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REDUCIBILITY OF QUASIPERIODIC COCYCLES UNDER A BRJUNO-RÜSSMANN ARITHMETICAL CONDITION

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ABSTRACT. The arithmetics of the frequency and of the rotation number play a fundamental role in the study of reducibility of analytic quasiperiodic cocycles which are sufficiently close to a constant. In this paper we show how to generalize previous works by L.H. Eliasson which deal with the diophantine case so as to implement a Brjuno-Rüssmann arithmetical condition both on the frequency and on the rotation number. Our approach adapts the Pöschel-Rüssmann KAM method, which was previously used in the problem of linearization of vector fields, to the problem of reducing cocycles.

1. INTRODUCTION

Quasiperiodic cocycles are the fundamental solutions of quasiperiodic linear systems

$$\forall(\theta,t)\in\mathbb{T}^d\times\mathbb{R},\ X'(\theta,t)=A(\theta+t\omega)X(\theta,t) \tag{1}$$

where A is a continuous matrix-valued function on a torus $\mathbb{T}^d$ and ω is a rationally independent vector of some space $\mathbb{R}^d$ (the space of frequencies). Although the dynamics of such a system can be quite complicated, they are easily studied in case the cocycle is reducible, i.e when there is a map Z, continuous on the double torus $2\mathbb{T}^d=\mathbb{R}^d/2\mathbb{Z}^d$, taking its values in the group of invertible matrices and such that

$$\forall\theta\in 2\mathbb{T}^d,\ \frac{d}{dt}Z(\theta+t\omega)_{|t=0}=A(\theta)Z(\theta)-Z(\theta)B$$

for some matrix B not depending on θ. Since smoothness is an issue, given a class of functions $\mathscr{C}$, we will say that the cocycle is reducible in $\mathscr{C}$ if Z can be chosen in $\mathscr{C}$. Here we will focus on the case in which A takes its values in $sl(2,\mathbb{R})$, which is sufficient, for instance, for the study of the one-dimensional quasiperiodic Schrödinger equation. Moreover, we will consider solutions of (1) with $A\in C^\omega_r$, the space of functions on $\mathbb{T}^d$ having a holomorphic extension on $\{(z_1,\dots,z_d)\in\mathbb{C}^d,\ \forall j\ |\mathrm{Im}\, z_j|<r\}$, whose "weighted norm" $|\cdot|_r$ converges (see Section 2.1).

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The arithmetics of ω seem fundamental in the study of reducibility, as well as the arithmetics of the system's rotation number ρ (as it was defined in [7]; we recall the definition in Section 2.1). At least in the perturbative case, arithmetical conditions of diophantine type have long been used to obtain reducibility, which can be seen as the convergence of a certain sequence of analytic functions: a diophantine condition can be used to control small divisors and make sure that the sequence converges. This was achieved, in particular, by Eliasson in [4]:

THEOREM 1.1 (Eliasson). *Let $r > 0$, $V \in C_r^\omega(\mathbb{T}^d,\mathbb{R})$. Suppose ω is a diophantine vector. There exists ϵ_0 depending only on r,ω such that if $\sup_{|\mathrm{Im}\,\theta|<r}|V(\theta) - \hat{V}(0)| \le \epsilon_0$, then the cocycle which is solution of*

$$\frac{d}{dt}X(t,\theta) = \begin{pmatrix} 0 & V(\theta+t\omega)-E \\ 1 & 0 \end{pmatrix} X(t,\theta) \tag{2}$$

is reducible for all E for which the rotation number is rational or diophantine with respect to ω.

In this article, we will give a reducibility result for analytic cocycles under a weaker arithmetical condition than the diophantine one. In order to obtain an analytic reducibility result, we will have to pick a frequency and a rotation number with good approximation properties, in the sense of Rüssmann ([10]): ω will have to satisfy a strong irrationality condition controlled by an approximation function G, namely

$$\forall m \in \mathbb{Z}^d \smallsetminus \{0\}, |\langle m,\omega\rangle| \ge \frac{\kappa}{G(m)}$$

for some positive κ (Section 2.1), and ρ will have to satisfy a further arithmetical condition: its approximations by means of linear combinations of the frequencies are controlled by an approximation function g, i.e

$$\forall m \in \mathbb{Z}^d \smallsetminus \{0\}, |\rho - \langle m,\omega\rangle| \ge \frac{\kappa'}{g(m)}$$

(we will say for short that ρ has g as approximation function with respect to ω with constant κ') with g, G satisfying some extra assumptions.

We will be particularly interested in the case of Brjuno frequency, i.e when

$$\int_1^\infty \frac{\log G(t)}{t^2}\, dt < \infty \tag{3}$$

and of $\frac{1}{2}$-Brjuno rotation number (with respect to ω), i.e when

$$\int_1^\infty \frac{\log g(t)}{t^{3/2}}\, dt < \infty \tag{4}$$

In dimension $d = 2$, Condition (3) coincides with the well-known Brjuno condition defined in terms of continued fraction expansion, which is closely related, as shown by Yoccoz, to the dynamical properties of the quadratic polynomial (see [11]). Classes of numbers defined by a condition analogous to (4) when $d = 2$, which is slightly stronger than Brjuno, were constructed in [8].

Condition (3) was introduced by Rüssmann in KAM theory, making it possible to deal with a vector of frequencies. Brjuno-Rüssmann conditions are already known to be central in the study of the linearization of vector fields (see e.g., [6, 9] and references therein).

The Brjuno condition on the frequency was also considered by Young in [12], who constructed examples of nonreducible discrete cocycles in this case. For discrete cocycles with one frequency, Zhou and Wang recently obtained in [13] a positive measure reducibility result for non-Brjuno frequencies for nondegenerate one-parameters families of cocycles. Other results have been obtained on quasiperiodic cocycles regardless of any arithmetic condition on the frequency, worth mentioning although they are not reducibility results in our sense. In [1], it is shown that without any condition on the frequency, the Schrödinger cocycle (2) can be conjugated to a rotation-valued cocycle for a positive measure set of energies; in [13], Zhou and Wang showed that in the case $d = 2$, for any frequency, for a nondegenerate analytic one-parameter family which is close to a constant, the cocycle can be analytically diagonalized for a positive measure set of parameters.

Our main result states:

THEOREM 1.2. *Let ω be a Brjuno vector, $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C_r^{\omega}(\mathbb{T}^d, sl(2,\mathbb{R}))$. Let $\tilde{\rho}$ be a $\frac{1}{2}$-Brjuno number with respect to ω. There exists ϵ_0 depending only on $\omega, \tilde{\rho}, r$ such that if $|F|_r \le \epsilon_0$ and $A + F$ has rotation number $\tilde{\rho}$, then there exists $r' \in (0, r)$ such that $A + F$ is reducible in $C_{r'}^{\omega}$.*

This result gives an extension of Eliasson's theorem using Rüssmann's and Pöschel's formulation of arithmetic conditions by means of approximation functions and their use in KAM methods. It holds for cocycles with arbitrarily many frequencies and gives a quite explicit link between reducibility and the arithmetics of ω and ρ, what Zhou-Wang's result does not since they consider larger-dimensional systems with only two frequencies and formulate reducibility in terms of an abstract parameter. However, Zhou-Wang's article implies that it is impossible to find a lower bound for the function

$$(\omega, \rho) \mapsto \log \epsilon_0(\omega, \rho) + \int_1^{\infty} \frac{\log G(t)}{t^2} + \frac{\log g(t)}{t^{3/2}}\, dt.$$

Brjuno-Rüssmann conditions are therefore not optimal in this problem as they might be in other dynamical problems.

