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The recurrence time for interval exchange maps

Dong Han Kim[1] **and Stefano Marmi**[2]

[1] Department of Mathematics, University of Suwon, Hwaseong 445-743, Korea
[2] Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

E-mail: kimdh@suwon.ac.kr and marmi@sns.it

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Abstract
We consider the recurrence time to the r-neighbourhood for interval exchange maps (i.e.m.s). For almost every i.e.m. we show that the logarithm of the recurrence time normalized by $-\log r$ goes to 1. A similar result of the hitting time also holds for almost every i.e.m.

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1. Introduction

Let μ be a probability measure on a metric space (X, d) defined on Borel subsets of X and let T be a measure preserving transformation. Given a measurable subset E with positive measure and a point $x \in E$ one can consider the *first return time* $R_E(x) = \inf\{j \geqslant 1 | T^j(x) \in E\}$ of the point x in the set E. By the Poincaré recurrence theorem one has $R_E(x) < +\infty$ for almost every x and Kac's lemma states that $\int_E R_E(x)\, \mathrm{d}\mu \leqslant 1$ where equality holds for ergodic transformations.

Let $r > 0$ and let $\tau_r(x)$ be the return time to the r-neighbourhood of x

$$\tau_r(x) = \min\{j \geqslant 1 : d(T^j x, x) < r\},$$

and let $\tau_r(x, y)$ be the hitting time of an r-neighbourhood

$$\tau_r(x, y) = \min\{j \geqslant 1 : d(T^j x, y) < r\}.$$

For expanding maps on the interval we have that (e.g. [1, 6, 10])

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1 \qquad \text{a.e. } x,$$

$$\lim_{r\to 0} \frac{\log \tau_r(x, y)}{-\log r} = 1 \qquad \text{a.e. } x.$$

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For an irrational rotation ([3, 7]) we have that

$$\underline{\lim}_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=\frac{1}{\eta},\qquad \overline{\lim}_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=1,$$
$$\underline{\lim}_{r\to 0}\frac{\log\tau_r(x,y)}{-\log r}=1\quad\text{a.e. } x,\qquad \overline{\lim}_{r\to 0}\frac{\log\tau_r(x,y)}{-\log r}=\eta\quad\text{a.e. } x,$$

where η is the type of the rotation defined by

$$\eta=\sup\{\beta:\liminf_{j\to\infty} j^{\beta}||j\theta||=0\}.$$

If the rotation number is a Roth type number (see below) one has $\eta=1$.

Let $\mathcal{A}$ denote an alphabet with $d\geqslant 2$ elements. Let I be an interval and $(I_\alpha)_{\alpha\in\mathcal{A}}$ a partition of I into d subintervals. An interval exchange map (i.e.m.) T is an invertible map of I which is a translation on each I_α. Thus, T is orientation-preserving and preserves the Lebesgue measure.

In this paper we show that for almost every i.e.m. we have

$$\lim_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=1,\qquad\text{a.e. } x,$$

$$\lim_{r\to 0}\frac{\log\tau_r(x,y)}{-\log r}=1,\qquad\text{a.e. } x.$$

For an earlier work, it is shown [4] that for almost every i.e.m. the limit inferior of the hitting time was 1 using the criterion of [2].

2. Continued fraction algorithms for i.e.m.s

Following [9] we introduce here the basic notions about i.e.m.s needed in the following. We also recall the construction and the fundamental properties of the continued fraction algorithm for i.e.m.s. We refer to [9] and references therein for the proofs.

An i.e.m. is determined by combinatorial data on one side and, length data on the other side. The combinatorial data consist of a finite set $\mathcal{A}$ of names for the intervals and of two bijections (π_0,π_1) from $\mathcal{A}$ onto $\{1,\ldots,d\}$ (where d is the cardinality of $\mathcal{A}$): these indicate in which order the intervals are met before and after the map.

