Linearization of generalized interval exchange maps

Stefano Marmi

Scuola Normale Superiore, Pisa

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joint work with Pierre Moussa (CEA Saclay)
and Jean-Christophe Yoccoz (Collège de France)
The slides of this talk will be available on my webpage
http://homepage.sns.it/marmi/

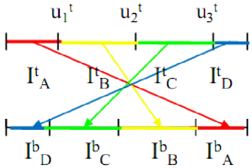
Plan of the talk

- Interval exchange maps
- Simple deformations and the linearization theorem
- Herman's Schwarzian derivative trick
- Combinatorial data, Keane property and Rauzy diagrams
- Rauzy-Veech algorithm
- The Kontsevich-Zorich cocycle
- Irrational i.e.m. and Poincaré's theorem
- Open Problems

Interval exchange maps

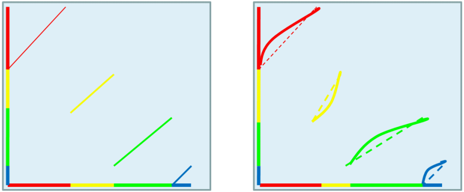
I bounded open interval, $I = \bigsqcup_{\mathcal{A}} I^t_\alpha = \bigsqcup_{\mathcal{A}} I^b_\alpha$ partitions (mod 0) in d subintervals, indexed by the alphabet $\mathcal{A}$.

A *standard (resp. generalized, resp. generalized C^r) interval exchange map* (i.e.m) with these data is a 1-to-1 map (mod 0) T on I sending each I^t_α onto I^b_α through a translation (resp. orientation-preserving homeomorphism, resp. a piecewise C^r diffeo $\overline{I^t_\alpha} \to \overline{I^b_\alpha}$ for each $\alpha \in \mathcal{A}$). Parameter space for standard normalized i.e.m is $(d-1)$-dimensional. If $d = 2$ then a standard i.e.m. is a rotation a generalized i.e.m. is an orientation-preserving circle homeomorphism.



Standard vs. generalized interval exchange maps

All translation surfaces obtained by suspension from standard i.e.m. with a given Rauzy diagram have the same genus g, and the same number s of marked points; these numbers are related to the number d of intervals of continuity by the formula $d = 2g + s - 1$.



A standard i.e.m. with $g = 2, s = 1, d = 4$; an affine (broken line) and a generalized i.e.m. with the same combinatorial data.

Question: What part of the theory of circle diffeomorphisms generalize to i.e.m.s?

Overview of results I

By means of the Rauzy–Veech "continued fraction" algorithm and its extension to generalized i.e.m. one can define an analogue of the rotation number.
When the i.e.m. has no *connection*, (i.e. no finite orbit starts and ends at a discontinuity of the map) the RV algorithm associates to it an infinite path in a Rauzy diagram that can be viewed as a "rotation number".

One can characterize the infinite paths associated to standard i.e.m. with no connections (∞–complete paths). One says that a generalized i.e.m. T is *irrational* if its associated path is ∞–complete; then T is semi–conjugated to any standard i.e.m. with the same rotation number (Poincaré's theorem)
Regarding Denjoy's theorem, partial results (Camelier-Gutierrez, Bressaud, Hubert and Maas, MMY) go in the negative direction, suggesting that topological conjugacy to a standard i.e.m. has positive codimension in genus $g \geq 2$.

Overview of results II

A first step in the direction of extending small divisor results beyond the torus case was achieved by Forni's theorem on the existence and regularity of solutions cohomological equation associated to linear flows on surfaces of higher genus.

In [MMY1], we considered the cohomological equation $\psi \circ T_0 - \psi = \varphi$ for a standard i.e.m. T_0. We found explicitly in terms of the Rauzy-Veech algorithm a full measure class of standard i.e.m. (which we called Roth type i.e.m.) for which the cohomological equation has bounded solution provided that the datum φ belongs to a finite codimension subspace of the space of functions having on each continuity interval a continuous derivative with bounded variation.

The cohomological equation is the linearization of the conjugacy equation $T \circ h = h \circ T_0$ for a generalized i.e.m. T close to the standard i.e.m. T_0.

Simple deformations and a linearization theorem

We say that a generalized i.e.m. T is a *simple deformation of class* C^r of a standard i.e.m. T_0 if

- T and T_0 have the same discontinuities;
- T and T_0 coincide in the neighborhood of the endpoints of I and of each discontinuity;
- T is a C^r diffeomorphism on each continuity interval onto its image.