In fact, our method gives this more explicit theorem:

THEOREM 1.3. *Let $\kappa > 0$ and let G, g be positive increasing functions such that*

- $G(1) \ge 1, g(1) \ge 1,$
- $\displaystyle\int_1^{+\infty} \frac{\log G(t) + \log g(t)}{t^2}\, dt < +\infty,$
- *the map* $t \mapsto \dfrac{g(t^2)}{G(t)}$ *is bounded.*

Suppose ω has G as an approximation function with constant κ. Let $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^{\omega}_{r}(\mathbb{T}^d, sl(2,\mathbb{R}))$. Let $n_0 \in \mathbb{N}$. There exist ϵ_0 depending only on g, κ, G, n_0, r such that if

1. $|F|_r \le \epsilon_0$,
2. *$\rho(A+F)$ has g as an approximation function with respect to ω with constant $\kappa' > \kappa \sup_{t\ge n_0} \frac{g(t^2)}{G(t)}$,*

then there exists $r' \in (0,r)$ such that $A+F$ is reducible in $C^{\omega}_{r'}(2\mathbb{T}^d, sl(2,\mathbb{R}))$.

A discussion on the dependence of ϵ_0 on g, G and the other parameters is given in Subsection 3.4.

As an application, we consider the case when g and G look like exponentials (Section 3.4):

THEOREM 1.4. *Let $\kappa > 0$, $\kappa' > 0$ and let $G(t) = e^{\frac{t}{(\log t)^{\delta}}}$, $g(t) = e^{t^{\alpha}}$, $\delta > 1$, $\alpha < 1$. Suppose ω has G as an approximation function with constant κ. Let $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^{\omega}_{r}(\mathbb{T}^d, sl(2,\mathbb{R}))$. There exist ϵ_0 depending only on α, κ, δ, κ', r such that if*

1. $|F|_r \le \epsilon_0$,
2. *$\rho(A+F)$ has g as an approximation function with respect to ω with constant κ',*

then there exists $r' \in (0,r)$ such that $A+F$ is reducible in $C^{\omega}_{r'}(2\mathbb{T}^d, sl(2,\mathbb{R}))$.

Our aim is to adapt the Pöschel-Rüssmann method (see [10] and [9]), which was used in the problem of linearization for vector fields, to the problem of reducing cocycles. It is a KAM-type method in which the speed of convergence is linear.

First of all, we will build a setup in which a system $A+F$ with A constant and F small is conjugate to another system which is arbitrarily close to a constant, in an analytic class which, however, cannot be well controlled: this follows the technique used in [5] and is obtained by iterating (as in Subsection 3.1) infinitely many steps (described in Section 2) in which one conjugates a system $A_n + F_n$ to a system $A_{n+1} + F_{n+1}$ where A_n, A_{n+1} are constant and $|F_{n+1}|_{r_{n+1}} \le C|F_n|_{r_n}$, with $C < 1$ being independent of n and r_n being a decreasing sequence controlling how analytic a function is. Thus, if r_n tends to a nonzero limit, we have analytic reducibility.

At each step, in order to proceed, the constant part has to be nonresonant, and if it is resonant, then we will have to remove the resonances, as explained in Subsection 2.2.

Our context makes sure that r_n tends to a nonzero limit whenever there is only a small enough number of steps at which one has to remove resonances in the constant part. The Brjuno-Rüssmann condition on the frequency and on the rotation number of the cocycle is required exactly at this stage.

2. THE BASIC STEP

In this section, we will prove the iterative step, which consists in conjugating a system to another one with a smaller nonconstant part, whether the constant part be resonant or not.

2.1. Definitions and notations.

We will adopt the following conventions:

DEFINITION. Let $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$. We denote by $|m|$ its modulus:

$$|m| = \sum_{j=1}^{d} |m_j|.$$

DEFINITION. Let $F \in C^0(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$; we denote by $\hat{F}(k)$ the k-th Fourier coefficient of F, given by $\hat{F}(k) = \int_{\mathbb{T}^d} e^{-2i\pi\langle k,\theta\rangle} F(\theta)\, d\theta$.

DEFINITION. Let $F \in C^0(\mathbb{T}^d)$ and $r > 0$; we say that $F \in C_r^\omega(\mathbb{T}^d)$ if there exists an analytic continuation of F on a product of strips $\{(z_1, \dots, z_d) \in \mathbb{C}^d,\ \forall j\ |\mathrm{Im}\, z_j| < r\}$ and if the weighted norm

$$|F|_r = \sum_{k\in\mathbb{Z}^d} \|\hat{F}(k)\| e^{2\pi|k|r}$$

where $\|.\|$ is the relevant norm for $\hat{F}(k)$ (for matrices, we use the operator norm), is finite.

Note that $C_r^\omega(\mathbb{T}^d)$ is a Banach space.

NOTATION. For $F \in C_r^\omega(\mathbb{T}^d)$, we denote its truncation by

$$F^N(\theta) = \sum_{|m|\le N} \hat{F}(m) e^{2i\pi\langle m,\theta\rangle}.$$

REMARK. The weighted norms are particularly convenient since they satisfy, for any integer N,

$$|F - F^N|_r = \sum_{k\in\mathbb{Z}^d, |k|>N} \|\hat{F}(k)\| e^{2\pi|k|r} = |F|_r - |F^N|_r.$$

Moreover, they are related (although not equivalent) to the usual sup norms since it is easy to see that

$$\sup_{|\mathrm{Im}\,\theta|<r} \|F(\theta)\| \le |F|_r.$$

For $r' < r$, we still have

$$|F|_{r'} \le C(r - r') \sup_{|\mathrm{Im}\,\theta|<r} \|F(\theta)\|$$

where $C(r - r')$ does not depend on F but depends on $r - r'$.

DEFINITION. If $A \in C^0(\mathbb{T}^d, sl(2,\mathbb{R}))$ and X is the solution of $\frac{d}{dt}X(t,\theta) = A(\theta + t\omega)X(t,\theta);\ X(0,\theta) = Id$, the rotation number $\rho(A)$ is the quantity

$$\rho(A) = \lim_{t\to+\infty} \frac{1}{t} \mathrm{Arg}(X(t,\theta)\phi - \phi)$$

where $\phi \in \mathbb{R}^2 \simeq \mathbb{C}$, $\theta \in \mathbb{T}^d$ and Arg stands for the variation of the complex argument ($\rho(A)$ is independent of θ, ϕ).

Here and in what follows, we will fix $\omega \in \mathbb{R}^d$ rationally independent (i.e such that for all nonzero $m \in \mathbb{Z}^d, \langle m, \omega\rangle \neq 0$): the vector ω will be the frequency of the cocycles we will consider. We will always assume that $\kappa, \kappa' > 0$ and that G, g are two positive continuous and strictly increasing functions such that $G(1) \geq 1, g(1) \geq 1$.

DEFINITION. $NR(\kappa, G) = \{\omega \in \mathbb{R}^d, \forall m \in \mathbb{Z}^d \setminus \{0\}, |\langle m, \omega\rangle| \geq \frac{\kappa}{G(|m|)}\}$.

REMARK. There is a positive increasing and unbounded function $G \in C^0(\mathbb{R}^{*+})$ with $G(1) \geq 1$ and $\kappa > 0$ such that $\omega \in \mathrm{NR}(\kappa, G)$.

Indeed, one can take $\kappa = \min_i |\omega_i|$ and $G(N) = \max_{|m|\leq N} \dfrac{\kappa}{|\langle m, \omega\rangle|}$.

