Nonlinearity 5 (1992) 743–761. Printed in the UK

# On the standard map critical function

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Received 10 July 1991
Accepted by I C Percival

**Abstract.** A critical function for an area-preserving map associates with each fixed irrational rotation number $\omega$ the breakdown threshold $K(\omega)$ of the corresponding KAM invariant circle. Understanding the structure of such a function and obtaining good estimates and approximations to it is a problem of fundamental theoretical importance and also has relevance to many applications. We present strong numerical evidence that a purely arithmetic function, the Brjuno function $B(\omega)$, which only depends on the nearest-integer continued fraction expansion of the rotation number, $\omega$, provides a good approximation to $\log(K(\omega))$ for the standard map, which is one of the most commonly studied area-preserving maps. We discuss the relationship of the Brjuno function to critical functions in other small-divisor problems, remark on the relevance of our results in explaining the modular smoothing technique of Buric and Percival and prove that $K(\omega) > 0$ for all complex $\omega$ with a non-zero imaginary part.

AMS classification scheme numbers: 58F05, 58F13, 70K50

## 1. Introduction

The problem of obtaining accurate estimates for the breakdown threshold of KAM invariant tori of close-to-integrable Hamiltonian systems, area-preserving maps, or complex analytic maps has recently been the subject of considerable interest by many authors [1–6]. This problem is important both from the point of view of obtaining a better understanding of the (complicated) dynamics of these systems, and from its relevance to physical applications. Some of the techniques used are the construction of computer-assisted proofs of KAM theorems [1–3], the renormalization group approach [4, 5] and the stability analysis of neighbouring periodic orbits [6]. Sometimes it is also possible to exploit simple analytical properties of the conjugation to the irrational translation existing on the given torus [7].

The *critical function*, also sometimes called the *fractal diagram*, associates with each rotation number $\omega$ the breakdown threshold $K(\omega)$ of the corresponding invariant circle and has been numerically computed for several models, especially complex analytic and complex area-preserving maps [7–13]. It has a complicated

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nonlinar fractal structure, vanishes at all the rationals and it is almost everywhere discontinuous. These properties are a consequence of the fact that one is dealing with a difficult *small-divisor* problem: the fundamental obstruction to convergence of classical perturbation theory series expansions is number theoretical. These facts have also motivated an attempt to construct approximations to the critical function by exploiting its transformation properties under the modular group PSL(2, $\boldsymbol{Z}$) [14, 15]. Note that the action of PSL(2, $\boldsymbol{Z}$) can be used to generate the continued fraction expansion of an irrational number.

In the context of Siegel discs for complex analytic maps, and later of circle diffeomorphisms, Yoccoz [16] has proved that there is a purely arithmetic function, the *Brjuno function* [17, 18], whose ratio to the logarithm of the critical function is bounded for all irrational rotation numbers. Some of Yoccoz's ideas have been applied to several models [19], including the semistandard map (which is the complexification of the standard map [9, 12]). In this case one can prove that the convergence of the Brjuno function for a given rotation number $\omega$ is a necessary and sufficient condition for the existence of an invariant circle of rotation number $\omega$. Furthermore, there is numerical evidence that $B(\omega)$ gives a bounded approximation $\log(K(\omega))$ for $\omega \in [0, \frac{1}{2}]$. Moreover, this fact can be used to explain the 'modular smoothing' approach of [14, 15].

In this paper we study the standard map critical function and present numerical evidence that once again the Brjuno function gives an excellent approximation to $\log(K(\omega))$ (see also [20] where the numerical results were not sufficient to draw this conclusion).

The paper is organized as follows. First we recall some elementary facts on irrational numbers and continued fractions. After having introduced the Brjuno function we show how to calculate it for some classes of irrational numbers. In the third section we explain how we computed the critical function for the standard map and we describe and discuss our numerical results. In the conclusion we suggest some possible directions for future investigations. In the two appendices we compare the Brjuno function interpolation with the modular smoothing approach of Buric and Percival, and we prove that $K(\omega) > 0$ for all complex $\omega$ with a non-zero imaginary part.

## 2. Continued fractions and the Brjuno function

Let $[a_0, a_1, \ldots]$ denote the coefficients of the continued fraction expansion of an irrational number $\omega \in \boldsymbol{R} \backslash \boldsymbol{Q}$, recursively determined by $a_j = [1/\omega_j]$ and $\omega_j = \{1/\omega_{j-1}\}$ for all $j \geqslant 1$, where [ ] and { } denote the integer part and the fractional part of a real number respectively and $\omega_0 = \omega - [\omega]$, $a_0 = [\omega]$. The partial fractions

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}} \tag{2.1}$$

verify the well-known recurrences

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2} \end{aligned} \tag{2.2}$$

for all $k \geqslant 0$, with initial data $q_{-2} = p_{-1} = 1$, $q_{-1} = p_{-2} = 0$, and the inequalities

$$\frac{1}{q_k(q_k + q_{k+1})} < (-1)^k\left(\omega - \frac{p_k}{q_k}\right) < \frac{1}{q_k q_{k+1}}. \tag{2.3}$$

The modular group PSL(2, $\boldsymbol{Z}$) acts on $\boldsymbol{R}$ as follows: let $T \in \mathrm{PSL}(2, \boldsymbol{Z})$. For all $x \in \boldsymbol{R}$

$$Tx := \frac{ax + b}{cx + d} \tag{2.4}$$

where $a, b, c, d \in \boldsymbol{Z}$, $ad - bc = 1$.

Note that two irrational numbers $\omega$ and $\omega'$ are equivalent (i.e. their continued fractions coincide apart from finitely many terms) if and only if $\omega = T\omega'$ with $T \in \mathrm{PSL}(2, \boldsymbol{Z})$. In particular, one has that

$$\omega = \frac{p_{n-1}\omega_n + p_n}{q_{n-1}\omega_n + q_n} \tag{2.5}$$

and $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$.

The Brjuno function $B: \boldsymbol{R}\backslash\boldsymbol{Q} \to \boldsymbol{R}^+ \cup \{\infty\}$ is defined by means of the 'nearest-integer' continued fraction expansion. Given a real number $x$ let $\langle x \rangle$ denote the nearest integer to $x$ and $\|x\|$ the distance from $x$ to $\langle x \rangle$. For $\omega$ irrational define

$$\begin{aligned} b_k &= \langle \theta_{k-1}^{-1} \rangle \\ \theta_k &= \|\theta_{k-1}^{-1}\| \end{aligned} \tag{2.6}$$

for all $k \geqslant 1$, with initial data $b_0 = \langle \omega \rangle$, $\theta_0 = \|\omega\|$. Moreover, let $\beta_k = \prod_{i=0}^{k} \theta_k$ if $k \geqslant 0$, and $\beta_{-1} = 1$. $B(\omega)$ is defined by

$$B(\omega) := -\sum_{k=0}^{+\infty} \beta_{k-1} \log \theta_k. \tag{2.7}$$

This 'nearest-integer' continued fraction expansion is closely related to the continued fraction expansion. Thus, for all $k$ there exists an integer $n(k) \geqslant k$ such that $\beta_k = (-1)^n(q_n\omega - p_n)$ [7]. In fact $n(k)$ is defined inductively by $n(-1) = -1$ and $n(k+1) = n(k) + 2$ if $a_{n(k)+2} = 1$, $n(k+1) = n(k) + 1$ if $a_{n(k)+2} \geqslant 2$ for all $k \geqslant -1$.