The length data $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$ give the length $\lambda_\alpha>0$ of the corresponding interval. More precisely, we set

$$\begin{aligned} I_\alpha &:= [0,\lambda_\alpha)\times\{\alpha\},\\ \lambda^* &:= \sum_{\alpha\in\mathcal{A}}\lambda_\alpha,\\ I &:= [0,\lambda^*). \end{aligned}$$

We then define, for $\varepsilon=0,1$, a bijection j_ε from $\sqcup_{\alpha\in\mathcal{A}}I_\alpha$ onto I:

$$j_\varepsilon(x,\alpha)=x+\sum_{\pi_\varepsilon(\beta)<\pi_\varepsilon(\alpha)}\lambda_\beta.$$

The i.e.m. T associated with these data is the bijection $T=j_1\circ j_0^{-1}$ of I and

$$T(x)=x+\sum_{\pi_1(\alpha)>\pi_1(\beta)}\lambda_\beta-\sum_{\pi_0(\alpha)>\pi_0(\beta)}\lambda_\beta,\qquad\text{for } x\in I_\alpha.$$

In the following, we will always consider only combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$ which are *admissible*, meaning that for all $k = 1, 2, \ldots, d-1$, we have

$$\pi_0^{-1}(\{1, \ldots, k\}) \neq \pi_1^{-1}(\{1, \ldots, k\}).$$

Moreover we will assume our maps to have the *Keane property*: there exists no finite orbit segment which starts and ends in a discontinuity of the map. More formally, if one defines a *connexion* for T to be a triple (α, β, m) where $\alpha, \beta \in \mathcal{A}$, $\pi_0(\beta) > 1$, m is a positive integer, and $T^m(j_0(0, \alpha)) = j_0(0, \beta)$ we say that T has the Keane property if there is no connexion for T.

The Keane property is the appropriate notion of irrationality for i.e.m. since, as Keane [8] himself proved,

- an i.e.m. with Keane's property is minimal (i.e. all orbits are dense);
- if the length data are rationally independent (and the combinatorial data are admissible) then T has Keane's property.

For admissible i.e.m.s with the Keane property we can introduce the generalization of continued fractions to i.e.m.s (see [13] for a more detailed discussion) due to the work of Rauzy [11], Veech [12] and Zorich [14, 15].

Let (π_0, π_1) be an admissible pair. We define two new admissible pairs $\mathcal{R}_0(\pi_0, \pi_1)$ and $\mathcal{R}_1(\pi_0, \pi_1)$ as follows: let α_0, α_1 be the (distinct) elements of $\mathcal{A}$ such that $\pi_0(\alpha_0) = \pi_1(\alpha_1) = d$; one has

$$\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0, \hat{\pi}_1),$$
$$\mathcal{R}_1(\pi_0, \pi_1) = (\hat{\pi}_0, \pi_1),$$

where

$$\hat{\pi}_1(\alpha) = \begin{cases} \pi_1(\alpha) & \text{if } \pi_1(\alpha) \leqslant \pi_1(\alpha_0), \\ \pi_1(\alpha) + 1 & \text{if } \pi_1(\alpha_0) < \pi_1(\alpha) < d, \\ \pi_1(\alpha_0) + 1 & \text{if } \alpha = \alpha_1, (\pi_1(\alpha_1) = d); \end{cases}$$

$$\hat{\pi}_0(\alpha) = \begin{cases} \pi_0(\alpha) & \text{if } \pi_0(\alpha) \leqslant \pi_0(\alpha_1), \\ \pi_0(\alpha) + 1 & \text{if } \pi_0(\alpha_1) < \pi_0(\alpha) < d, \\ \pi_0(\alpha_1) + 1 & \text{if } \alpha = \alpha_0, (\pi_0(\alpha_0) = d). \end{cases}$$

The *extended Rauzy class* of (π_0, π_1) is the set of admissible pairs obtained by saturation of (π_0, π_1) under the action of $\mathcal{R}_0$ and $\mathcal{R}_1$. The *extended Rauzy diagram* has for vertices the elements of the extended Rauzy class, each vertex (π_0, π_1) being the origin of two arrows joining (π_0, π_1) to $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$. The *name* of an arrow joining (π_0, π_1) to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ (with $\varepsilon \in \{0, 1\}$) is the element $\alpha_\varepsilon \in \mathcal{A}$ such that $\pi_\varepsilon(\alpha_\varepsilon) = 1$.