As noticed by H. Rüssmann ([10]), a condition $\mathrm{NR}(\kappa, G)$ with G such that

$$\int_1^{\infty} \frac{\log G(t)}{t^2}\, dt < \infty \tag{5}$$

is fulfilled by all Bruno vectors (see [2]), i.e vectors satisfying:

$$B := \sum_{k\geq 1} \frac{|\log \alpha_{2^k-1}|}{2^k} < +\infty \tag{6}$$

where

$$\alpha_k = \min_{l\leq k} \min_{j=1,\dots,d} \min_{|m|=l+1} |\langle m, \omega\rangle - \omega_j|.$$

In [6] it is shown that condition (6) is equivalent to

$$\Gamma := \sum_{k\geq 1} \frac{|\log \alpha_k|}{k(k+1)} < +\infty. \tag{7}$$

More precisely, it is shown that

$$\Gamma \leq B \leq 2\left(\Gamma - \frac{\log \alpha_1}{2}\right).$$

Now condition (5) is equivalent to condition (7), which can be seen as follows: for each $k \in \mathbb{N}$,

$$\min_{|m|\leq k+2} |\langle m, \omega\rangle| \leq \alpha_k \leq \min_{|m|\leq k+1} |\langle m, \omega\rangle|;$$

therefore $G(k+2) \leq \frac{\kappa}{\alpha_k} \leq G(k+1)$. Thus

$$\frac{\log G(k+2)}{k(k+1)} \leq |\frac{\log \alpha_k}{k(k+1)}| + \frac{\log \kappa}{k(k+1)} \leq \frac{\log G(k+1)}{k(k+1)}$$

so that the convergence of Γ is equivalent to the convergence of the integral in (5). Therefore (5), (6) and (7) are equivalent. Moreover, in dimension $d = 2$, these conditions are equivalent to the usual Brjuno condition on $\frac{\omega_2}{\omega_1}$ (see [11]). This suggests the following definition:

DEFINITION. The vector ω is a Brjuno vector if $\omega \in \mathrm{NR}(\kappa, G)$ with G satisfying (5).

We now have to introduce another type of arithmetic condition, related to the well-known "second Melnikov condition".

DEFINITION. Let $N \in \mathbb{N} \setminus \{0\}$; we set

$$\mathrm{NR}_\omega^N(\kappa', g) = \left\{ \alpha \in \mathbb{C}, \forall m \in \mathbb{Z}^d \setminus \{0\}, 0 < |m| \le N \Rightarrow |\alpha - i\pi\langle m, \omega\rangle| \ge \frac{\kappa'}{g(|m|)} \right\}$$

and $\mathrm{NR}_\omega(\kappa', g) = \bigcap_{N\in\mathbb{N}} \mathrm{NR}_\omega^N(\kappa', g)$.

REMARK. If $g(t) = t^\tau$ for some $\tau > 1$, this is a diophantine condition.

DEFINITION. Let $\nu > 0$. The number α is a ν-Brjuno number with respect to ω if $\alpha \in \mathrm{NR}_\omega(\kappa', g)$ with g satisfying

$$\int_1^\infty \frac{\log g(t)}{t^{(1+\nu)}} < +\infty. \tag{8}$$

This extended Brjuno condition was first considered in [8].

2.2. **Elimination of resonances.** Now we shall prove the uniqueness of a resonance, i.e the situation of the spectrum being close to a number of the form $\langle m, \omega\rangle, m \in \mathbb{Z}^d$, when it exists.

LEMMA 2.1. *Let $\alpha \in \mathbb{C}$. Let $N \in \mathbb{N} \setminus \{0\}$. There exists $m \in \mathbb{Z}^d$ such that $|m| \le N$ and $\alpha - i\pi\langle m, \omega\rangle \in \mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$; if m is nonzero, then*

$$|\alpha - i\pi\langle m, \omega\rangle| < \frac{\kappa}{4G(N)g(|m|)}$$

and $\alpha - i\pi\langle m, \omega\rangle \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$.

Proof. Suppose α is not in $\mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$, i.e there exists $m \in \mathbb{Z}^d, 0 < |m| \le N$, such that

$$|\alpha - i\pi\langle m, \omega\rangle| < \frac{\kappa}{4G(N)g(|m|)}.$$

Then for all $m' \in \mathbb{Z}^d$ with $0 < |m'| \le N$,

$$|\alpha - i\pi\langle m + m', \omega\rangle| \ge |\pi\langle m', \omega\rangle| - |\alpha - i\pi\langle m, \omega\rangle| \ge \frac{\kappa}{G(|m'|)} - \frac{\kappa}{4G(N)g(|m|)}$$

so

$$|\alpha - i\pi\langle m + m', \omega\rangle| \ge \frac{\kappa}{G(N)g(|m'|)}$$

and so $\alpha - i\pi\langle m, \omega\rangle \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$. □

The following proposition explains how to eliminate resonances in the spectrum of a trace zero matrix.

PROPOSITION 2.2. *Let $A \in sl(2, \mathbb{R})$ with eigenvalues $\pm\alpha$. Let $N \in \mathbb{N}$. Suppose that α is not in $\mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$. There exists $\Phi \in \bigcap_{r'\ge 0} C_{r'}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$ and a numerical constant C' such that*

$$\forall r' \ge 0,\ |\Phi|_{r'} \le C' e^{\pi N r'};\ |\Phi^{-1}|_{r'} \le C' e^{\pi N r'} \tag{9}$$

and if $\tilde{A}$ *with eigenvalues* $\pm\tilde{\alpha}$ *is such that*

$$\partial_\omega \Phi = A\Phi - \Phi\tilde{A} \tag{10}$$

then $\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$.

Moreover,

$$|\tilde{\alpha}| < \frac{\kappa}{4G(N)}.$$

Proof. Lemma 2.1 gives a number $m, 0 < |m| \le N$, such that letting

$$\tilde{\alpha} = \alpha - i\pi\langle m, \omega\rangle$$

then $\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$. Let P be such that $P^{-1}AP$ is diagonal and $\|P\| = 1$. We define

$$\Phi(\theta) = P^{-1}\begin{pmatrix} e^{i\pi\langle m,\theta\rangle} & 0 \\ 0 & e^{-i\pi\langle m,\theta\rangle}\end{pmatrix}P.$$

Relation (10) gives

$$\tilde{A} = P^{-1}\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & -\tilde{\alpha}\end{pmatrix}P.$$

To obtain the estimate (9), we use an estimate shown for instance in [5, Lemma A]:

$$\|P^{-1}\| \le \max\left(1, \left(\frac{C.\|A\|}{2|\alpha|}\right)^6\right)$$

where C is a numerical constant, and since A is diagonalizable whenever its eigenvalues are nonzero,

$$\|P^{-1}\| \le \max\left(1, \left(\frac{C.|\alpha|}{2|\alpha|}\right)^6\right) \le C'$$

where C' is a numerical constant, which gives (9). □

2.3. **Solution of the linearized homological equation.** Our aim is to solve an equation of the form

$$\partial_\omega Z = (A+F)Z - Z(A'+F')$$

where A and F are known, $A \in sl(2,\mathbb{R})$ and F is analytic with values in $sl(2,\mathbb{R})$. If A is nonresonant, we first solve

$$\partial_\omega \tilde{X} = [A, \tilde{X}] + aF^N - a\hat{F}(0)$$

where F^N is some truncation of F and a is close enough to 1; then we define $A' = A + a\hat{F}(0)$ and F' by

$$\partial_\omega e^{\tilde{X}} = (A+F)e^{\tilde{X}} - e^{\tilde{X}}(A'+F')$$

and then we estimate F' to get $|F'|_{r'} \le \sqrt{1-a}|F|_r$. If A is resonant, we conjugate $A+F$ to a system $\tilde{A}+\tilde{F}$ where $\tilde{A}$ is nonresonant and we proceed in the same way as in the nonresonant case. So, from now on, to simplify the notations, we will assume that $A, \tilde{A}, A' \in sl(2,\mathbb{R})$, that F, $\tilde{F}$, $\tilde{X}$, F' have values in $sl(2,\mathbb{R})$ and Z, Φ have values in $SL(2,\mathbb{R})$.