By means of this relation between the two continued fractions one can show that the Brjuno function is finite if and only if $\omega$ verifies the *Brjuno condition* [17, 19]

$$\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} < +\infty. \tag{2.8}$$

Clearly $B(\omega) = B(\omega + 1) = B(-\omega)$. Moreover, for all $\omega \in \boldsymbol{R}\backslash\boldsymbol{Q} \cap (0, \frac{1}{2}]$, $B(\omega)$ verifies the functional equation

$$B(\omega) = -\log(\omega) + \omega B(\omega^{-1}). \tag{2.9}$$

From these relations one may read off immediately the behaviour of the Brjuno function under the action of the modular group PSL(2, $\boldsymbol{Z}$) which has generators $\omega \mapsto \omega + 1$ and $\omega \mapsto -1/\omega$.

The Brjuno function can easily be computed from some special classes of quadratic irrationals. Thus, consider the family $\omega_m = (\sqrt{m^2 + 4} - m)/2 = [0, m, m, \ldots]$. These irrationals are solutions of

$$m + \omega_m = \frac{1}{\omega_m} \tag{2.10}$$

and from (2.9) one immediately has that for all $m \geqslant 2$

$$B(\omega_m) = \frac{\log(\omega_m)}{\omega_m - 1} \tag{2.11}$$

whilst

$$B(\omega_1) = \frac{-\log(1-\omega_1)}{\omega_1} = \frac{-2\log(\omega_1)}{\omega_1}.$$

Another easily computed case is given by the so-called *noble numbers* which can be obtained by the application of a modular transformation to the golden mean $\omega_1 = (\sqrt{5}-1)/2 = [0, 1, 2, \ldots]$. Their continued fraction expansion $[0, a_1, \ldots, a_n, 1, 1, \ldots]$ differs from the expansion of the golden mean in only finitely many terms, i.e.

$$\omega_{a_1, a_2, \ldots, a_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \omega_1}}}}. \tag{2.12}$$

Let us denote by $b_{a_1, a_2, \ldots, a_n}$ the value of the Brjuno function $B(\omega_{a_1, a_2, \ldots, a_n})$. If $a_k \geqslant 2$ then $\omega_{a_k, a_{k+1}, \ldots, a_n} \in (0, \frac{1}{2})$ and

$$b_{a_k, a_{k+1}, \ldots, a_n} = -\log \omega_{a_k, a_{k+1}, \ldots, a_n} + \omega_{a_k, a_{k+1}, \ldots, a_n} b_{a_{k+1}, \ldots, a_n}. \tag{2.13}$$

If $a_k = 1$, so that $\omega_{a_k, a_{k+1}, \ldots, a_n} \in (\frac{1}{2}, 1)$, it suffices to observe that

$$\frac{1}{1 - \omega_{a_k, a_{k+1}, \ldots, a_n}} = a_k + a_{k+1} + \omega_{a_{k+2}, \ldots, a_n} \tag{2.14}$$

to find that

$$b_{a_k, a_{k+1}, \ldots, a_n} = -\log(1 - \omega_{a_k, a_{k+1}, \ldots, a_n}) + (1 - \omega_{a_k, a_{k+1}, \ldots, a_n}) b_{a_{k+2}, \ldots, a_n}. \tag{2.15}$$

A quadratic irrational [21] is any number of the form $\omega = (P \pm \sqrt{D})/Q$ where $P, Q, D$ are integers, $D$ is positive and not a perfect square. It is thus the solution of the quadratic equation $Q^2x^2 - 2PQx + P^2 - D = 0$. The continued fraction expansion of all quadratic irrationals is either purely periodic, like the expansion of the golden mean, or is periodic from some point onward. Thus, for example $\sqrt{2} = [1, 2, 2, 2] = [1, \bar{2}]$ and $\sqrt{9} = [4, \overline{2, 1, 3, 1, 2, 8}]$. The converse is also true, any purely periodic continued fraction, or any fraction which is periodic from some point onward, represents some quadratic irrational.

It should by now be clear that one can compute the Brjuno function for all quadratic irrationals too. For example, take the purely periodic case, $\omega = [\overline{a_0, a_1, \ldots, a_n}] = [a_0, a_1, \ldots, a_n, \omega]$, and suppose $a_1 \geqslant 2$. Then

$$B(\omega) = B(\omega - a_0) = -\log(\omega - a_0) + B([a_1, \ldots, a_n, \omega]) \tag{2.16}$$

and again one can iterate this procedure, leading to an explicit formula for $B(\omega)$.

## 3. The critical function for the standard map

The *standard map* is given by

$$\begin{aligned} y_{n+1} &= y_n + K \sin x_n \\ x_{n+1} &= x_n + y_{n+1} (\mathrm{mod}\, 2\pi) \end{aligned} \tag{3.1}$$

where $(y_n, x_n) \in \boldsymbol{R} \times \boldsymbol{R}/2\pi\boldsymbol{Z}$ and $K$ is a real-valued parameter. The best way of computing the critical function for the standard map—at least conceptually—would probably be by using a perturbative approach. Thus, given a 'strongly' irrational rotation number $\omega \in \boldsymbol{R}\backslash\boldsymbol{Q}$, the KAM theorem states that there exists a corresponding invariant circle for sufficiently small $K$. On this circle the standard map is analytically conjugate to the irrational translation

$$T_\omega : t \mapsto t + 2\pi\omega (\mathrm{mod}\, 2\pi). \tag{3.2}$$

This means [22] that there exists an analytic function $u(t, K)$ which satisfies the functional equation

$$u(t + 2\pi\omega, K) - 2u(t, K) + u(t - 2\pi\omega, K) = K \sin(t + u(t, K)) \tag{3.3}$$

such that

$$x = t + u(t, K) = t + \sum_{n=1}^{\infty} K^n \sum_{m=-n}^{n} u_{n,m} \mathrm{e}^{\mathrm{i}mt}. \tag{3.4}$$

This is—for the standard map case—what Poincaré called a *Lindstedt series* [23]. One may now define the standard map *critical function* by

$$K(\omega) := \inf_{t \in [0, 2\pi]} \left[ \limsup_{n \to \infty} |u_n(t)|^{1/n} \right]^{-1} \tag{3.5}$$

where $u_n(t) = \sum_{m=-n}^{n} u_{n,m} \mathrm{e}^{\mathrm{i}mt}$. $K(\omega)$ thus gives the parameter value at which the circle of rotation number $\omega$ ceases to be analytic. It is generally believed that typically this also corresponds to the complete breakdown of the circle. Clearly from (3.3) it follows that $K(\omega) = K(\omega + 1) = K(-\omega)$ and $K(\omega) = 0$ at all rationals. Moreover, one can derive a recurrence for the coefficients $u_n(t)$ of $u(t, K)$ (see appendix 2) and in principle obtain numerical estimates of $K(\omega)$. This approach has the disadvantage that it is extremely time consuming. One would like to have reasonably accurate estimates of the critical function (say to at least three significant digits as in [7]), which would imply computing the series expansion in (3.4) up to very high orders.

On the other hand, a variety of heuristic criteria for determining the existence or not of invariant circles do exist and can be used to obtain extremely good numerical approximations to $K(\omega)$. The most useful of these in practice is Greene's residue criterion [6]. This is based on the empirical observation that the stability of an invariant circle appears to be closely related to the linear stability of nearby periodic orbits.

