Dynamics and time series: theory and applications

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Scuola Normale Superiore
Lecture 1, Jan 13, 2009

- Lecture 1: An introduction to dynamical systems and to time series. Periodic and quasiperiodic motions. (Tue Jan 13, 2 pm - 4 pm Aula Bianchi)
- Lecture 2: Ergodicity. Uniform distribution of orbits. Return times. Kac inequality Mixing (Thu Jan 15, 2 pm - 4 pm Aula Dini)
- Lecture 3: Kolmogorov-Sinai entropy. Randomness and deterministic chaos. (Tue Jan 27, 2 pm - 4 pm Aula Bianchi)
- Lecture 4: Time series analysis and embedology. (Thu Jan 29, 2 pm - 4 pm Dini)
- Lecture 5: Fractals and multifractals. (Thu Feb 12, 2 pm - 4 pm Dini)
- Lecture 6: The rhythms of life. (Tue Feb 17, 2 pm - 4 pm Bianchi)
- Lecture 7: Financial time series. (Thu Feb 19, 2 pm - 4 pm Dini)
- Lecture 8: The efficient markets hypothesis. (Tue Mar 3, 2 pm - 4 pm Bianchi)
- Lecture 9: A random walk down Wall Street. (Thu Mar 19, 2 pm - 4 pm Dini)
- Lecture 10: A non-random walk down Wall Street. (Tue Mar 24, 2 pm – 4 pm Bianchi)

- Seminar I: Waiting times, recurrence times ergodicity and quasiperiodic dynamics (D.H. Kim, Suwon, Korea; Thu Jan 22, 2 pm - 4 pm Aula Dini)
- Seminar II: Symbolization of dynamics. Recurrence rates and entropy (S. Galatolo, Università di Pisa; Tue Feb 10, 2 pm - 4 pm Aula Bianchi)
- Seminar III: Heart Rate Variability: a statistical physics point of view (A. Facchini, Università di Siena; Tue Feb 24, 2 pm - 4 pm Aula Bianchi)
- Seminar IV: Study of a population model: the Yoccoz-Birkeland model (D. Papini, Università di Siena; Thu Feb 26, 2 pm - 4 pm Aula Dini)
- Seminar V: Scaling laws in economics (G. Bottazzi, Scuola Superiore Sant'Anna Pisa; Tue Mar 17, 2 pm - 4 pm Aula Bianchi)
- Seminar VI: Complexity, sequence distance and heart rate variability (M. Degli Esposti, Università di Bologna; Thu Mar 26, 2 pm - 4 pm Aula Dini)
- Seminar VII: Forecasting (TBA)

Examples of dynamical systems in natural and social sciences

- The Solar System
- Atmosphere (meteorology)
- Human body (heart, brain cells, lungs, ...)
- Ecology (dynamics of animal populations)
- Epidemiology
- Chemical reactions

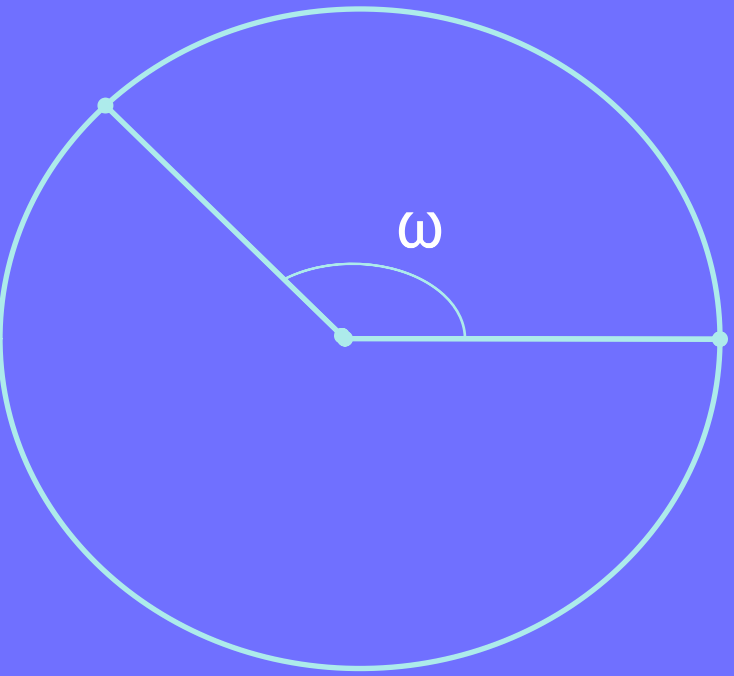
Dynamical systems not necessarirly deterministic

