $[G/H \times K]$ is not well-founded, and $X \subseteq [G/H \times K]$ is a closed substack with $\mathrm{H}^0(X, \mathscr{O}) = k$ that is not a gerbe, the embedding $X \subseteq \mathscr{B}_k H \times_{\mathscr{B}_k G} \mathscr{B}_k K$ can not factor through a gerbe, and the $\mathscr{C}$-foundamental gerbe $\Pi^{\mathscr{C}}_{X/k}$ can not exist, if G is in $\mathscr{C}$.

Here are some examples.

(1) If $\mathscr{C}$ is the class of finite group schemes, we get the Nori fundamental gerbe $\Pi^{\mathrm{N}}_{X/k}$. For the existence of this we don't need to assume that $\mathrm{H}^0(X, \mathscr{O}) = k$, it is enough that X be geometrically connected.

(2) If $\mathscr{C}$ is the class of unipotent groups, we get the unipotent fundamental gerbe $\Pi^{\mathrm{U}}_{X/k}$.

(3) If $\mathscr{C}$ is the class of virtually unipotent groups, we get the virtually unipotent fundamental gerbe $\Pi^{\mathrm{VU}}_{X/k}$.

(4) If $\mathscr{C}$ is the class of virtually abelian groups, we get the virtually abelian fundamental gerbe $\Pi^{\mathrm{VA}}_{X/k}$.

We don't know whether well-founded group schemes form a stable family: the issue is whether products of well-founded group schemes is well-founded. If this is true, we get a "largest possible" fundamental gerbe dominating all the others (the One Gerbe, like Tolkien's One Ring).

Some of these fundamental gerbes have interesting tannakian interpretations.

If $\Gamma \to (\mathrm{Aff}/k)$ is an affine gerbe, the category of representations $\operatorname{Rep}\Gamma$ is the category of vector bundles on Γ. If G is an affine group scheme, $\operatorname{Rep}\mathscr{B}_k G$ is the category of finite-dimensional representations of G.

The category $\operatorname{Rep}\Gamma$ is a tannakian category: it is a k-linear categories with finite-dimensional Hom's, has a symmetric monoidal structure, given by tensor product, it has duals, and a fiber functor $\operatorname{Rep}\Gamma \to \mathrm{Vect}_{k'}$, where k' is an extension of k, obtained by pulling back along a morphism $\operatorname{Spec} k' \to \Gamma$.

The gerbe Γ can be recovered from $\operatorname{Rep}\Gamma$ as the stack of fiber functors; this gives an equivalence between the 2-category of affine gerbes and the 2-category of tannakian categories.

If $X \to \Pi^{\mathscr{C}}_{X/k}$ is a fundamental gerbe, pullback of vector bundles yiels an exact strongly monoidal functor $\operatorname{Rep} \Pi^{\mathscr{C}}_{X/k} \to \operatorname{Vect} X$.

Recall that an affine group scheme G over k is virtually unipotent if $G_{\overline{k}}$ contains a unipotent subgroup scheme of finite index.

Proposition. If $\mathscr{C}$ is a stable subclass of the class of virtually unipotent group schemes, the pullback $\operatorname{Rep} \Pi^{\mathscr{C}}_{X/k} \to \operatorname{Vect} X$ is fully faithful.

Thus, if $\mathscr{C}$ is a stable subclass of the category of virtually unipotent group schemes, $\operatorname{Rep} \Pi^{\mathscr{C}}_{X/k}$ is equivalent to a tannakian subcategory of the category of vector bundles.

Let us give a tannakian intepretation of the Nori fundamental group scheme $\Pi^{\mathrm{N}}_{X/k}$, extending Nori's famous theorem for proper varieties. Assume that X is *pseudo-proper*, that is, if E is a locally free sheaf on X, then $\dim_k \mathrm{H}^0(X, E) < \infty$.

If $f \in \mathbb{N}[x]$ is a polynomial and E is a vector bundle on X, we can define $f(E)$, interpreting the sum as a direct sum and the powers as tensor powers. A vector bundle E is *finite* if there exist f and g in $\mathbb{N}[x]$ with $f \neq g$ and $f(E) \simeq g(E)$.

A vector bundle on X is *essentially finite* if it is the kernel of a morphism between finite vector bundles; so our definition if different and much simpler than Nori's geometric definition in terms of semistable bundles, which only works for proper varieties. Essentially finite objects can be defined in arbitrary additive monoidal categories.

Theorem (Borne–V.). Let $X \to \Pi^{\mathrm{N}}_{X/k}$ be the Nori fundamental gerbe. The pullback functor $\operatorname{Rep} \Pi^{\mathrm{N}}_{X/k} \to \operatorname{Vect} X$ induces an equivalence between $\operatorname{Rep} \Pi^{\mathrm{N}}_{X/k}$ and the category of essentially finite bundles on X.

If char $k = 0$, every essentially finite object is finite.

Corollary. Essentially finite objects in a tannakian category form a tannakian subcategory.

The category $\operatorname{Rep} \Pi^{\mathrm{U}}_{X/k}$ is equivalent to the category of vector bundles E on X that have a filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_r = E$ in which every quotient E_i/E_{i-1} is trivial (this is due to Nori in the case of schemes with a fixed rational point).

In a recent preprint, Shusuke Otabe considers (in characteristic 0) what he calls *semifinite bundles*, vector bundles E on X that have a filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_r = E$ in which every quotient E_i/E_{i-1} is essentially finite.

An affine group scheme of finite type G over a field is *strongly virtually unipotent* if it contains a normal unipotent subgroup scheme G' such that G/G' is finite. In characteristic 0 every virtually unipotent group scheme is strongly virtually unipotent, but this is false in positive characteristic. The category of strongly virtually unipotent group schemes is stable; the corresponding fundamental gerbe is denoted by $\Pi^{\mathrm{SVU}}_{X/k}$.

Theorem (Borne–V.).

(a) The tannakian category $\operatorname{Rep} \Pi^{\mathrm{SVU}}_{X/k}$ is equivalent to the category of semifinite bundles on X.

(b) If $\operatorname{char} k > 0$, let $F\colon X \to X$ be the absolute Frobenius morphism. Then $\operatorname{Rep} \Pi^{\mathrm{VU}}_{X/k}$ is equivalent the tannakian category of vector bundles E such that $(F^n)^* E$ is semifinite in the sense of Otabe for $n \gg 0$.

We don't even have a conjecture about $\operatorname{Rep} \Pi^{\mathrm{VA}}_{X/k}$. Certainly the pullback $\operatorname{Rep} \Pi^{\mathrm{VA}}_{X/k} \to \operatorname{Vect} X$ is not fully faithful.

An example of a stable class is given by group schemes of multiplicative type. These are group schemes of finite type $G \to \operatorname{Spec} k$ such that $G_{\overline{k}}$ is diagonalizable, that is, isomorphic to a product of copies of $\mathbb{G}_{\mathrm{m}}$ and $\boldsymbol{\mu}_n$'s.

The corresponding fundamental gerbe is called the fundamental gerbe of multiplicative type, and is denoted by $\Pi^{\mathrm{MT}}_{X/k}$. If $\Pi^{\mathrm{MT}}_{X/k}(k) \neq \emptyset$, then we get a torsor on X, which is what Colliot-Thélène and Sansuc call a *universal torsor*.

The fundamental gerbe $\Pi^{\mathrm{MT}}_{X/k}$ can be constructed directly, starting from the action of the Galois group of a separable closure k^{sep} on the category of invertible sheaves on $X_{k^{\mathrm{sep}}}$, unsing a duality result between gerbes of multiplicative type and certain classes of Picard stacks.