It is convenient to measure the linear stability of a periodic orbit of period $q$ for an area-preserving map $f$ (such as the standard map (3.1)) in terms of its residue $R$ defined as

$$R = \frac{2 - \mathrm{Trace}(Df^q)}{4} \tag{3.6}$$

where $Df^q$ is the Jacobian matrix of $f^q$ at any point on the periodic orbit. Recall that by the Poincaré–Birkhoff theorem any area-preserving twist map has at least two periodic orbits of rotation number $p/q$ for every rational $p/q$. One orbit has a non-negative residue which we shall denote $R_{p/q}$ and the other orbit has a non-positive residue (see, for instance, [24, 25]).

Given an irrational rotation number $\omega$ we can look at the behaviour of the $R_{p_k/q_k}$ as $k \to \infty$ where $p_k/q_k$ is the $k$th convergent of $\omega$ given by (2.1). Greene observed numerically that there appears to be a close relationship between the asymptotic behaviour of $R_{p_k/q_k}$ and the existence of an invariant circle of rotation $\omega$:

(i) $R_{p_k/q_k} \to 0$ as $k \to \infty$ if and only if $f$ has a smooth rotational invariant circle of rotation number $\omega$;

(ii) $R_{p_k/q_k} \to \infty$ as $k \to \infty$ if and only if $f$ has no rotational invariant circle of rotation number $\omega$;

(iii) $R_{p_k/q_k}$ converges to neither 0 nor $\infty$ as $k \to \infty$ if and only if $f$ has a critical rotational invariant circle of rotation number $\omega$.

This gives rise to the following algorithm for computing the parameter value $K(\omega)$ at which an invariant circle of rotation number $\omega$ breaks in a one-parameter family $f_K$ of area-preserving maps such as the standard map. Choose some fixed residue $R_c > 0$ and for each $n$ find the parameter value $K_n$ such that $R_{p_n/q_n}(K_n) = R_c$ (typically this is unique). Then it appears that $K_n \to K(\omega)$ as $n \to \infty$.

There are two main problems with this scheme, particularly when the $a_n$ of the continued fraction expansion of $\omega$ are large or behave irregularly. The first is that the choice of $R_c$ is crucial for the rapid convergence of the $K_n$. Unfortunately, the optimal choice is different for each $\omega$ (or rather each equivalence class of $\omega$ under $\mathrm{PSL}(2, \boldsymbol{Z})$) and there is no *a priori* way of determining this choice. The second problem is the sheer difficulty of numerically finding the $p_n/q_n$ periodic orbits for large $n$. This is because for such orbits $R_{p_n/q_n}(K)$ varies very rapidly with $K$. Hence if the initial estimate for $K_n$ is not very accurate a numerical orbit finder may well end up searching for a periodic orbit with a very large residue. Such an orbit will be extremely unstable and hence the orbit finder is unlikely to succeed.

In [26] one of us showed how an approximate renormalization scheme due to MacKay [5] can be used to greatly improve the performance of the residue criterion. Two particular algorithms are described there. The first of these is not suitable for the purposes of this paper, since its results are dependent on the approximate renormalization. The second scheme, however, merely uses the approximate renormalization to provide good predictions for numerical root finders and hence the values it computes are completely independent of the approximate renormalization. This independence is extremely important in our case since the approximate renormalization bears a marked resemblance to the Brjuno functional equation (2.9) and hence if it was allowed to influence our numerical determination of $K(\omega)$ it would make our comparison with the Brjuno function highly suspect.

We shall give only a brief description of the numerical algorithm used; further details can be found in [26]. It turns out to be convenient to work in terms of the *intermediate approximants* to $\omega$ rather than the convergents. These are defined to be rationals of the form $[a_0, a_1, \dots, a_{n-1}, b]$ for $1 \leqslant b \leqslant a_n$. The intermediates are thus more frequent and more closely spaced than the convergents. As a consequence, the parameter values $K_i$ corresponding to successive intermediates are closer to each other. This both greatly aids the numerical determination of the $K_i$ and makes it easier to estimate their limit. Whilst it is possible to generate the intermediates in terms of the continued fraction, it is often more convenient to use Farey subdivision. As we shall see below this is also the most natural way in which to present the approximate renormalization. Thus, set $\omega_0 = \omega, p_0 = 1, q_0 = 0, P_0 = 1, Q_0 = 0$ and define $\omega_i$, $p_i/q_i$ and $P_i/Q_i$ inductively by

$$\begin{aligned} P_{i+1} &= p_i + P_i \\ Q_{i+1} &= q_i + Q_i \end{aligned} \tag{3.7}$$

if $\omega_i < 1$

$$\begin{aligned} \omega_{i+1} &= \frac{1}{\omega_i} - 1 \\ p_{i+1} &= P_i \\ q_{i+1} &= Q_i \end{aligned} \tag{3.8}$$

if $\omega_i > 1$

$$\begin{aligned} \omega_{i+1} &= \omega_i - 1 \\ p_{i+1} &= p_i \\ q_{i+1} &= q_i. \end{aligned} \tag{3.9}$$

Then successive $P_i/Q_i$ are precisely the intermediate approximants of $\omega$. The $P_i/Q_i$ corresponding to $\omega_i < 1$ give the convergents. Note that $\omega_i = -(Q_i\omega_0 - P_i)/(q_i\omega_0 - p_i)$ and $P_iq_i - p_iQ_i = \pm 1$ for all $i$. Thus, $\omega = (p_i\omega_i + P_i)/(q_i\omega_i + Q_i)$ and hence $\omega$ belongs to the interval whose end-points are $p_i/q_i$ and $P_i/Q_i$ for all $i$.

The approximate renormalization scheme seeks to model the exact renormalization (which acts on the space of all area-preserving twist maps [15]) by only four parameters $(\omega_i, \nu_i, r_i, R_i)$. Here $\omega_i$ is the 'renormalized' rotation number defined as above, whilst $\nu_i = (Q_i\nu_0 + P_i)/(q_i\nu_0 + p_i)$ encodes the past history of renormalizations. It can also be defined inductively by $\nu_{i+1} = 1 + 1/\nu_i$ if $\omega_i < 1$ and $\nu_{i+1} = \nu_i + 1$ if $\omega_i > 1$. The initial choice of $\nu_0$ is arbitrary but since $P_i \to \omega Q_i$ and $p_i \to \omega q_i$ we have $\nu_i \to Q_i/q_i$ as $i \to \infty$ independently of $\nu_0$. The $r_i$, $R_i$ correspond to the residues of the $P_i/Q_i$ and $p_i/q_i$ periodic orbits respectively, that is, $r_i = R_{p_i/q_i}$ and $R_i = R_{P_i/Q_i}$. Note that more usually the scheme is expressed in terms of the logarithms of $r_i$ and $R_i$ and that $\nu_i$ is often denoted by $k_i$.