Let T be an i.e.m. given by data (π_0, π_1), $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$. For $\varepsilon \in \{0, 1\}$, define $\alpha_\varepsilon \in \mathcal{A}$ by $\pi_\varepsilon(\alpha_\varepsilon) = d$ as above.

We say that T is of *type* ε if one has $\lambda_{\alpha_\varepsilon} \geqslant \lambda_{\alpha_{1-\varepsilon}}$; we then define a new i.e.m. $\mathcal{V}(T)$ by the following data: the admissible pair $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ and the lengths $(\hat{\lambda}_\alpha)_{\alpha \in \mathcal{A}}$ given by

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_\varepsilon, \\ \hat{\lambda}_{\alpha_\varepsilon} = \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}} & \text{otherwise,} \end{cases}$$

i.e. the length data of T, are obtained from those of $\mathcal{V}(T)$ as follows:

$$\lambda = V(T)\hat{\lambda}, \tag{1}$$

where the matrix $V(T)$ has all diagonal entries equal to 1 and all off-diagonal entries equal to 0 except the one corresponding to $(\alpha_\varepsilon, \alpha_{1-\varepsilon})$ which is also equal to 1.

The i.e.m. $\mathcal{V}(T)$ is the first return map of T on $[0, \sum_{\alpha\neq\alpha_{1-\varepsilon}} \lambda_\alpha)$. We also associate with T the arrow in the extended Rauzy diagram joining (π_0, π_1) to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$. Iterating this process, we obtain a sequence of i.e.m $(\mathcal{V}^k(T))_{k\geqslant 0}$ and an infinite path in the extended Rauzy diagram starting from (π_0, π_1). In fact, a further property of irrational i.e.m.s (i.e. with the Keane property) is that every name is taken infinitely many times in the infinite path (in the Rauzy diagram) associated with T.

This property is fundamental in order to be able to group together several iterations of $\mathcal{V}$ to obtain the accelerated Zorich continued fraction algorithm introduced in [9].

Starting from $T = T(0)$, we define a sequence $T(n) = \mathcal{V}^{k(n)}(T)$ by the following property: for $n \geqslant 0$, $k(n+1)$ is the largest integer $k > k(n)$ such that not all names in $\mathcal{A}$ are taken by arrows associated with iterations of $\mathcal{V}$ from $T(n)$ to $\mathcal{V}^k(T)$.

Let $T = T(0)$ be an i.e.m. satisfying the Keane property. Let $(T(n))_{n\geqslant 0}$ be the sequence of i.e.m. obtained by the accelerated Zorich algorithm, with associated lengths $(\lambda_\alpha(n))_{\alpha\in\mathcal{A}}$.

Iterating formula (1) gives a matrix $Z(n) \in \mathrm{SL}(d, \mathbb{Z})$ with non-negative entries such that

$$\lambda(n-1) = Z(n)\lambda(n). \tag{2}$$

We will write, for $m < n$,

$$\begin{aligned} Q(m,n) &= Z(m+1)\cdots Z(n),\\ \lambda(m) &= Q(m,n)\lambda(n) \end{aligned}$$

and say $Q(n) = Q(0, n)$.