For simple deformations our main result can be summarized as follows:

Theorem. *For almost all standard i.e.m.* T_0 *and for any integer* $r \geq 2$*, amongst the* C^{r+3} *simple deformations of* T_0*, those which are* C^r*-conjugate to* T_0 *by a diffeomorphism* C^r *close to the identity form a* C^1 *submanifold of codimension* $d^* = (g-1)(2r+1)+s$.

Some remarks

The standard i.e.m. T_0 considered in the theorem are the Roth type i.e.m. for which the Lyapunov exponents of the KZ-cocycle are non zero (we call this *restricted Roth type*). They still form a full measure set by a theorem of Forni.

To extend this result to generalized i.e.m. T of class C^r which are not simple deformations of a standard i.e.m. T_0, there are gluing problems of the derivatives of T at the discontinuities. *Indeed there is a conjugacy invariant which is an obstruction to linearization*.

An earlier result is presented in an unpublished manuscript of De La Llave and Gutierrez. They consider standard i.e.m. with periodic paths for the Rauzy-Veech algorithm (for $d = 2$, this corresponds to rotations by a quadratic irrational). They prove that, amongst piecewise analytic generalized i.e.m. , the bi-Lipschitz conjugacy class of such a standard i.e.m. contains a submanifold of finite codimension.

Herman's Schwarzian derivative trick I

The Schwarzian derivative of a C^3 orientation preserving diffeomorphism f is

$$Sf := D^2 \mathrm{Log} Df - \frac{1}{2}(D \mathrm{Log} Df)^2 .$$

The Schwarzian derivative vanishes on linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$, $ad - bc = \pm 1$.

The composition rule for Schwarzian derivatives is

$$S(f \circ g) = Sf \circ g \; (Dg)^2 + Sg .$$

Let f be a circle diffeomorphisms of rotation number ω. Let R_ω be the corresponding rotation of the circle. Assume that ω is diophantine: $|\omega - p/q| \geq \gamma q^{-2-\tau}$ for some $\gamma > 0$, $\tau < 1$.

Herman's Schwarzian derivative trick II

Taking Schwarzian derivatives, the conjugacy equation

$$f \circ h = h \circ R_\omega$$

becomes

$$(Sh) \circ R_\omega - Sh = ((Sf) \circ h)(Dh)^2$$

This is a linear difference equation in the Schwarzian derivative Sh of the conjugacy (but the r.h.s. depends also on h).

Given a diffeomorphism h, one computes the r.h.s. $((Sf) \circ h)(Dh)^2$, solves the equation $\psi \circ R_\omega - \psi = ((Sf) \circ h)(Dh)^2$ and then finds a diffeomorphism $\tilde{h} = \Phi(h)$ as smooth as h with $S\tilde{h} = \psi$. One can use the contraction principle to conclude (at the cost of one more derivative for f) as we do in our proof.

Combinatorial data and genus of an i.e.m.

The order in which the I^t_α, I^b_α appear is encoded by a pair $\pi = (\pi_t, \pi_b)$ of bijections from $\mathcal{A}$ to $\{1, \cdots, d\}$, the *combinatorial data* of the i.e.m T. t=TOP, b=BOTTOM

Irreducibility: for any $1 \leqslant k < d$

$$\pi_t^{-1}(1, \cdots, k) \neq \pi_b^{-1}(1, \cdots, k).$$

Otherwise, the recurrent part of the dynamics occurs in two disjoint intervals.

The antisymmetric matrix Ω given by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t(\beta) > \pi_t(\alpha)\ ,\ \pi_b(\beta) < \pi_b(\alpha), \\ -1 & \text{if } \pi_t(\beta) < \pi_t(\alpha)\ ,\ \pi_b(\beta) > \pi_b(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

has rank $\Omega = 2g$, g is the *genus of the map*. This is also the genus of all translation surfaces obtained from T by Veech's zippered rectangles construction

Singularities and connections

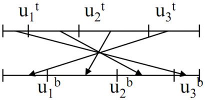
- The *singularities* of T are the $d-1$ points $u_1^t < \cdots < u_{d-1}^t$ separating the I_α^t.
- The *singularities* of T^{-1} are the $d-1$ points $u_1^b < \cdots < u_{d-1}^b$ separating the I_α^b.
- A *connection* is a relation $T^m(u_i^b) = u_j^t$ with $1 \leqslant i,j < d$ and $m \geqslant 0$.

Theorem (Keane) If the length data are rationally independent, a standard i.e.m. T has no connections.
If a standard i.e.m. T has no connections, T is minimal.

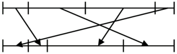
Definition A (generalized) i.e.m. T has the *Keane property* if it has no connections.