The renormalization scheme can be used to give a good estimate of $K_{i+1}$ based on previously found orbits. Thus, suppose we have found $K_i$ such that $R_{P_i/Q_i}(K_i) = R_c$ and wish to compute $K_{i+1}$ such that $R_{P_{i+1}/Q_{i+1}}(K_{i+1}) = R_c$. Then, assuming that $K$ is reasonably close to $K(\omega)$, the approximate renormalization predicts that

$$R_{P_{i+1}/Q_{i+1}}(K) = U(\nu_i)R_{P_i/Q_i}(K)R_{p_i/q_i}(K) \tag{3.10}$$

where

$$U(\nu) = A\left(\frac{1}{2\pi^2}\frac{(1+\nu_i)^6}{\nu_i^3}\right)^{\alpha}. \tag{3.11}$$

Originally, based on a perturbation analysis of a two-wave Hamiltonian [4], MacKay [5] used $A = \alpha = 1$. However, subsequently [17, 28] much better agreement was found when $A \approx 1.97 \approx \exp(0.677)$ and $\alpha \approx 0.537$. With this choice the approximation (3.10) is remarkably accurate as long as the continued fraction terms of $\omega$ are less than about 20. Now $p_i/q_i = P_j/Q_j$ for some $j < i$ and hence we have already found the $p_i/q_i$ orbit at some nearby parameter value. We can approximate $R_{P_i/Q_i}$ and $R_{p_i/p_i}$ by their linear parts, that is,

$$\begin{aligned} R_{P_i/Q_i}(K) &\approx R_{P_i/Q_i}(K_i) + \partial_K R_{P_i/Q_i}(K_i)\cdot(K-K_i) \\ R_{p_i/p_i}(K) &\approx R_{P_j/Q_j}(K_j) + \partial_K R_{P_j/Q_j}(K_j)\cdot(K-K_j) \end{aligned} \tag{3.12}$$

where $\partial_K$ indicates differentiation with respect to the parameter $K$ and can either be computed directly or obtained from a finite difference estimate. Using (3.10) and (3.12) we can obtain an estimate $\tilde{K}_{i+1}$ for the parameter value such that $R_{P_i/Q_i}(\tilde{K}_{i+1}) \approx R_c$. More precisely, $\delta = \tilde{K}_{i+1} - K_i$ will be the solution of the quadratic

$$\partial_K R_i \partial_K R_j \delta^2 + (\partial_K R_j R_{P_i/Q_i}(K_i) + \partial_K R_i \tilde{R}_j)\delta + \tilde{R}_j R_i - \frac{R_c}{U(\nu_i)} = 0 \tag{3.13}$$

where

$$\begin{aligned} \tilde{R}_j &= R_{P_j/Q_j}(K_j) + \partial_K R_{P_j/Q_j}(K_j)\cdot(K_i - K_j) \\ \partial_K R_i &= \partial_K R_{P_i/Q_i}(K_i) \\ \partial K R_j &= \partial_K R_{P_j/Q_j}(K_j). \end{aligned}$$

This can be used as an initial guess for finding $K_{i+1}$. Since in practice $\tilde{K}_{i+1}$ seems to be an extremely good estimate for $K_{i+1}$ usually only three or four iterations of a root finder are required to converge to $K_{i+1}$.

Finally note that for reversible area-preserving maps, which are the cases usually studied (e.g. [29]) and include the standard map, the existence of symmetries implies that finding a $p_n/q_n$ periodic orbit is only a one-dimensional search and hence determining $K_i$ is a two-dimensional one. This greatly eases the computational load of these algorithms.

Most of our numerical work was carried out using the above version of Greene's residue criterion. In addition we have also used a method (described in [20]) based on converse KAM theory [29–31] and the Lindstedt series (3.4). Whilst this is less precise, it can be used closer to the resonances. The results obtained by both algorithms were in good agreement. However, due to its better precision, all the results obtained below were obtained using the residue criterion.

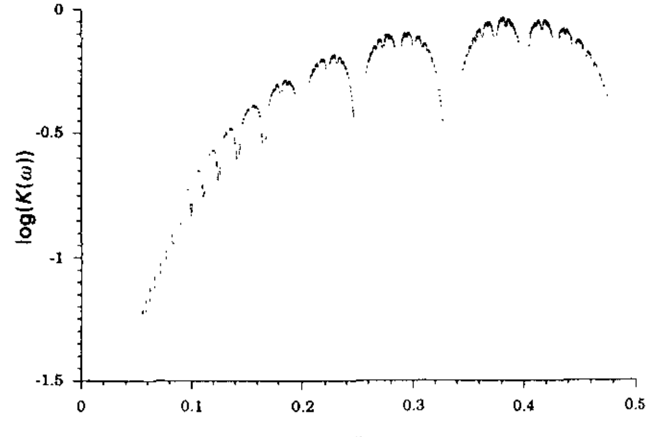


**Figure 1.** Plot of the logarithm of the critical function of the standard map for some 2300 rotation numbers.

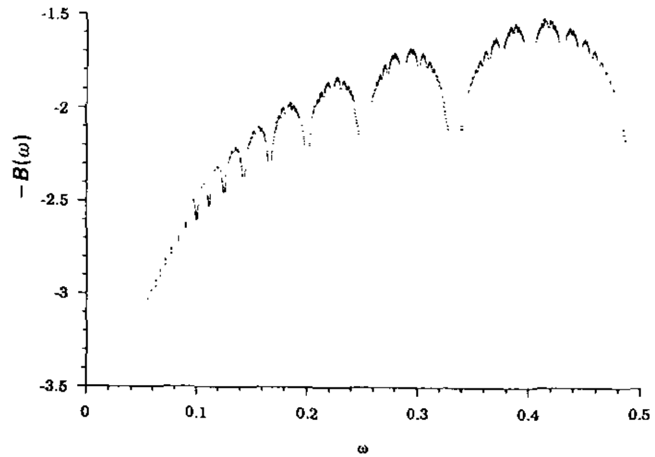


**Figure 2.** The Brjuno function $B(\omega)$ for the same rotation numbers as figure 1.

Figure 1 shows the logarithm of the critical function for the standard map for some 2300 rotation numbers $\omega \in [0, \frac{1}{2}]$. These include all nobles up to a depth of 12 in the Farey tree as well as a selection of other rotation numbers chosen to explore more closely the resonances at 0, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Figure 2 gives the Brjuno function for the same set of rotation numbers. There is a great deal of qualitative similarity between the two figures. To further investigate the relationship between the two functions figure 3 shows a plot of $\log(K(\omega))$ against $B(\omega)$, again for the same set of $\omega$. Clearly $B(\omega)$ gives a remarkably good approximation to $\log(K(\omega))$, especially when one considers that $B(\omega)$ is purely number-theoretic function whilst $K(\omega)$ depends on the full nonlinear behaviour of the standard map.

The form of figure 3 suggests a relationship between the two functions of the form $\log(K(\omega)) \approx \alpha - \beta B(\omega)$. *A priori*, there are strong reasons for suggesting that $\beta = 2$. This is because the standard map takes the form of a second-order recurrence (i.e. $x_{n+1} - 2x_n + x_{n-1} = K \sin x_n$) and previous work with other small-divisor