For $m < n$, $T(n)$ is the first return map of $T(m)$ on $I(n) = [0, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(n))$; the return time of $I_\beta(n)$ in $I(n)$ is $Q_\beta(m,n) := \sum_\alpha Q_{\alpha\beta}(m,n)$ and the time spent in $I_\alpha(m)$ is $Q_{\alpha\beta}(m,n)$, where

$$I_\beta(n) = [0, \lambda_\beta(n)) + \sum_{\pi_0^{(n)}(\alpha)<\pi_0^{(n)}(\beta)} \lambda_\alpha(n).$$

Note that

$$I(n) = \bigcup_{\beta\in\mathcal{A}} I_\beta(n).$$

The following inequality is not difficult to have

$$\min_{\alpha\in\mathcal{A}} \lambda_\alpha(n) \leqslant \frac{\lambda^*}{\|Q(n)\|} \leqslant \max_{\alpha\in\mathcal{A}} \lambda_\alpha(n), \tag{3}$$

where $\|A\| = \sum_{i,j} |a_{ij}|$ for a matrix $A = (a_{i,j})$. The most important virtue of the accelerated Zorich algorithm is the following.

There exists an integer $m(d) > 0$ such that for $n \geqslant m + m(d)$, one has $Q_{\alpha\beta}(m,n) > 0$ for all $\alpha, \beta \in \mathcal{A}$.

Therefore, since $Q(n) = Q(m)Q(m,n)$, we have for $n \geqslant m + m(d)$

$$\min_{\beta\in\mathcal{A}} Q_\beta(n) \geqslant \max_{\beta\in\mathcal{A}} Q_\beta(m). \tag{4}$$

AQ1 In [9] it is shown that for almost every i.e.m. the following 'diophantine condition holds. *(A) for any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that for all $n \geqslant 1$ we have*

$$\|Z(n+1)\| \leqslant C_\varepsilon \|Q(n)\|^\varepsilon.$$

By proposition 1.3.1 in [9], if the i.e.m. satisfies condition (A), then for all $\varepsilon > 0$ there is $C'_\varepsilon > 0$ such that for all $n \geqslant 0$

$$\max_{\alpha\in\mathcal{A}} \lambda_\alpha(n) \leqslant C'_\varepsilon \|Q(n)\|^\varepsilon \min_{\alpha\in\mathcal{A}} \lambda_\alpha(n). \tag{5}$$

3. The first return time of i.e.m.

Let $\mathcal{P}_n$ be the partition of $I = [0, 1)$ consisting of

$$T^i(I_\alpha(n)), \qquad 0 \leqslant i < Q_\alpha(n).$$

Note that $\sum_{\beta\in\mathcal{A}} Q_\beta(n)\lambda_\beta(n) = 1$ and $R_{I(n)}(x) = Q_\alpha(n)$, for each $x \in I_\alpha(n)$.

Let $P_n(x)$ be the element of $\mathcal{P}_n$ which contains x. Define $R_n(x)$ by the first return time to $P_n(x)$, i.e. $R_n(x) = R_{P_n(x)}(x)$.

Proposition 3.1. *We have always, for $n \geqslant 1$,*

$$\min_{\beta\in\mathcal{A}} Q_\beta(n) \leqslant R_n(x) < 2\max_{\beta\in\mathcal{A}} Q_\beta(n+m(d)) + \max_{\beta\in\mathcal{A}} Q_\beta(n).$$

Proof. Let $x \in T^i(I_\alpha(n)) = P_n(x)$ for some α and i with $0 \leqslant i < Q_\alpha(n)$. Then $T^{-j}(x) \notin I(n)$ for any $0 \leqslant j < i$ and $T^{-i}(x) \in I_\alpha(n)$.

By the construction of $\mathcal{P}_n$ we have

$$R_n(x) = i + R_{I_\alpha(n)}(x) \geqslant i + R_{I(n)}(x).$$

Since

$$T^{Q_\alpha(n)}(I_\alpha(n)) \subset I(n),$$

we have

$$R_{I(n)}(x) = Q_\alpha(n) - i.$$

Hence, we have

$$R_n(x) \geqslant Q_\alpha(n).$$

Let $x \in T^j(I_\beta(n+m(d)))$ for some β and j with $0 \leqslant j < Q_\beta(n+m(d))$. Then