PROPOSITION 2.3. *Let $N \in \mathbb{N}$, $r, r' > 0$, and suppose that $\tilde{A}$ has eigenvalues $\pm\tilde{\alpha} \in NR_\omega^N(\frac{\kappa}{4G(N)}, g)$. Let $\tilde{F} \in C_r^\omega(\mathbb{T}^d)$. Then the equation*

$$\forall \theta \in \mathbb{T}^d,\ \partial_\omega \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + \tilde{F}^N(\theta) - \hat{\tilde{F}}(0);\ \hat{\tilde{X}}(0) = 0 \tag{11}$$

has a unique solution $\tilde{X} \in C_{r'}^\omega(\mathbb{T}^d)$ such that

$$|\tilde{X}|_{r'} \leq \frac{4}{\kappa} G(N) g(N) |\tilde{F}^N|_{r'}$$

Proof. In Fourier series, equation (11) can be written:

$$\begin{aligned} &\forall m \in \mathbb{Z}^d, 0 < |m| \leq N \Rightarrow 2i\pi\langle m, \omega\rangle \hat{\tilde{X}}(m) = [\tilde{A}, \hat{\tilde{X}}(m)] + \hat{\tilde{F}}(m); \\ &|m| \in \{0\} \cup [N+1, +\infty[\Rightarrow 2i\pi\langle m, \omega\rangle \hat{\tilde{X}}(m) = [\tilde{A}, \hat{\tilde{X}}(m)]. \end{aligned} \tag{12}$$

So for $|m| \in \{0\} \cup [N+1, +\infty[$, $\hat{\tilde{X}}(m) = 0$ is a solution (not necessarily unique).

For $0 < |m| \leq N$, the solution is formally written as

$$\hat{\tilde{X}}(m) = \mathscr{L}_m^{-1} \hat{\tilde{F}}(m)$$

where $\mathscr{L}_m$ is the operator

$$\mathscr{L}_m : sl(2,\mathbb{R}) \to sl(2,\mathbb{R}),\ M \mapsto 2i\pi\langle m, \omega\rangle M - [\tilde{A}, M].$$

Its spectrum is $\{2i\pi\langle m, \omega\rangle - 2\tilde{\alpha}, 2i\pi\langle m, \omega\rangle + 2\tilde{\alpha}, 2i\pi\langle m, \omega\rangle\}$.

Now, $\omega \in NR(\kappa, G)$ and $\tilde{\alpha} \in NR_\omega^N(\frac{\kappa}{4G(N)}, g)$, so $\mathscr{L}_m$ is invertible, and if $m \in \mathbb{Z}^d$ with $|m| \in (0, N]$, then

$$\|\mathscr{L}_m^{-1}\| \leq \max\left\{\frac{G(|m|)}{\kappa}, \frac{4G(N)g(|m|)}{\kappa}\right\} = \frac{4G(N)g(|m|)}{\kappa},$$

hence

$$\|\hat{\tilde{X}}(m)\| \leq 4G(N)\frac{g(|m|)}{\kappa}\|\hat{\tilde{F}}(m)\|.$$

Thus,

$$|\tilde{X}|_{r'} \leq 4G(N) \sum_{m\in\mathbb{Z}^d \smallsetminus\{0\}, |m|\leq N} \frac{g(|m|)}{\kappa} \|\hat{\tilde{F}}(m)\| e^{2\pi|m|r'} \leq 4\frac{G(N)g(N)}{\kappa}|\tilde{F}^N|_{r'}. \qquad \square$$

2.4. **Solution of the full homological equation without resonances.** This section explains the basic step in case the constant part is nonresonant, i.e when its eigenvalues are far from all $\langle m, \omega\rangle$, $m \in \mathbb{Z}^d \smallsetminus \{0\}$.

PROPOSITION 2.4. *Let $0 < r' \leq r$, $a' \in (0,1]$, $N \in \mathbb{N}$, $\tilde{F} \in C_r^\omega(\mathbb{T}^d)$, $\tilde{A} \in sl(2,\mathbb{R})$. If $\sigma(\tilde{A}) = \{\pm\alpha\}$, $\alpha \in NR_\omega^N(\frac{\kappa}{4G(N)}, g)$, then there exists $\tilde{X}, F' \in C_{r'}^\omega(\mathbb{T}^d)$ and $A' \in sl(2,\mathbb{R})$ such that*

$$|A' - \tilde{A}| \leq \|\hat{\tilde{F}}(0)\| \tag{13}$$

$$\partial_\omega e^{\tilde{X}} = (\tilde{A} + \tilde{F})e^{\tilde{X}} - e^{\tilde{X}}(A' + F') \tag{14}$$

$$|\tilde{X}|_{r'} \leq 4a'\frac{G(N)g(N)}{\kappa}|\tilde{F}^N|_{r'} \tag{15}$$

and

$$|F'|_{r'} \le e^{|\tilde X|_{r'}}(1-a')|\tilde F|_{r'} + e^{|\tilde X|_{r'}} a' |\tilde F - \tilde F^N|_{r'} + e^{|\tilde X|_{r'}}|\tilde F|_{r'}|\tilde X|_{r'}(e^{|\tilde X|_{r'}} + a' + a' e^{|\tilde X|_{r'}}). \tag{16}$$

Proof. Let $\tilde X$ be a solution of

$$\forall \theta \in \mathbb{T}^d,\ \partial_\omega \tilde X(\theta) = [\tilde A, \tilde X(\theta)] + a' \tilde F^N(\theta) - a' \hat{\tilde F}(0);\ \hat{\tilde X}(0) = 0 \tag{17}$$

as given by Proposition 2.3 (so it satisfies (15)). Let $A' = \tilde A + a' \hat{\tilde F}(0)$ so that (13) holds, and let F' be defined by

$$\partial_\omega e^{\tilde X} = (\tilde A + \tilde F) e^{\tilde X} - e^{\tilde X}(A' + F').$$

We have

$$F' = e^{-\tilde X}(\tilde F - a' \tilde F^N) + e^{-\tilde X}\tilde F(e^{\tilde X} - Id) + a'(e^{-\tilde X} - Id)\hat{\tilde F}(0) - e^{-\tilde X}\sum_{k\ge 2}\frac{1}{k!}\sum_{l=0}^{k-1}\tilde X^l (a' \tilde F^N - a' \hat{\tilde F}(0))\tilde X^{k-1-l}.$$

Since

$$|\tilde F - a' \tilde F^N|_{r'} \le a' |\tilde F - \tilde F^N|_{r'} + (1-a')|\tilde F|_{r'}.$$

one easily obtains (16). □

REMARK. Denote $\epsilon = |\tilde F|_r$. Suppose

$$2G(N)g(N)\epsilon \le \frac{\kappa(1-a')}{2}. \tag{18}$$

Then (15) implies $|\tilde X|_{r'} \le a'(1-a')$ and thus $e^{|\tilde X|_{r'}} \le 2$. By (16), if one assumes moreover that

$$e^{-2\pi N(r-r')} \le 1-a'$$

with $r' > 0$, then

$$|\tilde F - \tilde F^N|_{r'} \le (1-a')|\tilde F - \tilde F^N|_r,$$

and hence

$$|F'|_{r'} \le 2(1-a')\epsilon + 2a'(1-a')\epsilon + 2\epsilon a'(1-a')(2+3a').$$

Thus, if a' is close enough to 1 (i.e larger than $1 - \frac{1}{14^2}$), then

$$|F'|_{r'} \le (1-a')^{\frac{1}{2}}\epsilon.$$

2.5. **Solution of the full homological equation with resonances.** This section presents the basic step when there are resonances in the constant part, i.e when its eigenvalues are too close to some $\langle m, \omega\rangle, m \in \mathbb{Z}^d \setminus \{0\}$.