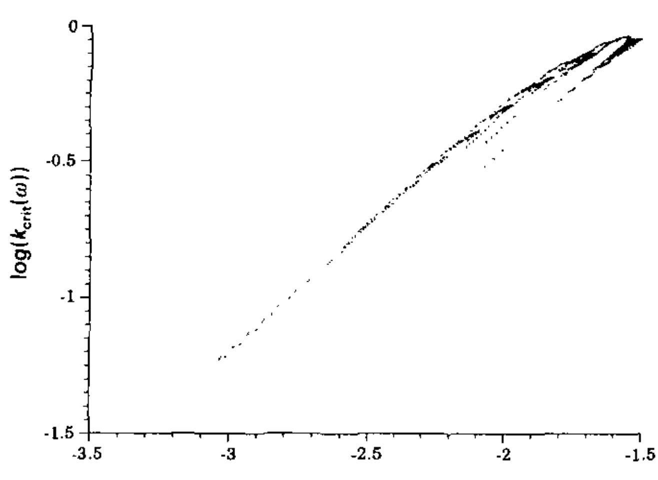


**Figure 3.** Plot of $\log(K(\omega))$ against $B(\omega)$ for the same rotation numbers as figure 1.

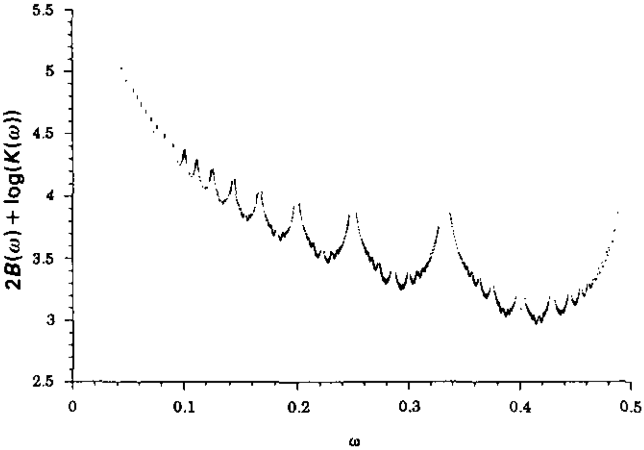


**Figure 4.** Plot of $\log(K(\omega)) + 2B(\omega)$ for the same rotation numbers as figure 1.

problems has always given a value of $\beta$ equal to the order of the recurrence. Thus, in the case of Siegel discs and circle diffeomorphisms (both first-order recurrences) Yoccoz [16, 32] proves that $\log(K(\omega)) + B(\omega)$ is a bounded continuous function, whilst one of us [19] has given strong numerical evidence that for the semistandard map (a second-order recurrence) it is $\log(K(\omega)) + 2B(\omega)$ which is continuous and bounded. We thus plotted $\log(K(\omega)) + 2B(\omega)$ for the standard map (figure 4). Unfortunately, this does not appear to have bounded behaviour and in fact seems very similar to the plot of $B(\omega)$ itself, suggesting a value of $\beta$ closer to 1 would be more appropriate. A more detailed examination of figure 3 and in particular a linear fit of $\log(K(\omega))$ against $B(\omega)$ for several sequences of $\omega$ converging to various resonances finally led us to a value of about 0.9. This is illustrated in figure 5, plotted at the same scale as figure 4, which shows that $1.5–0.9B(\omega)$ gives an excellent approximation to $\log(K(\omega))$ for the standard map. This suggests the following three conjectures.

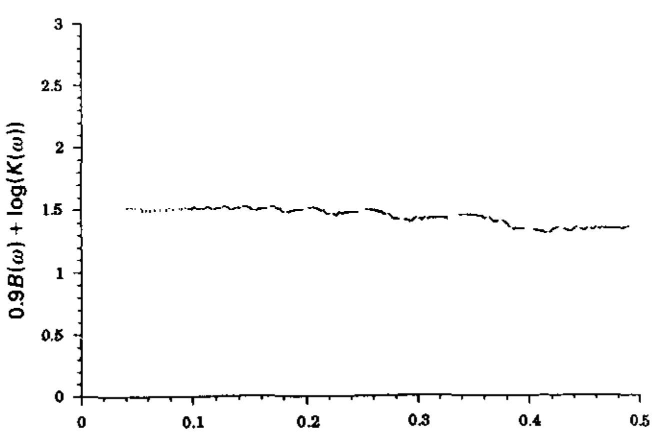


**Figure 5.** Plot of $\log(K(\omega)) + 0.9B(\omega)$ for the same rotation numbers as figure 1.

*Conjecture 3.1.* The standard map has an analytic rotational invariant circle of rotation number $\omega$ for $K>1$ if and only if $B(\omega)<\infty$, in other words, if and only if $\omega$ verifies the Brjuno condition (2.8).

*Conjecture 3.2.* There is some value $\beta>0$ such that $C_\beta = \log(K(\omega)) + \beta B(\omega)$ is a bounded function for $\omega \in [0, \frac{1}{2}]$.

*Conjecture 3.3.* $C_\beta$ for this value of $\beta$ is continuous.

Each conjecture implies the one before. Due to the properties of $K(\omega)$ and $B(\omega)$, $C_\beta(\omega)$ can be extended to an even periodic function defined on $\boldsymbol{R}$ by means of the relations $C_\beta(\omega) = C_\beta(\omega+1) = C_\beta(-\omega)$. Figure 4 suggests that conjecture 3.2 holds with $\beta \approx 0.9$. However, the real value of $\beta$, if it indeed exists, depends on the behaviour of $(K(\omega))$ as $\omega$ approaches rational values. Since we have only been able to determine $(K(\omega))$ for $\omega$ which are still quite far from rationals, it is entirely possible that a substantially different value of $\beta$ is appropriate, even possibly equal to 2 as we had originally conjectured. Note that since $B(\omega)\to\infty$ as $\omega$ tends to any rational, there is at most one value of $\beta$ such that $C_\beta$ is bounded. We are somewhat more hesitant in suggesting that conjecture 3.3 holds, though note its close relationship to modular smoothing (see appendix 1). Also, we believe that it is almost inconceivable that $C_\beta$ should be anything better than continuous.

In order to investigate the behaviour of $C_\beta$ near resonances we computed $K(\omega)$ for several sequences of rotation numbers converging to the resonances at 0, 1 (which is the same as 0 for the standard map), $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. These were of the form $\omega = [m_\infty]$, $[m, 1_\infty]$, $[q, m_\infty]$, $[q, m, 1_\infty]$, $[q-1, 1, m_\infty]$ and $[q-1, 1, m, 1_\infty]$. Here $m_\infty$ denotes an infinite sequence of $m$s, $q \in \{1, 2, 3, 4\}$ and $m \to \infty$. Figure 6 shows a typical example. We see that asymptotically we get an extremely good fit of $\log(K(\omega))$ against $1.5099 - 0.89715B(\omega)$. All the other cases studied gave similar fits, though with different values for $\alpha$ and $\beta$. These are listed in table 1. From the point of view of the above conjectures it is then crucial to decide whether the values

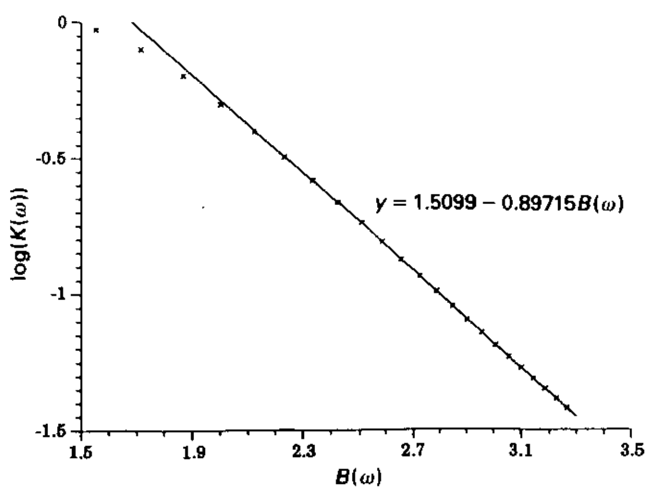


**Figure 6.** Plot of $\log(K(\omega_n))$ against $B(\omega_n)$ for $\omega_n = [n, 1_\infty]$ with $n$ from 1 to 24. The straight line represents the best linear fit in the range $n = 9$–24.