- Stockmarket
- Electric grid
- Internet

Dynamical systems

- A dynamical system is a couple (phase space, time evolution law)
- The phase space is the set of all possible states (i.e. initial conditions) of our system
- Each initial condition uniquely determines the time evolution (determinism)
- The system evolves in time according to a fixed law (iteration of a map, differential equation, etc.)
- Often (but not necessarily) the evolution law is not linear

The simplest dynamical systems

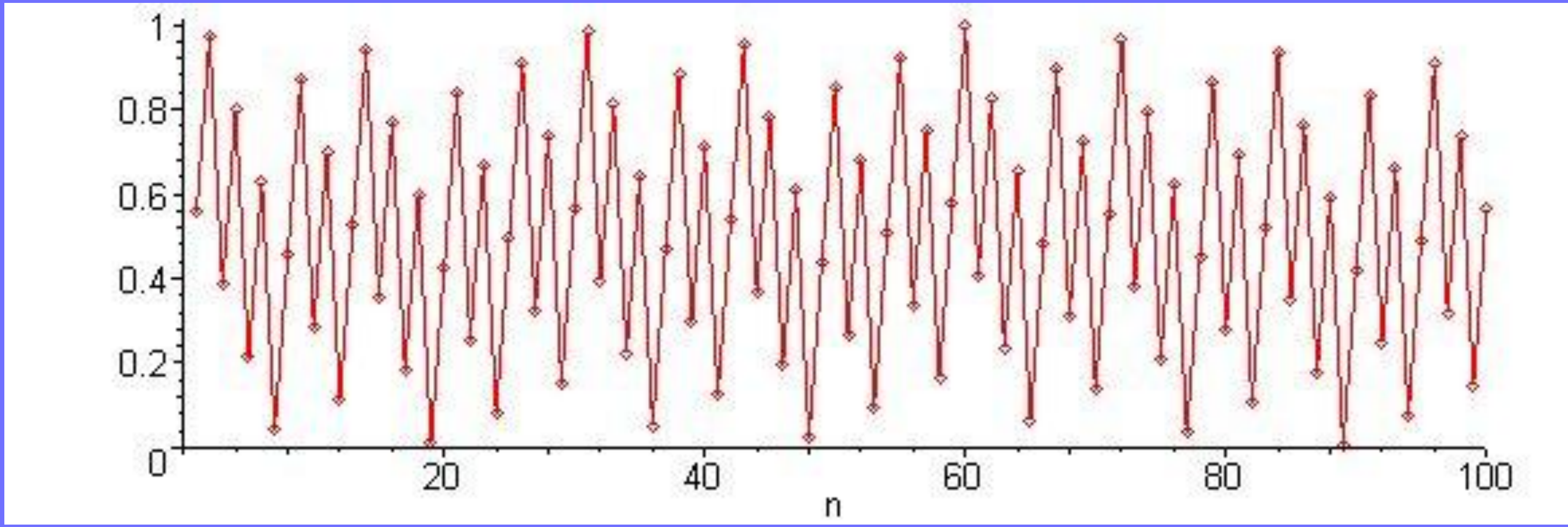


- The phase space is the circle: **S=R/Z**
- Case 1: quasiperiodic dynamics

  $\theta(n+1)=\theta(n)+\omega \pmod 1$

  (ω irrational)
- Case 2: chaotic dynamics

  $\theta(n+1)=2\theta(n) \pmod 1$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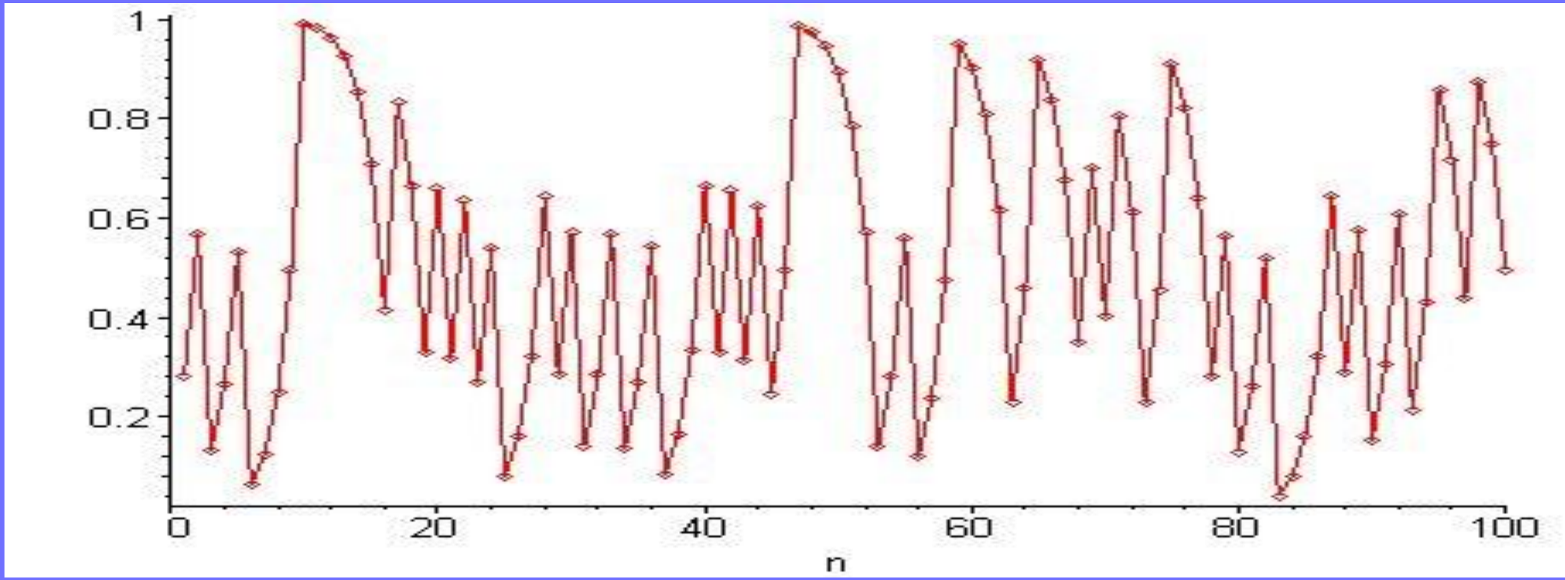
1
0.8
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100
n