PROPOSITION 2.5. *Suppose*

- $a \in (0,1)$,
- $c_0 > 0$,
- *C' is as in Proposition 2.2,*
- $N \in \mathbb{N}$,

- $r > \dfrac{2\log(g(N)G(N))}{\pi N}$,
- $F \in C^\omega_r(\mathbb{T}^d)$,
- $|F|_r = \epsilon$,
- $A \in sl(2,\mathbb{R})$,
- *the eigenvalues* $\pm\alpha$ *of* A *are not in* $\mathrm{NR}^N_\omega(\frac{\kappa}{4G(N)}, g)$.

If

$$2G(N)^2 g(N)^2 \epsilon \le \frac{(1-a)^2}{2}\kappa^2 \tag{19}$$

and

$$eC'(G\cdot g)(N+1)^{-c_0} \le 1-a \tag{20}$$

then letting $r' = \dfrac{r}{2} - c_0 \dfrac{\log(G\cdot g)(N+1)}{4\pi N}$, *there exist* $F' \in C^\omega_{r'}(\mathbb{T}^d)$, $A' \in sl(2,\mathbb{R})$ *and* $Z \in C^\omega_{r'}(2\mathbb{T}^d)$ *such that*

$$\partial_\omega Z = (A+F)Z - Z(A'+F')$$

and

$$|F'|_{r'} \le (1-a)\epsilon. \tag{21}$$

Proof. One first applies Proposition 2.2 on A. Let $\Phi, \tilde{A}$ be as in Proposition 2.2 so that $\sigma(\tilde{A}) = \pm\tilde{\alpha}$, $\tilde{\alpha} \in \mathrm{NR}^N_\omega(\frac{\kappa}{G(N)}, g)$ and $|\tilde{\alpha}| \le \frac{\kappa}{4G(N)}$; let $\tilde{F} = \Phi F \Phi^{-1}$. Note that, by construction of Φ, the map $\tilde{F}$ remains continuous on $\mathbb{T}^d$. Apply Proposition 2.4 with $a' = 1$ and with $r' = \dfrac{r}{2} - c_0 \dfrac{\log(G\cdot g)(N+1)}{4\pi N}$ to get $\tilde{X} \in C^\omega_{r'}(\mathbb{T}^d)$, $A', F' \in C^\omega_{r'}(\mathbb{T}^d)$ such that (14) and (16) hold as well as

$$|\tilde{X}|_{r'} \le 4\frac{G(N)g(N)}{\kappa}|\tilde{F}^N|_{r'}$$

and let $Z = \Phi e^{\tilde{X}} \in C^\omega_{r'}(2\mathbb{T}^d)$ so that Z satisfies (2.5). Condition (19) implies that

$$(G\cdot g)(N)|\tilde{X}|_{r'} \le \frac{(1-a)^2|\tilde{F}^N|_{r'}}{\epsilon}$$

so (16) with $a' = 1$ gives

$$|F'|_{r'} \le eC'|F - F^N|_r e^{-2\pi N(r-2r')} + eC'|F^N|_{r'} e^{2\pi N r'} \frac{(1-a)^2}{(G\cdot g)(N)}(2e+1)$$

and by the choice of r',

$$|F'|_{r'} \le eC'|F|_r (G\cdot g)(N+1)^{-c_0}.$$

This implies, by assumption (20), that

$$|F'|_{r'} \le (1-a)|F|_r. \qquad \square$$

3. ITERATION, REDUCIBILITY AND ARITHMETICAL CONDITIONS

3.1. **Iteration.** In this section we will first introduce the Brjuno-Rüssmann condition and then show how one can use it to control the convergence of the KAM iteration scheme.

ASSUMPTION 1. The functions g and G satisfy

$$\int_1^\infty \frac{\log[g(t)G(t)]}{t^2}\, dt < \infty. \tag{22}$$

In order to iterate the basic step, we will now fix the parameters as follows: let C' be as in Proposition 2.2. Furthermore, let $r_0 > 0$, $n_0 \in \mathbb{N}$ and choose

$$c_0 = \frac{r_0}{4^{n_0+3}(\sup_{t\in[1,n_0]} \frac{\log(G\cdot g)(t+1)}{t} + 1)}$$

Let $a \in [1-\bar{a}, 1)$ where $\bar{a} = \min(\frac{1}{14^2}, \frac{1}{(G\cdot g)(2)^2})$. Let $\epsilon_0 > 0$ be small enough to assure that

$$\int_{(G\cdot g)^{-1}\left(\frac{\kappa}{2(1-a)^{(n_0-5)/4}\epsilon_0^{1/2}}\right)}^\infty \frac{\log(G\cdot g)(t)}{t^2}\, dt \le \frac{r_0}{4^{n_0+2}} \tag{23}$$

and

$$eC'\epsilon_0^{\frac{c_0}{4}} \le (1-a)^2\kappa^2. \tag{24}$$

For all $n \in \mathbb{N}$, let $\epsilon_n = (1-a)^{\frac{n}{2}}\epsilon_0$ and let N_n be the biggest integer such that

$$(G\cdot g)(N_n)^2 \le \frac{(1-a)^2}{4\epsilon_n}\kappa^2$$

(N_n exists since $\epsilon_n \le \frac{(1-a)^2\kappa^2}{4e(G\cdot g)(1)^2}$).

The above choices of the sequences ϵ_n and N_n are made in such a way that the following holds:

$$\int_{N_{n_0}}^\infty \frac{\log(G\cdot g)(t)}{t^2}\, dt \le \frac{r_0}{4^{n_0+2}}.$$

REMARK. The number ϵ_0 will then only depend on a, κ, g, G, n_0 and r_0 (the larger n_0 and the smaller r_0 are, the smaller ϵ_0 will be).

To simplify the notations, from now on the functions A_n are understood to be in $sl(2,\mathbb{R})$, while the F_n have their values in $sl(2,\mathbb{R})$ and Z'_n, Z_n have their values in $SL(2,\mathbb{R})$.

PROPOSITION 3.1. *Let $A \in sl(2,\mathbb{R})$ and $F \in C^\omega_{r_0}(\mathbb{T}^d)$. If $|F|_{r_0} \le \epsilon_0$, then there exist sequences $(r_n)_{n\in\mathbb{N}}, r_n > 0$, $Z_n, \in C^\omega_{r_n}(2\mathbb{T}^d)$, A_n with spectrum $\pm\alpha_n$, $F_n \in C^\omega_{r_n}(2\mathbb{T}^d)$, and $m_n \in \mathbb{Z}^d$, such that*

1. *if all m_n are zero when $n \ge n_0$, then r_n has a positive limit;*
2. *if $m_n \ne 0$ then $|\alpha_n - \pi\langle m_n, \omega\rangle| \le \frac{\kappa}{4G(N_n)}$;*
3. *m_n has modulus less than N_n,*
4. *$|F_n|_{r_n} \le \epsilon_n$;*

5. $\partial_\omega Z_n = (A+F)Z_n - Z_n(A_n + F_n)$;
6. $|\alpha_{n-1} - i\pi\langle m_{n-1}, \omega\rangle - \alpha_n| \le \sqrt{\epsilon_{n-1}}$.