The elementary step of the Rauzy–Veech algorithm

Let T be a g.i.e.m with no connection. Then $u^t_{d-1} \neq u^b_{d-1}$. Set $\widehat{u}_d := \max(u^t_{d-1}, u^b_{d-1})$, $\widehat{I} := (u_0, \widehat{u}_d)$, and denote by $\widehat{T}$ the first return map of T in $\widehat{I}$. The return time $= 1$ or 2.



T

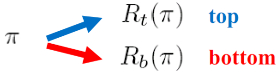


$\widehat{T}$

$\widehat{T}$ is a g.i.e.m on $\widehat{I}$ with combinatorial data $\widehat{\pi}$ labeled by the same alphabet $\mathcal{A}$. Moreover $\widehat{T}$ has no connection $\Longrightarrow$ one can iterate the algorithm! We say that $\widehat{T}$ is deduced from T by an elementary step of the Rauzy–Veech algorithm. We say that the step is of top (resp. bottom) *type* if $u^t_{d-1} < u^b_{d-1}$ (resp. $u^t_{d-1} > u^b_{d-1}$). One then writes $\widehat{\pi} = R_t(\pi)$ (resp. $\widehat{\pi} = R_b(\pi)$).

Rauzy diagrams

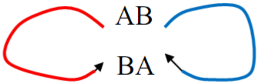
Rauzy class: nonempty set of irreducible combinatorial data invariant under R_t, R_b and minimal w.r.t. this property. *Rauzy diagram*: graph whose vertices are elements of a Rauzy class and whose arrows connect a vertex π to its images $R_t(\pi)$ and $R_b(\pi)$. Each vertex is therefore the origin and the endpoint of two arrows.



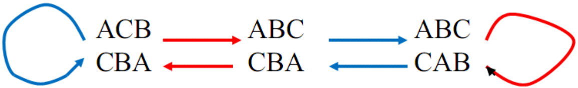
$\pi = (\pi_t, \pi_b)$; denote by α_t, α_b the letters such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$.

- The *winner* of the arrow of top (resp. bottom) type from π is α_t (resp. α_b); the *loser* is α_b (resp. α_t).
- The winner also gives a name to the arrow
- A path in the diagram is a word in the alphabet
- A path γ in a Rauzy diagram is *complete* if each letter in $\mathcal{A}$ appears in the corresponding word
- γ is *k–complete* if it is the concatenation of k complete paths.
- An infinite path is ∞–*complete* if it is the concatenation of infinitely many complete paths.