**Table 1.** Coefficients of the best linear fits of $\log(K(\omega))$ against $\alpha - \beta B(\omega)$ for a selection of sequences of rotation numbers.

| Continued fraction | $\alpha$ | $\beta$ | Range of $n$ for fit |
|---|---|---|---|
| $[n_\infty]$ | 1.6278 | 0.94572 | 7–16 |
| $[n, 1_\infty]$ | 1.5099 | 0.89715 | 9–24 |
| $[1, n_\infty]$ | 1.6249 | 0.94418 | 6–16 |
| $[2, n_\infty]$ | 1.4048 | 0.92849 | 7–16 |
| $[2, n, 1_\infty]$ | 1.2960 | 0.87113 | 8–24 |
| $[3, n_\infty]$ | 1.5261 | 0.94545 | 7–18 |
| $[2, 1, n_\infty]$ | 1.5153 | 0.92721 | 7–16 |
| $[3, n, 1_\infty]$ | 1.4176 | 0.88900 | 8–18 |
| $[2\,1, n, 1_\infty]$ | 1.4642 | 0.90015 | 8–18 |
| $[4, n_\infty]$ | 1.5555 | 0.93456 | 7–16 |
| $[3, 1, n_\infty]$ | 1.5875 | 0.94136 | 7–16 |
| $[4, n, 1_\infty]$ | 1.4707 | 0.89234 | 8–18 |
| $[3, 1, n, 1_\infty]$ | 1.4970 | 0.89587 | 8–18 |

of $\beta$ are significantly different or not, and similarly for all the values of $\alpha$ corresponding to the same resonance. On the whole we believe that this will turn out to be true, even though there do appear to be significant differences in table 1. We believe that these are due to the fact that we are not sufficiently far into the asymptotic regime. Thus, for instance, the smallest value of $K$ we have calculated is around 0.24 whilst of course the full range of $K$ in the standard map is from 0 to approximately 0.971.

Finally, we point out one other similarity between $B(\omega)$ and $\log(K(\omega)$. Let $\omega_i$ be a sequence of rotation numbers such that $\omega_{i+1} = 1/(1+\omega_i)$, with $\omega_0$ arbitrary. Thus, passing from $\omega_i$ to $\omega_{i+1}$ corresponds to adding an extra 1 at the front of the continued fraction expansion of $\omega_i$. Clearly $\omega_i \to \gamma$ as $i \to \infty$ where $\gamma$ is the golden mean $(\sqrt{5}+1)/2$. Then, asymptotically,

$$|B(\omega_i) - B(\gamma)| \sim |\omega_i - \gamma|^{1/2}.$$

In other words, at the golden mean the envelope of $B(\omega)$ takes the form of a square root cusp. The same results holds at any other noble rotation number. On the other hand a renormalization analysis shows that the analogous result for $\log(K(\omega)$ is

$$|\log(K(\omega_i)) - \log(k(\gamma))| \sim |\omega_i - \gamma|^{\eta}$$

with $\eta \simeq 0.5063$ (e.g. [28]), which is indeed very close to $\frac{1}{2}$.

## 4. Conclusion

We state four open problems:

(i) First of all to prove (or disprove) our conjectures, i.e.

(*a*) *The standard map has an analytic rotational invariant circle of rotation number $\omega$ for $K>1$ if and only if $B(\omega)<\infty$, in other words, if and only if $\omega$ verifies the Brjuno condition* (2.8).

(*b*) *There is some value $\beta>0$ such that $C_\beta = \log(K(\omega)) + \beta B(\omega)$ is a bounded function for $\omega \in [0, \frac{1}{2}]$.*

(*c*) *$C_\beta$ is continuous.*

As regards the first conjecture, the convergence of the Lindsted series (3.4) when $\omega$ verifies the Brjuno condition can probably be proved by adapting Russmann's version of KAM theory [33–35]. The difficult part of this conjecture is to show that the Brjuno condition is not only sufficient but also necessary. One possible approach would be to try to adapt Eliasson's proof of KAM theory [36–38], obtained by means of the majorant series method and Siegel's lemma [39], so as to include Brjuno's counting lemma [17, 18]. This is unlikely to be easy since Eliasson's argument is very subtle and not yet completely understood.

Support of this conjecture comes also from the fact that the stable manifold of the approximate renormalization operator of MacKay exists for $\log(K) > -\infty$ if and only if the Brjuno condition is verified [5]. The existence of a stable manifold for a given $\omega$ is exactly analogous to the existence of an invariant circle for an area-preserving map.

The second and third conjectures (and their analogous statements in [16] and [19] for complex analytic and/or area-preserving maps) seem to be beyond the range of currently available techniques.

(ii) In [28], MacKay and Stark present evidence that for the standard map (and using a universality argument for a wide class of other area-preserving maps) the local maxima of $K(\omega)$ occur at very special values of $\omega$. More precisely, suppose that $p/q$ and $P/Q$ are neighbouring rationals (i.e. $pQ - Pq = \pm 1$). Then if $1/qQ$ is sufficiently small the maximum of $K(\omega)$ in the interval $[p/q, P/Q]$ occurs at $\omega = (p + \mu P)/(q + \mu Q)$ with $\mu$ one of $1 + \gamma$, $\gamma$, $1/\gamma$ and $1/(1 + \gamma)$ where $\gamma$ is the golden mean $(\sqrt{5} + 1)/2$. The same holds for the approximate renormalization, through the set of allowable $\mu$ has to be expanded to include $2 + \gamma$ and $1/(2 + \gamma)$. It would be interesting to see if one could prove something similar for $B(\omega)$.

(iii) As suggested at the end of appendix 1, the relation $\log(K(\omega)) = \beta B(\omega) - C_\beta(\omega)$ seems to be the beginning of an (asymptotic) expansion of $K(\omega)$ of the critical function. It would be nice to devise ways of computing higher-order terms. One approach is clearly via modular smoothing [14, 15].

(iv) Does the Brjuno function admit a regular extension to some domain in the complex $\omega$ plane? This should be reasonable since, if the imaginary part of $\omega$ is different from zero, the convergence of the perturbation series—solution of (3.3)—is assured, as one can see by applying directly a standard majorant series argument (a sketch of the proof is given in the appendix 2). Moreover, one can use standard KAM arguments to prove the convergence of the perturbation series uniformly with respect to rotation numbers $\omega$ which belong to a complex cone, whose vertex is a real diophantine number (see [40] for more details). These two facts suggest that $B(\omega)$, for real $\omega$, might actually be the boundary value of some regular function of the complex upper-half plane, probably intimately related to modular forms.

It is interesting to remark that the first analysis of the behaviour of small-divisor problems and Lindstedt series for complex rotation numbers dates back to Kolmogorov [41] and Arnol'd [42], and to their studies about the linearization of vector fields on the two-dimensional torus. They conjectured that the Lindstedt series, regarded as a function of $\omega$, were *uniform monogenic functions* in the sense of E Borel [43]. These generalize the notions of analytic function and of the process of analytic continuation to functions which have a dense set of zero measure of singularities. In our case this set corresponds to those real rotation numbers not satisfying the Brjuno condition.

## Acknowledgments

One of us (SM) is grateful to INFN (sections of Bologna and Firenze) for the provision of computer facilities.

## Appendix 1. Modular smoothing and the Brjuno function

In this appendix we discuss the relationship between the Brjuno function and the modular smoothing approach of Buric, Percival and Vivaldi [6]. They consider the functions

$$L_1(\omega) = \log K(\omega) - \omega \log k(\omega^{-1}) \tag{A1.1}$$

and

$$L_2(\omega) = (\omega + 1)L_1(\omega + 1) - \omega L_1(\omega). \tag{A1.2}$$

defined for $\omega \in \boldsymbol{R}_+$. They provide numerical and analytic evidence that:

(i) $L_1$ is continuous on $\boldsymbol{R}_+$ and the limits as $\omega \to 0$ and $\omega \to +\infty$ exist and are respectively equal to $-\infty$ and $+\infty$. In particular, $L_1$ is bounded at all rational $\omega \neq 0$.