$$T^{Q_\beta(n+m(d))-j}(x) \in I(n+m(d)).$$

Since $Q_{\alpha\gamma}(n, n+m(d)) > 0$, we have that the orbit $T^\ell(y)$, $y \in I_\gamma(n+m(d))$ visits $I_\alpha(n)$ before $\ell \leqslant Q_\gamma(n+m(d))$. Therefore, we have

$$\begin{aligned} R_n(x) &= R_{I_\alpha(n)}(x) + i \\ &< Q_\beta(n+m(d)) - j + \max_{\gamma\in\mathcal{A}} Q_\gamma(n+m(d)) + Q_\alpha(n) \\ &\leqslant 2\max_{\beta\in\mathcal{A}} Q_\beta(n+m(d)) + \max_{\beta\in\mathcal{A}} Q_\beta(n). \end{aligned}$$

□

Let $W_n(x, y)$ be the hitting time from x to $P_n(y)$, i.e.

$$W_n(x, y) = R_{P_n(y)}(x) = \min\{j \geqslant 1 : T^j(x) \in P_n(y)\}.$$

Proposition 3.2. *For any x and y we have the following inequality for all $n \geqslant 0$.*

$$W_n(x, y) < 2\max_{\beta\in\mathcal{A}} Q_\beta(n+m(d)) + \max_{\beta\in\mathcal{A}} Q_\beta(n).$$

Note that $W_n(x, y) \leqslant i$ for $x = T^{-i}(y)$, $i \geqslant 1$ and there is no general lower bound for $W_n(x, y)$ as in proposition 3.1.

Proof. Let $x \in T^i(I_\alpha(n+m(d)))$ for some α and i with $0 \leqslant i < Q_\alpha(n+m(d))$ and $y \in T^j(I_\beta(n)) = P_n(y)$ for some β and j with $0 \leqslant j < Q_\beta(n)$.

By the construction of $\mathcal{P}_n$ we have

$$W_n(x, y) \leqslant R_{I_\beta(n)}(x) + j$$

and

$$T^{Q_\alpha(n+m(d))-i}(x) \in I(n + m(d)).$$

Since $Q_{\beta\gamma}(n, n+m(d)) > 0$, we have that the orbit $T^\ell(y)$, $y \in I_\gamma(n+m(d))$ visits $I_\beta(n)$ before $\ell \leqslant Q_\gamma(n + m(d))$. Therefore, we have

$$\begin{aligned} W_n(x, y) &\leqslant R_{I_\alpha(n)}(x) + j \\ &< Q_\alpha(n + m(d)) - i + \max_{\gamma\in\mathcal{A}} Q_\gamma(n + m(d)) + Q_\beta(n) \\ &\leqslant 2 \max_{\beta\in\mathcal{A}} Q_\beta(n + m(d)) + \max_{\beta\in\mathcal{A}} Q_\beta(n). \end{aligned}$$ □

Now we have some lemmas which hold for the i.e.m. with condition (A).

Lemma 3.3. *If the i.e.m. satisfies (A), then for any $\varepsilon > 0$ we can find $C > 0$ such that for any $n \geqslant m(d)$*

$$C\|Q(n)\|^{1-\varepsilon} \leqslant \min_{\beta\in\mathcal{A}} Q_\beta(n).$$

Proof. If $n = m + m(d)$, for any $\varepsilon > 0$ we have

$$\begin{aligned} \|Q(n)\| &= \sum_{\beta,\gamma} Q_\beta(m) Q_{\beta\gamma}(m, n) \\ &\leqslant \max_\beta Q_\beta(m)\|Q(m, n)\| \\ &\leqslant \min_\beta Q_\beta(n)\|Q(m, n)\|, && \text{by (4)}, \\ &\leqslant \min_\beta Q_\beta(n) C\|Q(n)\|^\varepsilon, && \text{by condition (A)}. \end{aligned}$$ □

Lemma 3.4. *If the i.e.m. satisfies condition (A), then we have*

$$\lim_{n\to\infty} \frac{\log|P_{n+1}(x)|}{\log|P_n(x)|} = 1$$

and the convergence is uniform.

