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On the cohomological equation for interval exchange maps

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Abstract We exhibit an explicit full measure class of minimal interval exchange maps T for which the cohomological equation $\Psi - \Psi \circ T = \Phi$ has a bounded solution Ψ provided that the datum Φ belongs to a finite codimension subspace of the space of functions having on each interval a derivative of bounded variation.
The class of interval exchange maps is characterized in terms of a diophantine condition of "Roth type" imposed to an acceleration of the Rauzy–Veech–Zorich continued fraction expansion associated to T. ***To cite this article: S. Marmi, P. Moussa, J-C. Yoccoz, C. R. Acad. Sci. Paris, Ser. I projet de Note (2003).***

Sur l'équation cohomologique pour les échanges d'intervalles

Résumé On présente une classe explicite d'échanges d'intervalles T, de mesure pleine, pour laquelle l'équation cohomologique $\Psi - \Psi \circ T = \Phi$ admet une solution bornée Ψ, à condition que la donnée Φ appartienne à un sous–espace de codimension finie de l'espace des fonctions dont la dérivée sur chaque intervalle est de variation bornée.
Cette classe est définie par une condition diophantienne "de type Roth" exprimé dans une variante du développement en fraction continue de Rauzy–Veech–Zorich associé à T. ***Pour citer cet article : S. Marmi, P. Moussa, J-C. Yoccoz, C. R. Acad. Sci. Paris, Ser. I projet de Note (2003).***

Version française abrégée

Soit $\mathcal{A}$ un alphabet constitué de $d \geq 2$ lettres. Soit (π_0, π_1) un couple de *bijections* de $\mathcal{A}$ sur $\{1, \ldots, d\}$, qu'on supposera toujours *admissible*: pour tout $1 \leq k < d$, on a $\pi_0^{-1}(\{1, \ldots, k\}) \neq \pi_1^{-1}(\{1, \ldots, k\})$. Si on se donne aussi des longueurs $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ (avec $\lambda_\alpha > 0$), on définit un échange d'intervalles T [V1] (avec discontinuités marquées) par la formule suivante: pour $\alpha \in \mathcal{A}$, $0 \leq x < \lambda_\alpha$

$$T\left(x + \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta\right) = x + \sum_{\pi_1(\beta)<\pi_1(\alpha)} \lambda_\beta \, .$$

On posera $I_\alpha = \left[\sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \sum_{\pi_0(\beta)\leq\pi_0(\alpha)} \lambda_\beta\right)$. Comme T est une translation sur chaque I_α, T préserve l'orientation et la mesure de Lebesgue. On peut penser $\mathcal{A}$ comme l'ensemble des orbites des points de discontinuité de T.

Note presented by Etienne Ghys

Notons Δ le simplexe standard de dimension $d-1$ dans R^d. Alors le vecteur normalisé des longueurs $\left(\lambda_\alpha\left(\sum_{\beta\in\mathcal{A}}\lambda_\beta\right)^{-1}\right)_{\alpha\in\mathcal{A}}$ appartient à Δ: on notera $\Delta(\pi_0,\pi_1)$ le simplexe des longueurs normalisées associé à un couple admissible (π_0,π_1).

On notera $\mathrm{BV}(\sqcup I_\alpha)$ l'espace des fonctions φ sur $[0,1)$ dont la restriction à chaque I_α est de variation bornée, et $\mathrm{BV}_*(\sqcup I_\alpha)$ l'hyperplan formé des fonctions de moyenne nulle.

Théorème. *Soit* (π_0,π_1) *un couple admissible. Il existe une partie* $D(\pi_0,\pi_1)\subset\Delta(\pi_0,\pi_1)$ *de* mesure totale *telle que, pour tout* $T\in D(\pi_0,\pi_1)$*, tout* $\varphi\in\mathrm{BV}_*(\sqcup I_\alpha)$ *on puisse trouver*

(1) une fonction Φ*, Lipschitzienne sur chaque* I_α*, dont la dérivée sur chaque* I_α *est* φ *et de moyenne nulle sur* $\sqcup I_\alpha$*;*

(2) une fonction bornée Ψ*, de moyenne nulle sur* $\sqcup I_\alpha$*,*
vérifiant l'équation cohomologique $\Psi-\Psi\circ T=\Phi$.