Rauzy diagrams for 2 and 3 intervals

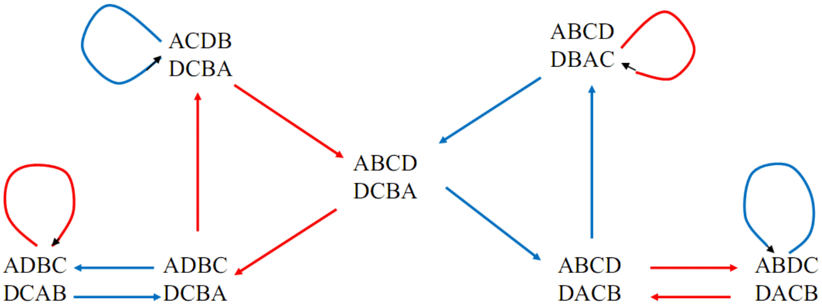


g=1, s=1,d=2



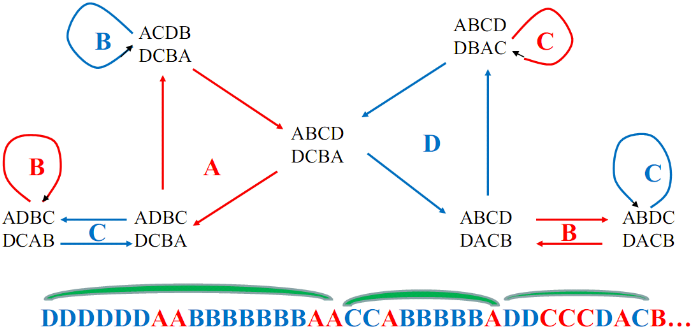
g=1,s=2,d=3

Rauzy diagrams for 4 intervals and genus 2



g=2,s=1,d=4

Rauzy diagram for d=4 and acceleration of the Rauzy-Veech algorithm



The Rauzy-Veech algorithm

Let $T = T^{(0)}$ be an i.e.m. with no connection, $\pi^{(0)}$ its combinatorial data and $\mathcal{D}$ the Rauzy diagram on $\mathcal{A}$ having $\pi^{(0)}$ as a vertex.
The i.e.m. $T^{(1)}$, with combinatorial data $\pi^{(1)}$, is deduced from $T^{(0)}$ by the elementary step of the Rauzy–Veech algorithm has also no connection.
Iterating the procedure one gets an infinite sequence $T^{(n)}$ of i.e.m. with combinatorial data $\pi^{(n)}$, acting on a decreasing sequence $I^{(n)}$ of intervals and a sequence $\gamma(n, n+1)$ of arrows in $\mathcal{D}$ from $\pi^{(n)}$ to $\pi^{(n+1)}$ associated to the successive steps of the algorithm.
If $m < n$, we also write $\gamma(m, n)$ for the path from $\pi^{(m)}$ to $\pi^{(n)}$ made of the concatenation of the $\gamma(l, l+1)$, $m \leqslant l < n$.
We write $\gamma(T)$ for the infinite path starting from $\pi^{(0)}$ formed by the $\gamma(n, n+1)$, $n \geqslant 0$.

Irrational generalized i.e.m.

If T is a s.i.e.m. with no connection, then $\gamma(T)$ is ∞-complete and, conversely, an ∞-complete path is equal to $\gamma(T)$ for some s.i.e.m. with no connection. However for a g.i.e.m. T with no connection, the path $\gamma(T)$ is not always ∞-complete.

A g.i.e.m. T is *irrational* if it has no connection and $\gamma(T)$ is ∞-complete. We then call $\gamma(T)$ the *rotation number* of T.

When $d = 2$ (circle) the Rauzy diagram has one vertex and two arrows. If the rotation number

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

the associated ∞-complete path takes a_1 times the first arrow, then a_2 times the second arrow, a_3 times the first arrow, ...

Poincaré's theorem

A standard i.e.m. is irrational iff it has no connection and two s.i.e.m. with no connection are topologically conjugated iff they have the same rotation number.

More generally, if T is an irrational generalized i.e.m. with the same rotation number of a standard i.e.m T_0, then there is, as in the circle case, a semiconjugacy from T to T_0, i.e. a continuous nondecreasing surjective map h from the interval I of T onto the interval I_0 of T_0 such that $T_0 \circ h = h \circ T$ (Poincaré's theorem).

The Kontsevich-Zorich cocycle

$\mathcal{D}$, Rauzy diagram; γ arrow of $\mathcal{D}$; (α, β) =(loser,winner) of γ.

$$B_\gamma = \mathbb{I} + E_{\alpha\,\beta}, \in SL(\mathbb{Z}^{\mathcal{A}})$$

For a path $\gamma = \gamma_1 \dots \gamma_l$ in $\mathcal{D}$ we associate the product $B_\gamma = B_{\gamma_l} \dots B_{\gamma_1} \in SL(\mathbb{Z}^{\mathcal{A}})$. It has nonnegative coefficients.

Let $\widehat{T}$ be deduced from T by Rauzy-Veech renormalization along the path γ. Γ = functions constant on each I_α^t; $\widehat{\Gamma}$ same for $\widehat{T}$. $\Gamma\,,\,\widehat{\Gamma} \approx \mathbb{R}^{\mathcal{A}}$. B_γ is the matrix of $S : \Gamma \to \widehat{\Gamma}$:

$$S\chi(x) = \sum_{0 \leqslant i < r(x)} \chi(T^i(x))$$

where $\chi \in \Gamma$, $x \in \widehat{I}$ and $r(x)$ is the return time of x in $\widehat{I}$.
Let $\mathcal{R}$ be the Rauzy class associated to $\mathcal{D}$. Restricted to standard i.e.m (up to affine conjugacy), the Rauzy-Veech algorithm defines a map Q_{RV} on the parameter space $\mathcal{R} \times \mathbb{P}(\mathbb{R}^{\mathcal{A}})$. The operator S is a cocycle over Q_{RV} called the *(extended) Kontsevich-Zorich cocycle*.