Proof. For any $\varepsilon > 0$ we have

$$\begin{aligned} \min_{\beta\in\mathcal{A}} \lambda_\beta(n+1) &\geqslant \frac{\max_{\beta\in\mathcal{A}} \lambda_\beta(n+1)}{C'_\varepsilon\|Q(n+1)\|^\varepsilon} \geqslant \frac{\max_{\beta\in\mathcal{A}} \lambda_\beta(n+1)}{C'_\varepsilon\|Q(n)\|^\varepsilon\|Z(n+1)\|^\varepsilon} && \text{by (5)} \\ &\geqslant \frac{\max_{\alpha\in\mathcal{A}} \lambda_\alpha(n)}{C'_\varepsilon\|Q(n)\|^\varepsilon\|Z(n+1)\|^{1+\varepsilon}} && \text{by (2)} \\ &\geqslant \frac{\max_{\alpha\in\mathcal{A}} \lambda_\alpha(n)}{C'_\varepsilon C_\varepsilon^{1+\varepsilon}\|Q(n)\|^{\varepsilon(2+\varepsilon)}} && \text{by condition (A)} \\ &\geqslant \frac{\lambda^*}{C'_\varepsilon C_\varepsilon^{1+\varepsilon}\|Q(n)\|^{1+\varepsilon(2+\varepsilon)}} && \text{by (3)} \end{aligned}$$

and

$$\begin{aligned} \max_{\alpha\in\mathcal{A}} \lambda_\alpha(n) &\leqslant C'_\varepsilon\|Q(n)\|^\varepsilon \min_{\alpha\in\mathcal{A}} \lambda_\alpha(n) && \text{by (5)} \\ &\leqslant C'_\varepsilon\|Q(n)\|^{-1+\varepsilon}\lambda^* && \text{by (3)}. \end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{\log|P_{n+1}(x)|}{\log|P_n(x)|} &\leqslant \frac{-\log(\min_\alpha \lambda_\alpha(n+1))}{-\log(\max_\alpha \lambda_\alpha(n))}\\ &\leqslant \frac{(1+\varepsilon(2+\varepsilon))\log\|Q(n)\| + \log C'_\varepsilon + (1+\varepsilon)\log C_\varepsilon - \log\lambda^*}{(1-\varepsilon)\log\|Q(n)\| - \log C'_\varepsilon - \log\lambda^*},\end{aligned} \tag{6}$$

which goes to $\frac{1+\varepsilon(2+\varepsilon)}{1-\varepsilon}$ uniformly as n goes to infinity. □

Proposition 3.5. *For an i.e.m. with condition (A) we have for all x*

$$\lim_{n\to\infty} \frac{\log R_n(x)}{-\log|P_n(x)|} = 1.$$

Proof. By proposition 3.1 and lemma 3.3, if $n \geqslant m(d)$, then for any $\varepsilon > 0$ we can choose a constant $C > 0$ such that

$$R_n(x) \geqslant \min_\beta Q_\beta(n) \geqslant C\|Q(n)\|^{1-\varepsilon}.$$

Since by (3) and (5)

$$\|Q(n)\|^{1+\varepsilon} \geqslant \frac{\|Q(n)\|^\varepsilon \lambda^*}{\max_\alpha \lambda_\alpha(n)} \geqslant \frac{\lambda^*}{C'_\varepsilon \min_\alpha \lambda_\alpha(n)},$$

we have

$$R_n(x) \geqslant C\|Q(n)\|^{1-\varepsilon} \geqslant \frac{C\lambda^{*\frac{1-\varepsilon}{1+\varepsilon}}}{(C'_\varepsilon \min_\alpha \lambda_\alpha(n))^{\frac{1-\varepsilon}{1+\varepsilon}}} \geqslant CC'^{-\frac{1-\varepsilon}{1+\varepsilon}}_\varepsilon \lambda^{*\frac{1-\varepsilon}{1+\varepsilon}} |P_n(x)|^{-\frac{1-\varepsilon}{1+\varepsilon}}.$$