Remarques. (1) La partie $D(\pi_0,\pi_1)$ of $\Delta(\pi_0,\pi_1)$ est explicitement définie ci–dessous.

(2) Keane [Ke1] a montré que si aucune orbite de T ne passe plus d'une fois par une discontinuité de T, alors T est minimal. Les échanges d'intervalles dans $D(\pi_0,\pi_1)$ satisfont l'hypothèse de Keane. Masur [Ma] et Veech [V2] ont montré que presque tout échange d'intervalles est uniquement ergodique (pour $d\geq 4$, il existe des échanges d'intervalles qui ne sont pas uniquement ergodiques [KN, Ke2]). Les échanges d'intervalles dans $D(\pi_0,\pi_1)$ sont uniquement ergodiques.

(3) Notre résultat est clairement relié (via une suspension singulière) au théorème de Forni [Fo] sur l'équation cohomologique pour les champs de vecteurs préservant les aires sur les surfaces. Notre méthode est différente de celle de Forni: il construit des instruments d'analyse de Fourier sur les surfaces plates (singulières) et utilise le théorème de Fatou sur les valeurs au bord des fonctions holomorphes bornées. Nous mettons à profit un algorithme de fraction continue dû à Rauzy–Veech–Zorich pour obtenir une condition diophantienne explicite. La perte de différentiabilité est par ailleurs plus faible que dans son théorème.

Esquisse de preuve.
Soit $T=T(0)$ un échange d'intervalles vérifiant l'hypothèse de Keane. Une version accélérée de l'algorithme de Rauzy–Veech–Zorich ([Ra], [V2], [Z1], [Z2]) fournit une suite d'échanges d'intervalles $(T(n))_{n\geq 0}$ avec des longueurs associées $(\lambda_\alpha(n))_{\alpha\in\mathcal{A}\,,\,n\geq 0}$ qui satisfont aux propriétés suivantes:

(i) pour $m<n$, $T(n)$ est l'application de premier retour de $T(m)$ sur $I(n)=[0,\sum_{\alpha\in\mathcal{A}}\lambda_\alpha(n))$;

(ii) pour $m<n$, on a $\lambda(m)=Q(m,n)\lambda(n)$, avec une matrice $Q(m,n)\in\mathrm{SL}\,(d,Z)$ à coefficients ≥ 0;

(iii) pour $m<n$,$\alpha,\beta\in\mathcal{A}$, le temps passé par $I_\beta(n)$ dans $I_\alpha(m)$ avant retour dans $I(n)$ est $Q_{\alpha\beta}(m,n)$, et le temps de retour est $Q_\beta(m,n)=\sum_{\alpha\in\mathcal{A}}Q_{\alpha\beta}(m,n)$;

(iv) on a $Q_{\alpha\beta}(m,n)>0$ pour $\alpha,\beta\in\mathcal{A}$, $n\geq m+m(d)$.

Soit φ une fonction sur $\sqcup_{\alpha\in\mathcal{A}}I_\alpha(m)$; pour $n>m$, $\beta\in\mathcal{A}$, $x\in I_\beta(n)$, on définit

$$S(m,n)\varphi(x)=\sum_{0\leq l<Q_\beta(m,n)}\varphi((T(m))^l(x))\ .$$

Notons $\Gamma^{(m)}$ l'espace de dimension d des fonctions constantes sur chaque $I_\alpha(m)$; alors $S(m,n)$ envoie $\Gamma^{(m)}$ sur $\Gamma^{(n)}$, sa matrice par rapport aux bases canoniques étant ${}^tQ(m,n)$. On posera

$$\Gamma_s^{(m)}=\{\chi\in\Gamma^{(m)}\,,\,\limsup_{n\to\infty}\frac{\log\|S(m,n)\chi\|}{\log\|Q(m,n)\|}<0\}\ .$$