(ii) $L_2$ is everywhere differentiable, and the limits as $\omega \to 0$ and $\omega \to +\infty$ exist and are respectively equal to 0 and to $+\infty$.

The properties of $L_1$ all follow from our conjecture 3.3, though unfortunately the converse is not true. Furthermore, the Brjuno function approach yields considerable insight into why modular smoothing should work. Thus, observe that

$$L_1(\omega) = C_\beta(\omega) - \omega C_\beta(\omega^{-1}) - \beta L_B(\omega)$$

where

$$L_B(\omega) = B(\omega) - \omega B(\omega^{-1}).$$

Thus, $L_B$ is the result of applying modular smoothing to the Brjuno function. Figure 7 shows a plot of $L_B$ on the interval $[\frac{1}{4}, 1]$. As we shall see below, $L_B$ has all the properties claimed above for $L_1$. This implies that the properties of $L_1$ follow as an immediate corollary to conjecture 3.3. So it remains to show the following.

*Lemma A1.1.* $L_B$ is continuous on $\boldsymbol{R}_+$ and the limits as $\omega \to 0$ and $\omega \to +\infty$ exist and are respectively equal to $-\infty$ and to $+\infty$.

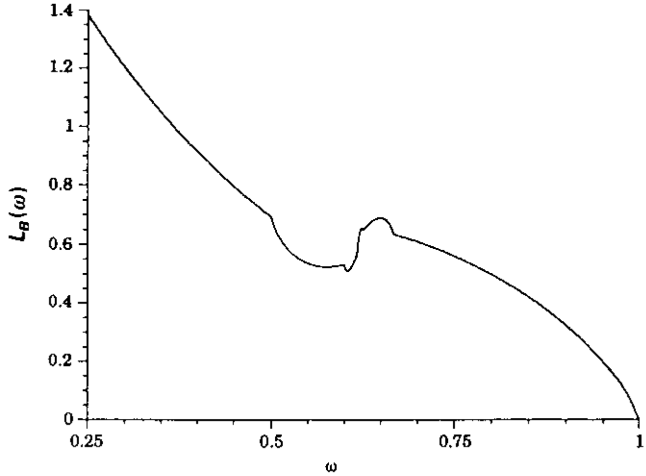


**Figure 7.** The function $L_B(\omega) = B(\omega) - \omega B(\omega^{-1})$ in the interval $\omega \in (\frac{1}{4}, 1]$. Note that for $\omega > 1$ we have $L_B(\omega) = -\omega L_B(\omega^{-1})$ and for $\omega \in (0, \frac{1}{2}] L_B(\omega) = -\log(\omega)$.

*Proof.* A simple calculation shows that

$$L_B(\omega)=\begin{cases}-\log(\omega) & \text{for } \omega\in[0,\tfrac{1}{2}]\\ -\log(1-\omega)-\omega L_B(\omega^{-1}-1) & \text{for } \omega\in[\tfrac{1}{2},1]\\ -\omega L_B(\omega^{-1}) & \text{for } \omega\in[1,\infty].\end{cases} \tag{A1.3}$$

Now define the map $T$: $[\frac{1}{2}, 1]\rightarrow[0, 1]$ by $T(\omega)=\omega^{-1}-1$. Let $F_i$ be the $i$th Fibonacci number defined by $F_0=0$, $F_1=1$ and $F_{i+1}=F_i+F_{i-1}$. Let $z_i$ be the point $z_i=F_i/F_{i+1}$, $I_i$ the interval between $z_i$ and $z_{i+1}$ and $J_i$ the interval between $z_i$ and $z_{i+2}$. Note that $I_i=J_i\cup I_{i+1}$. Since $I_i\subseteq[\frac{1}{2}, 1]$ for $i\geqslant 1$, $T$ is defined on $I_i$ for all $i\geqslant 1$. In fact, $T(I_i)=I_{i-1}$ and $T(z_i)=z_{i-1}$. Now consider $\omega\in[\frac{1}{2}, 1]$. By induction we get that

$$L_B(\omega)=-\log(\omega)+\sum_{i=1}^{n}\prod_{j=0}^{i}(-T^j(\omega))\log(T^i(\omega)$$
$$+\prod_{j=0}^{n-1}(-T^j(\omega))[\log(T^n(\omega))+L_B(T^n\omega))] \tag{A1.4}$$

for all $n$ such that $T^i(\omega))\in[\frac{1}{2}, 1]$ for $i=0,\ldots, n-1$. Then if $T^n(\omega))\in[0,\frac{1}{2}]$ we get

$$L_B(\omega)=-\log(\omega)+\sum_{i=1}^{n}\prod_{j=0}^{i}(-T^j(\omega))\log(T^i(\omega)). \tag{A1.5}$$

Note that $T^i(\omega)\in[\frac{1}{2}, 1]$ for $i=0,\ldots, n-1$ and $T^n(\omega)\in[0,\frac{1}{2}]$ if and only if $\omega\in J_n$ and if and only if precisely the first $n$ terms of the continued fraction expansion of $\omega$ are equal to 1 (and the $(n+1)$th term is larger than 1). Equation (A1.3) defines $L_B(\omega)$ for all $\omega\in[\frac{1}{2}, 1]$ except for $\omega=\omega_1=(\sqrt{5}-1)/2$. This can either be computed directly from the value of $B(\omega_1)$ or from (A1.3). In either case we get $L_B(\omega_1)=(1-\omega_1)B(\omega_1)=-2\beta_1\log(\omega_1)$. It is now clear that $L_B$ is continuous, except possibly at the points $z_i$ and $1/z_i$ for $i>0$ and $\omega_1$ and $1/\omega_1$.

Now either (A1.3) or (A1.4) gives $L_B(\omega)=-\omega\log(\omega)-(1-\omega)\log(1-\omega)$ for $\omega\in[\frac{2}{3}, 1]$. Applying (A1.3) we get $L_B(\omega)=-\omega\log(\omega)+(\omega-1)\log(\omega-1)$ for $\omega\in[1,\frac{3}{2}]$. Thus, $L_B$ is well defined and continuous at $\omega=1$. Similarly, again from either A1.3 or A1.4 we have $L_B(\omega)=-\omega\log(\omega)-\omega\log(1-\omega)+(2\omega-1)\log(2\omega-1)$ for $\omega\in[\frac{1}{2},\frac{2}{3}]=J_2$. Hence, $L_B(\frac{1}{2})=\log(2)$ and $L_B$ is continuous at $\omega=\frac{1}{2}$. Now observe that $T^n(z_{n+2})=z_2=\frac{1}{2}$ and that $T^n$ is continuous in the neighbourhood of $z_{n+2}$. Hence, $\log(T^n(\omega))+L_B(T^n\omega))$ is continuous in the neighbourhood of $z_{n+2}$ and of course so is $\log(T^i(\omega))$ for $0<i<n$. Hence, from (A1.4) we conclude that $L_B$ is continuous at $z_{n+2}$ for all $n>0$. Since we have already proved continuity at $z_2=\frac{1}{2}$ and $z_1=1$ we have continuity at all $z_n$. From (A1.3) we immediately get continuity at $1/z_n$. Thus, it remains to prove the continuity of $L_B$ at $\omega_1$. In fact we get the following.