REMARK. Proposition 3.1 implies that $A+F$ is reducible in $C^\omega_{r'}$ for some $r'>0$ if all m_n are zero for $n \ge n_0$.

Proof. This proposition is shown by recurrence. Suppose these sequences are defined up to some $n \in \mathbb{N}$ and suppose that for all $n' \le \min(n-1, n_0)$, $r_{n'+1} \ge \frac{r_{n'}}{4}$. We must distinguish two cases according to the possibility that the spectrum of A_n is resonant or not.

First case: $\alpha_n \in NR^{N_n}_\omega(\frac{\kappa}{4G(N_n)}, g)$**.** Let $r_{n+1} = r_n - c_0 \frac{|\log(1-a)|}{2\pi N_n}$, so that $r_{n+1} \ge r_n/2$ if $n \le n_0$. One can apply Proposition 2.4 with $r = r_n$, $r' = r_{n+1}$, $N = N_n$, $\tilde F = F_n$ and $\tilde A = A_n$ and obtain $Z'_n = e^{\tilde X_n} \in C^\omega_{r_{n+1}}(\mathbb{T}^d)$, $F_{n+1} \in C^\omega_{r_{n+1}}(\mathbb{T}^d)$, A_{n+1} such that

$$\partial_\omega Z'_n = (A_n + F_n)Z'_n - Z'_n(A_{n+1} + F_{n+1})$$

and

$$|F_{n+1}|_{r_{n+1}} \le (1-a)^{\frac{1}{2}} \epsilon_n = \epsilon_{n+1}.$$

One then takes $Z_{n+1} = Z_n Z'_n$.

Second case: $\alpha_n \notin NR^{N_n}_\omega(\frac{\kappa}{4G(N_n)}, g)$**.** Assumption (19) is satisfied by definition of N_n; assumption (20) is also satisfied since, by maximality of N_n,

$$(G\cdot g)(N_n+1)^{-c_0} \le \left(\frac{2\epsilon_n}{(1-a)^2\kappa^2}\right)^{c_0/2} \le \left(\frac{2\epsilon_0}{(1-a)^2\kappa^2}\right)^{c_0/2}$$

which, together with (24), implies that

$$(G\cdot g)(N_n+1)^{-c_0} \le \frac{1-a}{eC'}.$$

Therefore, one can apply Proposition 2.5 with

$$r = r_n, \qquad r' = r_{n+1} = \frac{r_n}{2} - c_0 \frac{\log(G\cdot g)(N_n+1)}{\pi N_n},$$

so that $r_{n+1} \ge r_n/4$ if $n \le n_0$, and $N = N_n$. It follows that there exist $A_{n+1} \in sl(2,\mathbb{R})$, $F_{n+1} \in C^\omega_{r_{n+1}}(\mathbb{T}^d)$ and $Z'_n \in C^\omega_{r_{n+1}}(2\mathbb{T}^d)$ such that

$$\partial_\omega Z'_n = (A_n + F_n)Z'_n - Z'_n(A_{n+1} + F_{n+1})$$

and

$$|F_{n+1}|_{r_{n+1}} \le (1-a)|F_n|_{r_n} \le \epsilon_{n+1}$$

One then takes $Z_{n+1} = Z_n Z'_n$.

To complete the proof, we now need to show that $(r_n)_n$ has a positive limit if all m_n are zero for $n \ge N_0$. We have

$$\lim_{n\to\infty} r_n = r_{n_0} - \sum_{k=n_0}^{\infty} (r_k - r_{k+1}) \ge \frac{r_0}{4^{n_0}} - \sum_{k\ge n_0} \frac{|\log(1-a)|}{2\pi N_k}.$$

Now, for all n,

$$N_n = E\left((G\cdot g)^{-1}\left(\frac{(1-a)\kappa}{2\sqrt{\epsilon_n}}\right)\right)$$

thus

$$\lim_{n\to\infty} r_n \geq \frac{r_0}{4^{n_0}} - \frac{|\log(1-a)|}{2\pi}\int_{n_0}^{\infty}\left[(G\cdot g)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n}{4}-1}}\right)\right]^{-1} dn.$$

Through the change of variables $X = \dfrac{\kappa}{2(1-a)^{\frac{n}{4}-1}}$,

$$\lim_{n\to\infty} r_n \geq \frac{r_0}{4^{n_0}} - \int_{(G\cdot g)(N_{n_0})}^{\infty}\frac{1}{\pi(G\cdot g)^{-1}(X)X}\,dX.$$

Letting now $Y = (G\cdot g)^{-1}(X)$, the integral becomes

$$\int_{(G\cdot g)(N_{n_0})}^{\infty}\frac{1}{\pi(G\cdot g)^{-1}(X)X}\,dX = \int_{N_{n_0}}^{\infty}\frac{1}{\pi Y(G\cdot g)(Y)}\,d(G\cdot g)(Y)$$
$$= \frac{-\log(G\cdot g)(N_{n_0})}{N_{n_0}} + \int_{N_{n_0}}^{\infty}\frac{\log(G\cdot g)(Y)}{Y^2}\,dY.$$

Therefore

$$\lim_{n\to\infty} r_n \geq \frac{r_0}{4^{n_0}} + \frac{\log(G\cdot g)(N_{n_0})}{\pi N_{n_0}} - \frac{1}{\pi}\int_{N_{n_0}}^{\infty}\frac{\log(G\cdot g)(Y)}{Y^2}\,dY$$

so, by (3.1),

$$\lim_{n\to\infty} r_n \geq \frac{r_0}{4^{n_0+1}},$$

which is positive. □

3.2. A link between the Brjuno sum and the allowed perturbation. It is easily seen that the condition on ϵ_0 can also be expressed more conveniently as the following sufficient condition:

$$\epsilon_0 \leq \exp\left(-\frac{r_0}{4^{n_0}} - \left|\log\frac{\kappa}{2(1-a)^{n_0}}\right| - 2\int_1^{\infty}\frac{\log(g\cdot G)(t)}{t^2}\,dt\right) \tag{25}$$

Indeed, we have the following bound:

$$\left|\frac{1}{2}\log\epsilon_0 + \int_1^{\infty}\frac{\log(g\cdot G)(t)}{t^2}\,dt)\right| = \left|\int_1^{\infty}\frac{\log(\sqrt{\epsilon_0}g\cdot G)(t)}{t^2}\,dt)\right|$$
$$\leq \frac{r_0}{4^{n_0}} + \int_1^{(g\cdot G)^{-1}\left(\frac{\kappa}{2(1-a)^{n_0}\sqrt{\epsilon_0}}\right)}\frac{|\log\sqrt{\epsilon_0}(g\cdot G)(t)|}{t^2}\,dt$$

and the conclusion follows easily from the upper bound on t.

3.3. **Reducibility theorem.** We will now need one more assumption on the approximation functions G and g.

ASSUMPTION 2. The map $t \mapsto \dfrac{g(t^2)}{G(t)}$ is bounded.

Now we can prove the main result:

THEOREM 3.2. *If $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C_r^\omega(\mathbb{T}^d)$, $n_0 \in \mathbb{N}$, $\rho(A+F) \in \mathrm{NR}_\omega(\kappa', g)$, $\kappa' > \kappa \sup_{t \geq n_0} \frac{g(t^2)}{G(t)}$, then under Assumptions 1 and 2 on the approximation functions g and G, there exists $\epsilon_0 > 0$ depending only on g, κ, G, n_0, r such that if*

$$|F|_r \leq \epsilon_0,$$

then there exists $r' \in (0, r)$ such that $A+F$ is reducible in $C_{r'}^\omega(2\mathbb{T}^d)$.