On dira que T est de type Roth si:

(a) pour tout $\varepsilon > 0$, on a $\|Q(n,n+1)\| \leq \|Q(0,n)\|^{\varepsilon}$ pour tout n assez grand;
(b) Il existe $\theta > 0$ tel qu'on ait $\|S(0,n)|_{\Gamma_*^{(0)}}\| \leq \|Q(0,n)\|^{1-\theta}$ pour tout n assez grand, $\Gamma_*^{(0)}$ désignant l'hyperplan de $\Gamma^{(0)}$ formé des fonctions de moyenne nulle;
(c) L'opérateur $S(m,n)_\flat : \Gamma^{(m)}/\Gamma_s^{(m)} \to \Gamma^{(n)}/\Gamma_s^{(n)}$ induit par $S(m,n)$ vérifie, pour tout $\varepsilon > 0$, $\|(S(m,n)_\flat)^{-1}\| \leq \|Q(0,n)\|^{\varepsilon}$ si n est assez grand.

Sous les conditions (a) et (b), on montre que, pour une fonction à variation bornée φ sur $\sqcup_{\alpha\in\mathcal{A}} I_\alpha(0)$, on a

$$\|S(0,n)(\varphi)\|_{L^\infty} \leq \|Q(0,n)\|^{1-\theta'} \|\varphi\|_{BV} \ ,$$

avec $\theta' = \theta'(\theta) > 0$, pour n assez grand. En utilisant aussi (c), on arrive à construire une fonction lipschitzienne Φ dont la dérivée sur chaque $I_\alpha(0)$ est φ, qui vérifie

$$\sum_{n\geq 0} \|Q(n,n+1)\| \|S(0,n)(\Phi)\|_{L^\infty} < +\infty \ .$$

On en déduit que les sommes de Birkhoff de Φ sont bornées et on conclut grâce à un théorème de Gottschalk–Hedlund.

1. Interval exchange maps and the cohomological equation. Main Theorem

1.1 Let $\mathcal{A}$ denote an alphabet with $d \geq 2$ elements. Consider a pair (π_0, π_1) of *bijections* of $\mathcal{A}$ on $\{1,\ldots,d\}$: we will always assume that the pair is *admissible*: for all $1 \leq k < d$ one has $\pi_0^{-1}(\{1,\ldots,k\}) \neq \pi_1^{-1}(\{1,\ldots,k\})$. One can associate the permutation $\pi = \pi_1 \circ \pi_0^{-1}$ of $\{1,\ldots,d\}$ to the pair (π_0, π_1). Given an admissible pair (π_0, π_1) and a vector of lengths of intervals $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$, an *interval exchange map* (i.e.m.) T [V1](*with marked discontinuities*) is defined through the formula: for $\alpha \in \mathcal{A}$, $0 \leq x < \lambda_\alpha$

$$T\left(x + \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta\right) = x + \sum_{\pi_1(\beta)<\pi_1(\alpha)} \lambda_\beta \ . \tag{1}$$

We write $I_\alpha = \left[\sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \sum_{\pi_0(\beta)\leq\pi_0(\alpha)} \lambda_\beta\right)$. We can think of $\mathcal{A}$ as the set of the orbits of the discontinuities, taking right limits (each interval is associated to the discontinuity at its left endpoint). Each i.e.m. is piecewise a translation, orientation–preserving and preserves Lebesgue measure.