*Lemma A1.2.* Given any $\eta$ such that $0<\eta<\frac{1}{2}$ there exists a constant $M>0$ such that

$$|L_B(\omega)-L_B(\omega_1)|<M\,|\omega-\omega_1|^{\eta}.$$

*Proof.* Choose $\varepsilon>0$ sufficiently small such that

$$(\omega_1+\varepsilon)\omega_1^{-2\eta}(1+\varepsilon)^{1+\eta}<1.$$

Choose $n$ sufficiently large such that for all $\omega \in J_i$ for all $i \geqslant n$ we have

$$\omega_1^2(1+\varepsilon)^{-1} < \frac{|\omega - \omega_1|}{|T(\omega) - \omega_1|} < \omega_1^2(1+\varepsilon)$$

$$|\log(1-\omega) - \log(1-\omega_1)| < \omega_1^{-2}(1+\varepsilon)\,|\omega - \omega_1|$$

$$|\omega - \omega_1| < \varepsilon.$$

Fix $i \geqslant n$ and choose $m > 0$ such that

$$|L_B(\omega) - L_B(\omega_1)| < M\,|\omega - \omega_1|^{\eta}$$

for all $\omega \in J_i$ and

$$[L_B(\omega_1) + \omega_1^{-2}(1+\varepsilon)]\varepsilon^{1-\eta} < M(1-(\omega_1+\varepsilon)\omega_1^{-2\eta}(1+\varepsilon)^{1+\eta}.$$

Then for all $\omega \in J_{i+1}$

$$\begin{aligned}
|L_B(\omega) - L_B(\omega_1)| &< |\log(1-\omega) - \log(1-\omega_1)| + \omega\,|T(\omega) - \omega_1| + |\omega - \omega_1|\,L_B(\omega_1) \\
&< (L_B(\omega_1) + \omega_1^{-2}(1+\varepsilon))\,|\omega - \omega_1| + M(\omega_1+\varepsilon)|\,T(\omega) - \omega_1|^{\eta} \\
&< \{M(\omega_1+\varepsilon)\omega_1^{-2\eta}(1+\varepsilon)^{1+\eta} + [L_B(\omega_1) + \omega_1^{-2}(1+\varepsilon)]\varepsilon^{1-\eta}\}\,|\omega - \omega_1|^{\eta} \\
&< M\,|\omega - \omega_1|^{\eta}.
\end{aligned}$$

Thus, by induction $|L_B(\omega) - L_B(\omega_1)| < M\,|\omega - \omega_1|^{\eta}$ for all $\omega \in J_i$ for all $i \geqslant n$. But $\bigcup_{i \geqslant n} J_i = J_n \backslash \{\omega_1\}$. Hence, $|L_B(\omega) - L_B(\omega_1)| < M\,|\omega - \omega_1|^{\eta}$ on a neighbourhood of $\omega_1$ as required.

Unfortunately applying the next modular smoothing procedure to $L_B$, i.e. forming $(\omega+1)L_B(\omega+1) - \omega L_B(\omega)$ does not result in a smooth function. In other words whilst our conjecture 3.3 suffices to explain the first smoothing procedure, to justify the second one needs further investigations into the nature of $C_\beta(\omega)$. In some ways we can thus think of the Brjuno function as a 'first-order' approximation to $\log(K(\omega))$. In this context our expression $\log(K(\omega)) = C_\beta(\omega) - \beta B(\omega)$ can be seen as the start of an 'asymptotic expansion' for $\log(K(\omega))$. One would like to hope that modular smoothing might help to compute the higher-order terms in this.

## Appendix 2. Complex invariant circles of the standard map

We prove, by means of a direct application of the majorant series method, that the formal series solution of (3.3) converges if the imaginary part of the frequency $\omega$ is strictly positive.

Let us denote by $D$ the finite difference operator defined by

$$Du(t, K) = u\left(t + \frac{\omega}{2}\right) - u\left(t - \frac{\omega}{2}\right) \tag{A2.1}$$

so that we can rewrite (3.3) as follows:

$$(D^2 u)(t, K) = K \sin(t + u(t, K)). \tag{A2.2}$$

Set

$$u(t, K) = \sum_{n=1}^{\infty} K^n u_n(t). \tag{A2.3}$$

Then (A2.2) gives rise to the recurrence

$$D^2 u_1 = \sin t$$
$$D^2 u_{n+1} = \sum_{j=1}^{n} \left(\frac{1}{j!}\frac{d^j \sin t}{dt^j}\right) \sum_{n_1+\ldots+n_j=n} u_{n_1} \ldots u_{n_j}. \tag{A2.4}$$

*Remark.* If one replaces $\sin t$ and $d^j \sin t/dt^j$ in (A2.4) with 1, one obtains the recurrence relation verified by the Lindstedt series for the semistandard map. This series—when $\omega$ is real—converges if and only if $\omega$ verifies the Brjuno condition (2.8) (see [19] for a proof).

Assume that $\omega \in \boldsymbol{C}$, $\mathrm{Im}\,\omega \geqslant \eta > 0$ for some positive constant $\eta$, and consider the operator $D$ acting on the space $\mathscr{A}_\rho$ of $2\pi$ periodic functions analytic in the interior of the complex strip

$$\boldsymbol{T}(\rho) = \{t \in \boldsymbol{C} \mid |\mathrm{Im}\, t| \leqslant \rho\} \tag{A2.5}$$

(for some positive constant $\rho$) and with a bounded continuous extension to its closure. It can then be immediately checked that $D$ maps $\mathscr{A}_\rho$ into itself and has bounded inverse. For example, if we consider the following norm for functions $f(t) \in \mathscr{A}_\rho$, $f(t) = \sum_{-\infty}^{+\infty} f_k \exp(ikt)$

$$\|f\|_\rho = \sum_{k=-\infty}^{+\infty} |f_k|\, e^{|k|\rho} \tag{A2.6}$$

then

$$\|D^{-2} f\|_\rho \leqslant \|f\|_\rho \left(\frac{2}{\eta}\right)^2. \tag{A2.7}$$

Since

$$\frac{1}{j!}\left\|\frac{d^j \sin t}{dt^j}\right\|_\rho \leqslant \frac{e^\rho}{j!} \leqslant e^\rho \tag{A2.8}$$

for all $j$, one has the following recurrence for the norms of the coefficients $u_n(t)$ of $u(t, K)$:

$$\|u_{n+1}\|_\rho \leqslant \frac{4e^\rho}{\eta^2} \sum_{j=1}^{n} \sum_{n_1+\ldots+n_j=n} \|u_{n_1}\|_\rho \ldots \|u_{n_j}\|_\rho. \tag{A2.9}$$

It is now easy to check by induction that for all $n$

$$\|u_n\|_\rho \leqslant \left(\frac{4e^\rho}{\eta^2}\right)^n b_n \tag{A2.10}$$

where the sequence $(b_n)_{n\geqslant 1}$ is recursively defined by $b_1 = 1$,

$$b_n = \sum_{k=1}^{n-1} b_k b_{n-k} \qquad n \geqslant 2 \tag{A2.11}$$

and it is well known that

$$b_n \leqslant \frac{1}{n}\binom{2n-2}{n-1} \leqslant 4^n. \tag{A2.12}$$

Thus

$$\|u_n\|_\rho \leqslant 4^n e^{n\rho}\eta^{-2n} b_n \leqslant 16^n e^{n\rho}\eta^{-2n} \tag{A2.13}$$

from which one reads that the series expansion of $u(t, K)$ converges uniformly to an analytic function on

$$T_\rho \times \left\{ K \in C \mid |K| \leqslant \frac{\eta^2 e^{-\rho}}{16} \right\}. \tag{A2.14}$$

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