Proof. Let $a \in [\max(1 - \dfrac{1}{14^2}, 1 - \dfrac{1}{G \cdot g(2)^2}), 1[$. Let $\epsilon_0 > 0$, $(\epsilon_n)_{n\in\mathbb{N}}$, $(N_n)_{n\in\mathbb{N}}$ as defined at the beginning of Section 3.1. Let (r_n), (α_n), (m_n), (A_n), (F_n), (Z_n) be the sequences given by Proposition 3.1. The sequence (A_n) is bounded in $sl(2,\mathbb{R})$ for the operator norm so taking a subsequence (A_{n_k}), we find that A_{n_k} tends to some $A_\infty \in gl(2,\mathbb{R})$. Now $\rho(A_\infty)$ is the limit of $\rho(A_{n_k})$ [4, Lemma A.3] which implies that for all n,

$$\rho(A_\infty) = \rho(A_{n+1}) - \lim_{k\to\infty} \sum_{j=n+1}^{n_k - 1} (\rho(A_j) - \rho(A_{j+1})).$$

Moreover,

$$\rho(A+F) = \rho(A_\infty) + \pi \sum_{j\geq 0} \langle m_j, \omega\rangle$$

(see also [4]). Therefore

$$\begin{aligned}
|\rho(A+F) - \pi \sum_{j\leq n} \langle m_j, \omega\rangle| &= |\rho(A_\infty) + \pi \sum_{j\geq n+1} \langle m_j, \omega\rangle| \\
&\leq |\alpha_{n+1}| + \sum_{j\geq n+1} |\alpha_j - \pi\langle m_j, \omega\rangle - \alpha_{j+1}| \\
&\leq |\alpha_{n+1}| + \sum_{j\geq n+1} \sqrt{\epsilon_j}.
\end{aligned}$$

Suppose $\rho(A+F)$ satisfies

$$\forall m \in \mathbb{Z}^d,\ |\rho(A+F) - \pi\langle m, \omega\rangle| \geq \frac{\kappa'}{g(|m|)}.$$

In particular,

$$|\rho(A+F) - \pi \sum_{j\leq n} \langle m_j, \omega\rangle| \geq \frac{\kappa'}{g(|\sum_{j\leq n} m_j|)},$$

and so

$$\frac{\kappa'}{g(|\sum_{j\leq n} m_j|)} \leq \sum_{j\geq n+1} \sqrt{\epsilon_j} + |\alpha_{n+1}|.$$

Let $n > n_0$. Assume $m_n \neq 0$. Then we have

$$|\alpha_{n+1}| \leq |\alpha_n - \pi\langle m_n, \omega\rangle| + |\alpha_n - \pi\langle m_n, \omega\rangle - \alpha_{n+1}| \leq \frac{\kappa}{4G(N_n)} + \sqrt{\epsilon_n},$$

so

$$\kappa' \leq \left[\sum_{j\geq n} \sqrt{\epsilon_j} + \frac{\kappa}{4G(N_n)}\right] g(|\sum_{j\leq n} m_j|). \tag{26}$$

Thus

$$\kappa' \leq \left[\sum_{j\geq n} \sqrt{\epsilon_j} + \frac{\kappa}{4G(N_n)}\right] g(\sum_{j\leq n} |m_j|).$$

Now

$$\sum_{j\geq n} \sqrt{\epsilon_j} = \frac{1}{1-(1-a)^{\frac{1}{4}}} \sqrt{\epsilon_n} \leq 2\sqrt{\epsilon_n},$$

and since, by definition of N_n,

$$\epsilon_n \leq \frac{(1-a)^2\kappa^2}{4G(N_n)^2 g(N_n)^2},$$

we also have

$$\kappa' \leq \frac{\kappa}{G(N_n)} g\Big(\sum_{j\leq n} |m_j|\Big).$$

Now, note that $\sum_{j\leq n} |m_j| \leq N_n^2$. This comes from the fact that, denoting by m_{j_k} the subsequence of nonzero m_j's, then for all k,

$$\begin{aligned}|\langle m_{j_{k+1}}, \omega\rangle| &< |\langle m_{j_{k+1}}, \omega\rangle - \alpha_{j_{k+1}}| + |\langle m_{j_k}, \omega\rangle - \alpha_{j_k} + \alpha_{j_{k+1}}| + |\langle m_{j_k}, \omega\rangle - \alpha_{j_k}| \\ &< \frac{\kappa}{4G(N_{j_{k+1}})} + 2\sqrt{\epsilon_{j_k}} + \frac{\kappa}{4G(N_{j_k})} \\ &\leq \frac{\kappa}{4G(N_{j_{k+1}})} + \frac{\kappa}{2G(N_{j_k})} + \frac{\kappa}{4G(N_{j_k})}\end{aligned}$$

which together with the arithmetic condition on ω implies that $N_{j_{k+1}} > N_{j_k} \geq k$. Therefore

$$\kappa' \leq \frac{\kappa}{G(N_n)} g(N_n^2)$$

and since by assumption $\frac{g(t^2)}{G(t)}$ is bounded,

$$\kappa' \leq \kappa \sup_{t\geq n} \frac{g(t^2)}{G(t)}.$$

In other words, if

$$\kappa' > \kappa \sup_{t\geq n} \frac{g(t^2)}{G(t)},$$

then $m_n = 0$ for all $n \geq n_0$; and so $A + F$ is analytically reducible. □

This proves Theorem 1.3. Here is an easy consequence of the main result:

COROLLARY 3.3. *Let $A \in sl(2, \mathbb{R})$, $r > 0$, $F \in C_r^\omega(\mathbb{T}^d)$. Assume*

1. *the map $t \mapsto \dfrac{g(t^2)}{G(t)}$ tends to* 0*,*

2. $\rho(A+F) \in \mathrm{NR}_\omega(\kappa', g)$ *for some* $\kappa' > 0$.

There exist ϵ_0 *depending only on* g, κ, G, $\rho(A+F)$, r *such that if*

$$|F|_r \le \epsilon_0,$$

then there exists $r' \in (0,r)$ *such that* $A+F$ *is reducible in* $C^\omega_{r'}(2\mathbb{T}^d)$.

Proof. Let n_0 be the smallest integer such that $\kappa' > \sup_{t\ge n_0} \frac{g(t^2)}{G(t)}$. Take ϵ_0 as in Theorem 3.2 so that it really depends on g, κ, G, $\rho(A+F)$, r and apply Corollary 3.2. □

Eliasson's Theorem is a particular case of Corollary 3.3 since we can take $g(t) = t^\mu$, $G(t) = t^{\mu'}$ with $\mu' \ge \frac{\mu}{2}$, $\mu \ge 1$, $\mu' \ge 1$, as we will see in the next section.

Our main result will be made more convenient by the following corollary:

COROLLARY 3.4. *Let* $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^\omega_r(\mathbb{T}^d)$. *Assume* ω *is a Brjuno vector and* $\rho(A+F)$ *is a* $\frac{1}{2}$*-Brjuno number with respect to* ω. *There exists* ϵ_0 *depending only on* $\rho(A+F), \omega, r$ *such that if* $|F|_r \le \epsilon_0$, *then there exists* $r' \in (0,r)$ *such that* $A+F$ *is reducible in* $C^\omega_{r'}$.

Proof. By assumption, there exists $\kappa' > 0$ and g positive increasing and continuous such that $\int_1^\infty \frac{\log g(t)}{t^{3/2}}\,dt < +\infty$ and $\rho(A+F) \in \mathrm{NR}_\omega(\kappa', g)$; there also exists $\kappa > 0$ and G' positive increasing and continuous such that $\int_1^\infty \frac{\log G'(t)}{t^2}\,dt < +\infty$ and $\omega \in \mathrm{NR}(\kappa, G')$. Now let for all $t, G(t) = t\max(G'(t), g(t^2))$. The function G is positive increasing and continuous and $\omega \in \mathrm{NR}(\kappa, G)$. Since $g(t^2)/G(t) \le 1/t$ for all t, we can apply the previous corollary. □

3.4. **Possible choices of approximation functions.** Here we give a few examples of approximation functions to which Theorem 3.2 can be applied.