1.2 Let Δ denote the standard $(d-1)$–dimensional simplex in R^d. The normalized vector $\left(\lambda_\alpha \left(\sum_{\beta\in\mathcal{A}} \lambda_\beta\right)^{-1}\right)_{\alpha\in\mathcal{A}}$ belongs to Δ: we denote $\Delta(\pi_0, \pi_1)$ the symplex of normalized lengths corresponding to the admissible pair (π_0, π_1). We will denote BV $(\sqcup I_\alpha)$ (resp. BV$_*$ $(\sqcup I_\alpha)$) the space of functions φ whose restriction to each of the intervals I_α is a function of bounded variation (resp. the hyperplane of BV $(\sqcup I_\alpha)$ made of functions whose integral on the disjoint union $\sqcup I_\alpha$ vanishes).

Our main result can be stated as follows:

Theorem. *Let (π_0, π_1) be admissible. There exists a subset $D(\pi_0, \pi_1) \subset \Delta(\pi_0, \pi_1)$ of* full measure *such that for all $T \in D(\pi_0, \pi_1)$ and for all function $\varphi \in$ BV$_*$ $(\sqcup I_\alpha)$ one can find*
(1) a function Φ, Lipschitz on each I_α, with $D\Phi = \varphi$ on each I_α and total mean value 0 on $\sqcup I_\alpha$;
(2) a bounded function Ψ, with total mean value 0 on $\sqcup I_\alpha$,
which satisfy the cohomological equation $\Psi - \Psi \circ T = \Phi$.

1.3 Remarks. (1) The subset $D(\pi_0, \pi_1)$ of $\Delta(\pi_0, \pi_1)$ will be explicitely defined below.

(2) Keane [Ke1] proved that if there is no orbit segment starting and ending with discontinuities, T is minimal. All the i.e.m. considered here are assumed to satisfy this hypothesis, which in particular holds when the lengths are rationally independent. Masur [Ma] and Veech [V2] proved that almost all i.e.m. are uniquely ergodic (for $d \geq 4$, there exist non uniquely ergodic i.e.m. [KN, Ke2]). The diophantine condition $D(\pi_0, \pi_1)$ is easily seen to imply unique ergodicity.

(3) Obviously, our result is closely connected (via singular suspension) to Forni's theorem [Fo] on the cocycle equation for area–preserving vector fields on surfaces. Our method is different from Forni's: he constructs some Fourier analysis on flat surfaces and gets his results through Fatou's theorem on boundary values of bounded holomorphic functions. Our method provides an explicit diophantine condition and a smaller loss of differentiability.

(4) We can prove similar results, with the same loss of differentiability, when we start with a more regular data Φ.

2. Rauzy–Veech–Zorich continued fraction algorithm and its acceleration.

In order to describe the set $D(\pi_0, \pi_1)$ we will make use of the generalization of continued fractions to i.e.m. 's due to the work of Rauzy [Ra], Veech [V2] and Zorich [Z1,Z2].

2.1 Let (π_0, π_1) be an admissible pair. We define two new admissible pairs $\mathcal{R}_0(\pi_0, \pi_1)$ and $\mathcal{R}_1(\pi_0, \pi_1)$ as follows: let α_0, α_1 be the (distinct) elements of $\mathcal{A}$ such that $\pi_0(\alpha_0) = \pi_1(\alpha_1) = d$; one has

$$\begin{aligned} \mathcal{R}_0(\pi_0, \pi_1) &= (\pi_0, \hat{\pi}_1) \ , \\ \mathcal{R}_1(\pi_0, \pi_1) &= (\hat{\pi}_0, \pi_1) \ , \end{aligned} \tag{2}$$

where

$$\begin{aligned} \hat{\pi}_1(\alpha) &= \begin{cases} \pi_1(\alpha) & \text{if } \pi_1(\alpha) \leq \pi_1(\alpha_0), \\ \pi_1(\alpha) + 1 & \text{if } \pi_1(\alpha_0) < \pi_1(\alpha) < d, \\ \pi_1(\alpha_0) + 1 & \text{if } \alpha = \alpha_1, \ (\pi_1(\alpha_1) = d); \end{cases} \\ \hat{\pi}_0(\alpha) &= \begin{cases} \pi_0(\alpha) & \text{if } \pi_0(\alpha) \leq \pi_0(\alpha_1), \\ \pi_0(\alpha) + 1 & \text{if } \pi_0(\alpha_1) < \pi_0(\alpha) < d, \\ \pi_0(\alpha_1) + 1 & \text{if } \alpha = \alpha_0, \ (\pi_0(\alpha_0) = d). \end{cases} \end{aligned} \tag{3}$$