Hence, we have for all x

$$\varliminf_{n\to\infty} \frac{\log R_n(x)}{-\log|P_n(x)|} \geqslant 1.$$

By condition (A) for all $\varepsilon > 0$ we can choose a constant D such that

$$\|Q(n, n+m(d))\| \leqslant \|Z(n+1)\| \cdots \|Z(n+m(d))\| \leqslant D\|Q(n)\|^\varepsilon.$$

Therefore we have

$$\begin{aligned}\max_\beta Q_\beta(n+m(d)) &< \|Q(n+m(d))\| \leqslant \|Q(n)\| \cdot \|Q(n, n+m(d))\|\\ &\leqslant D\|Q(n)\|^{1+\varepsilon} \leqslant D\left(\frac{C'_\varepsilon\lambda^*}{\max_\alpha \lambda_\alpha(n)}\right)^{\frac{1+\varepsilon}{1-\varepsilon}},\end{aligned} \tag{7}$$

where the last inequality is from the following inequality obtained by (3) and (5)

$$\|Q(n)\|^{1-\varepsilon} \leqslant \frac{\lambda^*}{\|Q(n)\|^\varepsilon \min_\alpha \lambda_\alpha(n)} \leqslant \frac{C'_\varepsilon\lambda^*}{\max_\alpha \lambda_\alpha(n)}.$$

Hence by proposition 3.1 we have

$$R_n(x) < 3\max_\beta Q_\beta(n+m(d)) < \frac{3D(C'_\varepsilon\lambda^*)^{\frac{1-\varepsilon}{1+\varepsilon}}}{\max_\alpha \lambda_\alpha(n)^{\frac{1+\varepsilon}{1-\varepsilon}}} \leqslant 3D(C'_\varepsilon\lambda^*)^{\frac{1-\varepsilon}{1+\varepsilon}} |P_n(x)|^{-\frac{1+\varepsilon}{1-\varepsilon}}.$$

Thus, we have for all x

$$\overline{\lim_{n\to\infty}} \frac{\log R_n(x)}{-\log |P_n(x)|} \leqslant 1. \qquad \square$$

Theorem 3.6. *For an i.e.m. with condition (A) we have*

$$\limsup_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} \leqslant 1, \quad \text{for all } x, \qquad \liminf_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} \geqslant 1, \quad \text{a.e. } x.$$

Proof. Since $\tau_{|P_n(x)|}(x) \leqslant R_n(x)$, by lemma 3.4 we have for all x

$$\overline{\lim_{r\to 0^+}} \frac{\log \tau_r(x)}{-\log r} = \overline{\lim_{n\to\infty}} \frac{\log \tau_{|P_n(x)|}(x)}{-\log |P_n(x)|} \leqslant \overline{\lim_{n\to\infty}} \frac{\log R_n(x)}{-\log |P_n(x)|} \leqslant 1.$$

Choose E_n the 'middle' part of each interval in $\mathcal{P}_n$ as $E_n = \{x : |\partial P_n(x) - x| > \frac{1}{n^2}|P_n(x)|\}$ (see also [5]). Then $\mu(E_n) = 1 - \frac{2}{n^2}$ and by the Borel–Cantelli lemma for almost every x $x \in E_n$ eventually. If $x \in E_n$ then we have

$$R_n(x) \leqslant \tau_{|P_n(x)|/n^2}(x).$$

By the way $Q_{\alpha\beta}(n, n + m(d)) > 0$ for all α, β implies the exponential decay of $|P_n(x)|$:

$$|P_{n+m(d)}(x)| < \frac{1}{d}|P_n(x)|. \tag{8}$$

Therefore, we have for almost every x

$$\begin{aligned} \underline{\lim}_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} &= \underline{\lim}_{n\to\infty} \frac{\log \tau_{|P_n(x)|/n^2}(x)}{-\log(|P_n(x)|/n^2)} \\ &\geqslant \underline{\lim}_{n\to\infty} \frac{\log R_n(x)}{-\log |P_n(x)|} = 1. \end{aligned} \qquad \square$$

Note that the almost everywhere sense upper bound of proposition 3.5 holds for general measure preserving systems. It is well known [6] that for any Lebesgue measure preserving transformation on interval

$$\limsup_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} \leqslant 1, \qquad \text{a.e.}$$

Now we have the limit theorem for the hitting time problem.