3.4.1. *Verification of Assumption 2.* Here are a few examples where Assumption 2 holds, i.e $g(t^2)/G(t)$ is bounded:

1. $g(t) = t^\mu$, $G(t) = t^{\mu'}$ with $\mu' \ge \frac{\mu}{2}$, $\mu \ge 1$, $\mu' \ge 1$;
2. $g(t) = e^{t^\alpha}$, $G(t) = e^{t^{\alpha'}}$ with $\alpha \le \frac{\alpha'}{2}$, $\alpha < 1$, $\alpha' < 1$;
3. $g(t) = e^{t^\alpha}$, $G(t) = e^{t/(\log t)^\delta}$, $\alpha < 1$, $\delta > 1$.

In Example 1, and if $\mu' > \mu/2$, then, as noted in Section 3.4, the condition on ϵ_0 does not depend on n_0 and κ' might be arbitrarily small, which corresponds to Eliasson's full-measure reducibility result in [4].

3.4.2. *Smallness conditions.* We shall make conditions (23) and (24) more explicit for the particular cases that we mentioned before, namely, when $(g\cdot G)(t) = t^{\mu+\mu'}$, $\mu, \mu' > 2$ (diophantine case), when $(g\cdot G)(t) = e^{t^\alpha + t^{\alpha'}}$, $\alpha, \alpha' < 1$ and when $(g\cdot G)(t) = e^{\frac{t}{(\log t)^\delta} + t^\alpha}$, $\delta > 1$, $\alpha < 1$.

Recall Condition (23):

$$\int_{(g\cdot G)^{-1}\left(\frac{\kappa}{2(1-a)^{(n_0-5)/4}\sqrt{\epsilon_0}}\right)}^{\infty} \frac{\log(g\cdot G)(t)}{t^2}\,dt \le \frac{r_0}{4^{n_0+2}} \tag{23}$$

LEMMA 3.5. *If* $(g\cdot G)(t) = t^{\mu+\mu'}$*, Condition* (23) *is satisfied if*

$$(1-a)^{\frac{1}{8(\mu+\mu')}} \le \frac{1}{2} \quad \text{and} \quad \epsilon_0 \le \left(\frac{r_0}{8(\mu+\mu')}\right)^{4\mu}(1-a)^{3/2}\kappa$$

or if

$$\epsilon_0 \le \left(\frac{r_0}{4^{n_0+3}(\mu+\mu')}\right)^{4(\mu+\mu')}\kappa. \tag{27}$$

Proof. Rewrite Condition (23) as

$$\int_b^{\infty} \frac{(\mu+\mu')\log t}{t^2}\,dt \le \frac{r_0}{4^{n_0+2}}$$

where $b = \left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right)^{\frac{1}{(\mu+\mu')}}$. Integrating by parts, this is

$$(\mu+\mu')\frac{\log b+1}{b} \le \frac{r_0}{4^{n_0+2}}.$$

It is enough that

$$2(\mu+\mu')\frac{1}{\sqrt{b}} \le \frac{r_0}{4^{n_0+2}},$$

that is,

$$2(\mu+\mu')\left(\frac{2}{\kappa}(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}\right)^{\frac{1}{2(\mu+\mu')}} \le \frac{r_0}{4^{n_0+2}}, \tag{28}$$

which is true if (27) is satisfied. If moreover

$$(1-a)^{\frac{1}{8(\mu+\mu')}} \le \frac{1}{2}$$

then (28) is satisfied as long as

$$\epsilon_0 \le \left(\frac{r_0}{8(\mu+\mu')}\right)^{4(\mu+\mu')}(1-a)^{3/2}.$$

□

LEMMA 3.6. *If* $(g\cdot G)(t) = e^{t^\alpha + t^{\alpha'}}$, $\alpha' < \alpha < 1$*, then* (23) *holds if*

$$\epsilon_0 \le \frac{\kappa}{4}\exp\left[-2\left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right]. \tag{29}$$

Proof. By plugging $e^{t^\alpha+t^{\alpha'}}$ into (23) and recalling that $g\cdot G$ is increasing, one finds that Condition (23) holds if

$$(1-a)^{\frac{n_0-3}{2}}\epsilon_0 \le \frac{\kappa}{4}\exp\left[-2\left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right]$$

so, in particular, (29) is a sufficient condition. □

LEMMA 3.7. *If* $(g\cdot G)(t) = e^{\frac{t}{(\log t)^\delta}+t^\alpha}$, $\alpha<1$, $\delta>1$, *then* (23) *holds if*

$$\epsilon_0 \le \frac{\kappa}{4}\left((g\cdot G)\circ\exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)(1-\alpha)}\right)^{\frac{1}{(\delta-1)(1-\alpha)}}\right]\right)^{-2}.$$

Proof. In this case, (23) can be rewritten

$$\int_b^\infty \frac{1}{t^{2-\alpha}}\,dt + \int_b^\infty \frac{1}{t(\log t)^\delta}\,dt \le \frac{r_0}{4^{n_0+2}}$$

with $b=(g\cdot G)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right)$. Integrating by parts, we compute

$$\int_b^\infty \frac{1}{t(\log t)^\delta}\,dt = \left[\frac{1}{(\log t)^{\delta-1}}\right]_b^\infty + \delta\int_b^\infty \frac{1}{t(\log t)^\delta}\,dt$$

which implies

$$(\delta-1)\int_b^\infty \frac{1}{t(\log t)^\delta}\,dt = \frac{1}{(\log b)^{\delta-1}}$$

so that (23) is equivalent to

$$\frac{b^{\alpha-1}}{1-\alpha} + \frac{1}{(\delta-1)(\log b)^{\delta-1}} \le \frac{r_0}{4^{n_0+2}}.$$

Now $\frac{1}{(\delta-1)(\log b)^{\delta-1}} \le \frac{r_0}{4^{n_0+3}}$ if

$$(g\cdot G)\circ\exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)}\right)^{\frac{1}{\delta-1}}\right]^2 \le \frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}$$

and $\frac{b^{\alpha-1}}{1-\alpha} \le \frac{r_0}{4^{n_0+3}}$ if

$$\epsilon_0 \le \frac{\kappa}{4}\exp\left[-2\left(\frac{4^{n_0+3}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right]$$

so that (23) holds if

$$\epsilon_0 \le \frac{\kappa}{4}\left((g\cdot G)\circ\exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)(1-\alpha)}\right)^{\frac{1}{(\delta-1)(1-\alpha)}}\right]\right)^{-2}. \qquad \square$$

Finally we consider Condition (24): first note that in these examples, one can drop the term $\sup_{t\in[1,n_0]}\frac{\log(G\cdot g)(t+1)}{t}$ in (24) since it is a nonincreasing function. Therefore (24) holds if, for instance,

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \le (1-a)^2\kappa$$

Thus in Example 1, (24) holds for a suitable a if

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \le \frac{1}{2^{2(\mu+\mu')}}\kappa.$$

In Example 2, (24) holds for a suitable a if

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \le \kappa\exp(-4\cdot 4^{\frac{1}{1-\alpha'}})$$

and finally in Example 3, it holds under the analogous condition

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \leq \kappa \exp(-4\cdot 4^{\frac{1}{1-\alpha}}).$$

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