The *extended Rauzy class* of (π_0, π_1) is the set of admissible pairs obtained by saturation of (π_0, π_1) under the action of $\mathcal{R}_0$ and $\mathcal{R}_1$. The *extended Rauzy diagram* has for vertices the elements of the extended Rauzy class, each vertex (π_0, π_1) being the origin of two arrows joining (π_0, π_1) to $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$. The *name* of an arrow joining (π_0, π_1) to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ (with $\varepsilon \in \{0, 1\}$) is the element $\alpha_\varepsilon \in \mathcal{A}$ such that $\pi_\varepsilon(\alpha_\varepsilon) = 1$.

2.2 Let T be an i.e.m. with marked discontinuities, given by data (π_0, π_1), $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$. For $\varepsilon \in \{0, 1\}$, define $\alpha_\varepsilon \in \mathcal{A}$ by $\pi_\varepsilon(\alpha_\varepsilon) = d$ as above.

We say that T is of *type* ε if one has $\lambda_{\alpha_\varepsilon} \geq \lambda_{\alpha_{1-\varepsilon}}$; we then define a new i.e.m. $\mathcal{V}(T)$ by the following data: the admissible pair $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ and the lengths $(\hat{\lambda}_\alpha)_{\alpha \in \mathcal{A}}$ given by

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_\varepsilon, \\ \hat{\lambda}_{\alpha_\varepsilon} = \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}} & \text{otherwise.} \end{cases} \tag{4}$$

The i.e.m. $\mathcal{V}(T)$ is the first return map of T on $\left[0, \sum_{\alpha \neq \alpha_{1-\varepsilon}} \lambda_\alpha\right)$. We also associate to T the arrow in the extended Rauzy diagram joining (π_0, π_1) to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$. Iterating this process, we obtain a sequence of i.e.m $(\mathcal{V}^k(T))_{k\geq 0}$ and an infinite path in the extended Rauzy diagram starting from (π_0, π_1).

2.3 The next step is to group together several iterations of $\mathcal{V}$ to obtain the Zorich and the accelerated Zorich algorithms.

Starting from $T = T(0)$, we define a sequence $T(n) = \mathcal{V}^{k(n)}(T)$ by the following property: for $n \geq 0$, $k(n+1)$ is the largest integer $k > k(n)$ such that not all names in $\mathcal{A}$ are taken by arrows associated to iterations of $\mathcal{V}$ from $T(n)$ to $\mathcal{V}^k(T)$. This definition is based on the following elementary lemma:

Lemma*Assuming that no orbit segment of T starts and ends with a discontinuity, every name is taken infinitely many times in the infinite path associated to T.*

Remark: We have defined above the accelerated Zorich algorithm, which is most convenient for us because of the lemma below. For the Zorich algorithm itself, we have $\tilde{T}(n) = \mathcal{V}^{\tilde{k}(n)}(T)$, $\tilde{k}(n+1)$ being the largest integer $\tilde{k} > \tilde{k}(n)$ such that all arrows associated to the iterations of $\mathcal{V}$ from $\tilde{T}(n)$ to $\mathcal{V}^k(T)$ have the same name.

2.4 Let $T = T(0)$ be an i.e.m. satisfying the hypotheses of the lemma above. Let $(T(n))_{n\geq 0}$ be the sequence of i.e.m. obtained by the accelerated Zorich algorithm, with associated lengths $(\lambda_\alpha(n))_{\alpha\in\mathcal{A}}$.