Theorem 3.7. *Assume that the i.e.m. satisfies condition (A). Then for any x and y we have*

$$\overline{\lim_{n\to\infty}} \frac{\log W_n(x, y)}{-\log |P_n(y)|} \leqslant 1 \qquad \text{and} \qquad \overline{\lim_{r\to 0^+}} \frac{\log \tau_r(x, y)}{-\log r} \leqslant 1.$$

Proof. The proof is similar to the upper bound part of proposition 3.5. By proposition 3.2 and (7) we have

$$W_n(x, y) < 3 \max_\beta Q_\beta(n + m(d)) \leqslant 3D \left(\frac{C'_\varepsilon \lambda^*}{\max_\alpha \lambda_\alpha(n)}\right)^{\frac{1+\varepsilon}{1-\varepsilon}} \leqslant 3D(C'_\varepsilon \lambda^*)^{\frac{1-\varepsilon}{1+\varepsilon}} |P_n(y)|^{-\frac{1+\varepsilon}{1-\varepsilon}}.$$

Thus we have for all x and y

$$\overline{\lim_{n\to\infty}} \frac{\log W_n(x, y)}{-\log |P_n(y)|} \leqslant 1.$$

If we choose $r_n = |P_n(y)|$, then $\tau_{r_n}(x, y) \leqslant W_n(x, y)$ and by lemma 3.4 we have

$$\overline{\lim_{r\to 0^+}} \frac{\log \tau_r(x, y)}{-\log r} = \overline{\lim_{n\to\infty}} \frac{\log \tau_{r_n}(x, y)}{-\log r_n} \leqslant \overline{\lim_{n\to\infty}} \frac{\log W_n(x, y)}{-\log |P_n(y)|} \leqslant 1. \qquad \square$$

It is well known that for any Lebesgue measure preserving transformation on interval (e.g. [4, 6])

$$\underline{\lim_{r\to 0^+}} \frac{\log \tau_r(x, y)}{-\log r} \geqslant 1, \qquad \text{a.e. } x.$$

Therefore we have the following corollary.

Corollary 3.8. *For an i.e.m. with condition (A) we have*

$$\lim_{r\to 0^+} \frac{\log \tau_r(x, y)}{-\log r} = 1, \qquad \text{a.e. } x.$$

Remark 3.9. For a fixed y and $\varepsilon > 0$, let

$$E_n = \{x : W_n(x, y) < |P_n(y)|^{-1+\varepsilon}\} = \bigcup_{1\leqslant i<|P_n(y)|^{-1+\varepsilon}} T^{-i}(P_n(y)).$$

Then we have

$$\mu(E_n) \leqslant |P_n(y)|^{-1+\varepsilon} \cdot |P_n(y)| = |P_n(y)|^{\varepsilon}.$$

Since by (8), $|P_n(y)|$ decreases to 0 exponentially, $\sum_n \mu(E_n) < \infty$ and by the first Borel–Cantelli lemma, for almost every x, $x \in E_n$ finitely many ns, which implies that

$$\underline{\lim_{n\to\infty}} \frac{\log W_n(x, y)}{-\log |P_n(y)|} \geqslant 1 - \varepsilon \qquad \text{a.e. } x.$$

If we choose $x = T^{-i}(y)$ for $i \in \mathbb{N}$, then $W_n(x, y) \leqslant i$ for any n and

$$\lim_{n\to\infty} \frac{\log W_n(x, y)}{-\log |P_n(y)|} = \lim_{r\to 0^+} \frac{\log \tau_r(x, y)}{-\log r} = 0.$$

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