Suspension and genus

Let T be a standard i.e.m with combinatorial data $\pi = (\pi_t, \pi_b)$. For $\alpha \in \mathcal{A}$ let $\lambda_\alpha = |I^t_\alpha| = |I^b_\alpha|$, $\tau_\alpha = \pi_b(\alpha) - \pi_t(\alpha)$, $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha$. In the complex plane, draw a top (resp. bottom) polygonal line from u_0 to u_d through $u_0 + \zeta_{\pi_t^{-1}(1)}, u_0 + \zeta_{\pi_t^{-1}(1)} + \zeta_{\pi_t^{-1}(2)}, \ldots$ (resp. $u_0 + \zeta_{\pi_b^{-1}(1)}, u_0 + \zeta_{\pi_b^{-1}(1)} + \zeta_{\pi_b^{-1}(2)}, \ldots$). These two polygonal lines bound a polygon. Gluing the ζ_α bottom and top sides of the polygon produces a translation surface M_T. After gluing the vertices of the polygon form a set of marked points Σ on M_T and $d = 2g + s - 1$. If one identifies $\mathbb{R}^{\mathcal{A}}$ with the relative homology group $H_1(M_T, \Sigma, \mathbb{R})$ via the basis defined by the sides ζ_α of the polygon, the image of Ω coincides with the absolute homology group $H_1(M_T, \mathbb{R})$. Another way to compute s (and thus g) consists in going around the marked points.

Open problems I

Prove the theorem for $r = 1$: for a.a. s.i.e.m. T_0, C^4 simple deformations which are C^1-conjugate to T_0 by a C^1-close-to-Id diffeo form a C^1 submanifold of codim$= d^* = 3g - 3 + s = (d-1) + (g-1)$.

$(d-1)$ parameters fix the rotation number

$(g-1)$ parameters fixed by requiring Birkhoff sums of $\log DT$ $(= \log DT^n)$ to be bounded. Needed in order T to be C^1-conjugate to T_0. Since $\int \log DT d\mu = 0 \Longrightarrow$ kill components w.r.t. the remaining $g-1$ positive exponents of the KZ-cocycle. If the derivatives DT^n are allowed to grow exponentially fast one may have wandering intervals.

A dichotomy between being C^1-conjugated (to a s.i.e.m) and having wandering intervals? If this is the case then locally for C^4 g.i.e.m. topological conjugacy implies C^1 conjugacy.

The two conjectures above have a more general formulation (not restricted to simple deformations) using our conjugacy invariant.

Open problems II

The local C^r conjugacy class of a standard i.e.m T_0 (of restricted Roth type) exhibited by our theorem can be considered as a *local stable manifold* for the renormalization operator $\mathcal{R}$ defined by the Rauzy-Veech induction (with rescaling) on generalized i.e.m.'s in a suitable functional space. By the standard techniques this local stable manifold extends to a global stable manifold

$$W^s(T_0) = \cup_{n\geq 0}\mathcal{R}^{-n}(W^s_{loc}(\mathcal{R}^n T_0)$$

which is the full C^r conjugacy class of T_0.

Is this stable manifold "properly embedded" in parameter space?

More precisely, given a sequence h_n in $\mathrm{Diff}^r(\bar{I})$ s.t. $h_n \to \infty$,

Is it possible that $h_n \circ T_0 \circ h_n^{-1} \to T_0$ in the C^{r+3} topology? Is it possible that $h_n \circ T_0 \circ h_n^{-1}$ stays bounded in the C^{r+3} topology?

When $d = 2$, the answer to both questions is no. For the second question, this follows from Herman's global conjugacy theorem for circle diffeos.

Open problems III

Describe the set of generalized C^r interval exchange maps which are semi-conjugate to a given standard i.e.m. T_0 (with no connections).

In the circle case, for a diophantine rotation number, one has a C^∞ submanifold of codimension 1. In the Liouville case one has still a topological manifold of codimension 1 which is transverse to all 1-parameter strictly increasing families. One can therefore dare to ask:

1. *Is the above set a topological submanifold of codimension $d-1$?*
2. *if the answer is positive, does there exist a (smooth) field of "transversal" subspaces of dimension $d-1$?*

The questions make sense for any T_0, but the answer could depend on the diophantine properties of T_0.

Open problems IV

In a generic smooth family of generalized i.e.m.'s, is the rotation number irrational with positive probability?
In the circle case: YES! by Herman's theorem. This is not very likely in higher genus.

Let $r \geq 1$. Describe exactly (in terms of the Rauzy-Veech renormalization algorithm) the set of rotation numbers such that the C^r conjugacy class of T_0 has finite codimension in the space of C^∞ generalized i.e.m.'s. Does this set depends on r?

In the circle case: for any $r \geqslant 1$ this is just the set of diophantine rotation numbers. In higher genus, our theorem (in the stronger form) guarantees that this set contains the restricted Roth type rotation numbers and therefore has full measure. Of course the codimension of the C^r conjugacy class will depend on r but the point here is that we only require the codimension to be finite. Note that the answer is not known even at the level of the cohomological equation!