Iterating formula (4) gives a matrix $Z(n) \in \mathrm{SL}\,(d, Z)$ with non negative entries such that

$$\lambda(n-1) = Z(n)\lambda(n) \; . \tag{5}$$

We will write, for $m < n$

$$Q(m, n) = Z(m+1) \cdots Z(n) \; , \tag{6}$$

$$\lambda(m) = Q(m, n)\lambda(n) \; . \tag{7}$$

For $m < n$, $T(n)$ is the first return map of $T(m)$ on $I(n) = \left[0, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha(n)\right)$; the return time of $I_\beta(n)$ in $I(n)$ is $\sum_\alpha Q_{\alpha\beta}(m, n)$ and the time spent in $I_\alpha(m)$ is $Q_{\alpha\beta}(m, n)$. From the definition of the accelerated Zorich algorithm, one easily gets the following:

Lemma*There exists an integer $m(d) > 0$ such that for $n \geq m + m(d)$, one has $Q_{\alpha\beta}(m, n) > 0$ for all $\alpha, \beta \in \mathcal{A}$.*

3. Special Birkhoff Sums

3.1 Let T as above, $m \leq n$, and φ be a function defined on the disjoint union $\sqcup_{\mathcal{A}} I_\alpha(m)$. We define a new function $S(m, n)\varphi$ on $\sqcup_{\mathcal{A}} I_\beta(n)$ as follows: for $x \in I_\beta(n)$

$$S(m, n)\varphi(x) = \sum_{0\leq l<Q_\beta(m,n)} \varphi(T(m)^l(x)) \; , \tag{8}$$

where $Q_\beta(m, n) = \sum_{\alpha\in\mathcal{A}} Q_{\alpha\beta}(m, n)$ is the return time of x in $I(n)$.

3.2 The operators $S(m, n)$ preserve regularity and commute with derivation. They also satisfy

$$\int_{\sqcup_{\mathcal{A}} I_\alpha(m)} \varphi = \int_{\sqcup_{\mathcal{A}} I_\beta(n)} S(m, n)\varphi \; . \tag{9}$$

If the restriction of φ to each of the intervals $I_\alpha(m)$ is a polynomial of degree $\leq \mu$ then the restriction of $S(m,n)\varphi$ to each of the intervals $I_\beta(n)$ is also a polynomial of degree $\leq \mu$.

We will denote $\Gamma^{(m)}$ the space of functions φ which are constant on each of the intervals $I_\alpha(m)$. The characteristic functions of the intervals $I_\alpha(m)$ form a basis of $\Gamma^{(m)}$. The operator $S(m,n)$ maps $\Gamma^{(m)}$ into $\Gamma^{(n)}$. In the bases we have chosen of $\Gamma^{(m)}$ and $\Gamma^{(n)}$ the matrix of $S(m,n)|_{\Gamma^{(m)}}$ is ${}^tQ(m,n)$. We denote $\Gamma_*^{(m)}$ the hyperplane of $\Gamma^{(m)}$ whose elements are the functions whose integral on the disjoint union $\sqcup I_\alpha(m)$ vanishes. It is sent by $S(m,n)$ into $\Gamma_*^{(n)}$.

3.3 Write $\mathrm{BV}_*(m)$ for the space of functions φ on $\sqcup_{\mathcal{A}} I_\alpha(m)$ which are of bounded variation on each $I_\alpha(m)$ and of mean value 0; write $\mathrm{BV}^1(m)$ for the space of functions Φ on $\sqcup_{\mathcal{A}} I_\alpha(m)$ which are lipschitzian on each $I_\alpha(m)$ and whose derivative belongs to $\mathrm{BV}_*(m)$.

We denote $E_s^{(m)}$ the space of functions $\Phi \in \mathrm{BV}^1(m)$ such that

$$\limsup_{n\to\infty} \frac{\log \|S(m,n)\Phi\|_\infty}{\log \|Q(m,n)\|} < 0 \,. \tag{10}$$

We obviously have $S(m,n)(E_s^{(m)}) \subset E_s^{(n)}$. We will denote $\Gamma_s^{(m)}$ the intersection of $E_s^{(m)}$ with $\Gamma^{(m)}$.

Since $S(m,n)(\Gamma_s^{(m)}) \subset \Gamma_s^{(n)}$ we can consider the induced operator

$$S(m,n)_\flat \,:\, \Gamma_*^{(m)}/\Gamma_s^{(m)} \to \Gamma_*^{(n)}/\Gamma_s^{(n)} \,. \tag{11}$$

4. The Diophantine condition

Here we introduce the notion of i.e.m. of "Roth type" which gives a full measure class of i.e.m. 's which satisfy the assumptions of our theorem.

4.1 An i.e.m. T is said to be of *"Roth type"* if its accelerated Zorich continued fraction verifies the following conditions:

(a) For any $\varepsilon > 0$, we have $\|Z(n+1)\| \leq \|Q(0,n)\|^\varepsilon$ for all large enough n;

(b) There exists $\theta > 0$ such that $\|S(0,n)|_{\Gamma_*^{(0)}}\| \leq \|S(0,n)|_{\Gamma^{(0)}}\|^{1-\theta} = \|Q(0,n)\|^{1-\theta}$ for all large enough n;

(c) For any $\varepsilon > 0$, $m < n$, with n large enough, we have $\|(S(m,n)_\flat)^{-1}\| \leq \|Q(0,n)\|^\varepsilon$.

It can be shown that for any admissible pair (π_0,π_1), conditions (a), (b), (c) are satisfied by a set $D(\pi_0,\pi_1)$ of full measure in $\Delta(\pi_0,\pi_1)$.

5. Sketch of the proof of the theorem

5.1 When $\hat{T}$ is a minimal homeomorphism of a compact space X, we know from a theorem of Gottschalk-Hedlund [GH] that a continuous function Φ on X is a $\hat{T}$–coboundary of some continuous function as soon as the Birkhoff sums of Φ at some point of X are bounded.

Let T be an i.e.m. with no orbit segment starting and ending with discontinuities; then T is minimal, but not continuous. Nevertheless, a Denjoy–like construction allows to apply Gottschalk-Hedlund's theorem and conclude that a continuous function whose Birkhoff sums at some point are bounded is the T–coboundary of a bounded function. Given $\varphi \in \mathrm{BV}_*(0)$, it is therefore sufficient to find a primitive Φ (determined by d constants of integration) whose Birkhoff sums at 0 are bounded.

5.2 Given $N > 0$ and a function Φ on $\sqcup_{\mathcal{A}} I_\alpha$, we can write the Birkhoff sums of Φ at 0 as a finite sum:

$$\sum_{i=0}^{N-1} \Phi \circ T^i(0) = \sum_j S(0, n_j)(\Phi)(x_j) \ , \tag{12}$$

where for every $n \geq 0$ we have

$$\mathrm{card}\,\{j \ , \ n_j = n\} \leq \|Z(n+1)\| := \max_\beta \sum_\alpha Z_{\alpha\beta}(n+1) \ . \tag{13}$$

Thus the Birkhoff sums of Φ at 0 are bounded as soon as

$$\sum_{n \geq 0} \|Z(n+1)\| \|S(0,n)(\Phi)\|_{\mathrm{L}^\infty} < +\infty \ . \tag{14}$$

5.3 Let $\varphi \in \mathrm{BV}_*(0)$; write $\varphi = \varphi_0 + \chi_0$, where $\chi_0 \in \Gamma_*^{(0)}$ and the mean value of φ_0 on every I_α vanishes. Write then inductively

$$S(n-1, n)(\varphi_{n-1}) = \varphi_n + \chi_n \ , \tag{15}$$

where $\chi_n \in \Gamma_*^{(n)}$ and the mean value of φ_n on every $I_\alpha(n)$ vanishes. We have, for every $n \geq 0$:

$$\|\varphi_n\|_{\mathrm{L}^\infty} \leq \max_\alpha \mathrm{Var}_{I_\alpha(n)} \varphi_n \leq \sum_\alpha \mathrm{Var}_{I_\alpha} \varphi := \mathrm{Var}\,\varphi \ , \tag{16}$$

and thus

$$\|\chi_0\| \leq \|\varphi\|_{\mathrm{L}^\infty} \ , \tag{17}$$

$$\|\chi_n\|_{\mathrm{L}^\infty} \leq \|S(n-1,n)(\varphi_{n-1})\|_{\mathrm{L}^\infty} \leq \|Z(n)\| \mathrm{Var}\,\varphi \ , \tag{18}$$

for $n > 0$. From this and conditions (a), (b) in the definition of Roth type, we get easily that there exists $\theta' = \theta'(\theta) > 0$ such that

$$\|S(0,n)(\varphi)\|_{\mathrm{L}^\infty} \leq \|Q(0,n)\|^{1-\theta'} \|\varphi\|_{\mathrm{BV}} \tag{19}$$

for all $n \geq 0$ large enough.

5.4 Given $\varphi \in \mathrm{BV}_*(0)$, we will find a primitive $\Phi \in \mathrm{BV}_*^1(0)$ that satisfies (14); for this, in view of condition (a) in Section 4.1, it is sufficient to find $\Phi \in E_s^{(0)}$. Thus, we only need to define Φ mod $\Gamma_s^{(0)}$.

For $n \geq 0$ and $\varphi \in \mathrm{BV}_*(n)$, let $P_0^{(n)} \varphi \in \mathrm{BV}_*^1(n)$ be the primitive of φ whose mean value over every interval $I_\alpha(n)$ vanishes; this is not functorial with respect to special Birkhoff sums. We define, for $n > 0$:

$$\begin{aligned} &\Lambda^{(n)} \ : \ \mathrm{BV}_*(n-1) \to \Gamma_*^{(n)} / \Gamma_s^{(n)} \\ &\Lambda^{(n)} = P_0^{(n)} S(n-1,n) - S(n-1,n) P_0^{(n-1)} \mathrm{mod}\,\Gamma_s^{(n)} \ . \end{aligned} \tag{20}$$

From condition (a) in Section 4.1, it is not difficult to show, for every $\varepsilon > 0$, $\varphi \in \mathrm{BV}_*(n-1)$, n large enough that:

$$\|\Lambda^{(n)} \varphi\|_{\mathrm{L}^\infty} \leq \|Q(0,n)\|^{-1+\varepsilon} \|\varphi\|_{\mathrm{L}^\infty} \ . \tag{21}$$

Joining (19), (21) and condition (c) of Section 4.1, we see that the series

$$\Delta P^{(m)}(\varphi) = \sum_{n>m} \Big((S(m,n)_\flat)^{-1} \circ \Lambda^{(n)} \circ S(m,n-1) \Big) (\varphi) \tag{22}$$

for $m \geq 0$, $\varphi \in \mathrm{BV}_*(m)$ converges in $\Gamma_*^{(m)}/\Gamma_s^{(m)}$.
Setting

$$P^{(m)}(\varphi) = P_0^{(m)}(\varphi) + \Delta P^{(m)}(\varphi) \tag{23}$$

we have now functoriality:

$$S(m,n)(P^{(m)}(\varphi)) = P^{(n)}(S(m,n)\varphi) \bmod \Gamma_s^{(n)} \, . \tag{24}$$

It is now not difficult to check that given $\varphi \in \mathrm{BV}_*(0)$, any primitive $\Phi \in \mathrm{BV}_*^1(0)$ whose class $\bmod \Gamma_s^{(0)}$ belongs to $P^{(0)}(\varphi)$ also belongs to $E_s^{(0)}$, and thus is the coboundary of a bounded function.

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