Quasiperiodic dynamics

- Quasiperiodic = periodic if precision is finite, but the period $\rightarrow\infty$ if the precision of measurements is improved
- More formally a dynamics f is quasiperiodic if

$$f^{n_k} \rightarrow \mathrm{id}$$

For some sequence $n_k \rightarrow \infty$

$$f^{n_k+1} \approx f$$

Renormalization approach

$$n_k + 1 \rightarrow \infty$$

return times

Sensitivity to initial conditions

For, in respect to the latter branch of the supposition, it should be considered that the most trifling variation in the facts of the two cases might give rise to the most important miscalculations, by diverting thoroughly the two courses of events; very much as, in arithmetic, an error which, in its own individuality, may be inappreciable, produces at length, by dint of multiplication at all points of the process, a result enormously at variance with truth.

(Egdar Allan Poe, The mistery of Marie Roget)

For the doubling map on the circle (case 2) one has

$\theta(N)-\theta'(N)=2^N(\theta(0)-\theta'(0))$ ➡ even if the initial datum is known with a 10 digit accuracy, after 40 iterations one cannot even say if the iterates are larger than ½ or not

In quasiperiodic dynamics this does not happen: for the rotations on the circle one has $\theta(N)-\theta'(N)=\theta(0)-\theta'(0)$ and long term prediction is possible

Ergodic theory

- The focus of the analysis is mainly on the asymptotic ditribution of the orbits, and not on transient phenomena. Ergodic theory is an attempt to study the statistical behaviour of orbits of

  dynamical systems restricting the attention to their asymptotic distribution. One waits until all transients have been wiped off and looks for an invariant probability measure describing the distribution of typical orbits.

Measure theory vs. probability theory

Table 1.1. Comparison of terminology

Measure Theory	Probability Theory
a probability measure space X	a sample space Ω
$x \in X$	$\omega \in \Omega$
a σ-algebra $\mathcal{A}$	a σ-field $\mathcal{F}$
a measurable subset A	an event E
a probability measure μ	a probability P
$\mu(A)$	$P(E)$
a measurable function f	a random variable X
$f(x)$	x, a value of X
a characteristic function χ_E	an indicator function 1_E
Lebesgue integral $\int_X f \, d\mu$	expectation $E[X]$
almost everywhere	almost surely, or with probability 1
convergence in L^1	convergence in mean
convergence in measure	convergence in probability
conditional measure $\mu_A(B)$	conditional probability $\Pr(B\|A)$

Ergodic theory vs. probability theory, i.e. statistics vs. a-priori probability

- There are few persons, even among the calmest thinkers, who have not occasionally been startled into a vague yet thrilling half-credence in the supernatural, by *coincidences* of so seemingly marvellous a character that, as *mere* coincidences, the intellect has been unable to receive them. Such sentiments --
for the half-credences of which I speak have never the full force of *thought* -- such sentiments are seldom thoroughly stifled unless by reference to the doctrine of chance, or, as it is technically termed, the Calculus of Probabilities. Now this Calculus is, in its essence, purely mathematical; and thus we have the anomaly of the most rigidly exact in science applied to the shadow and spirituality of the most intangible in speculation.
- (Egdar Allan Poe, The mistery of Marie Roget)

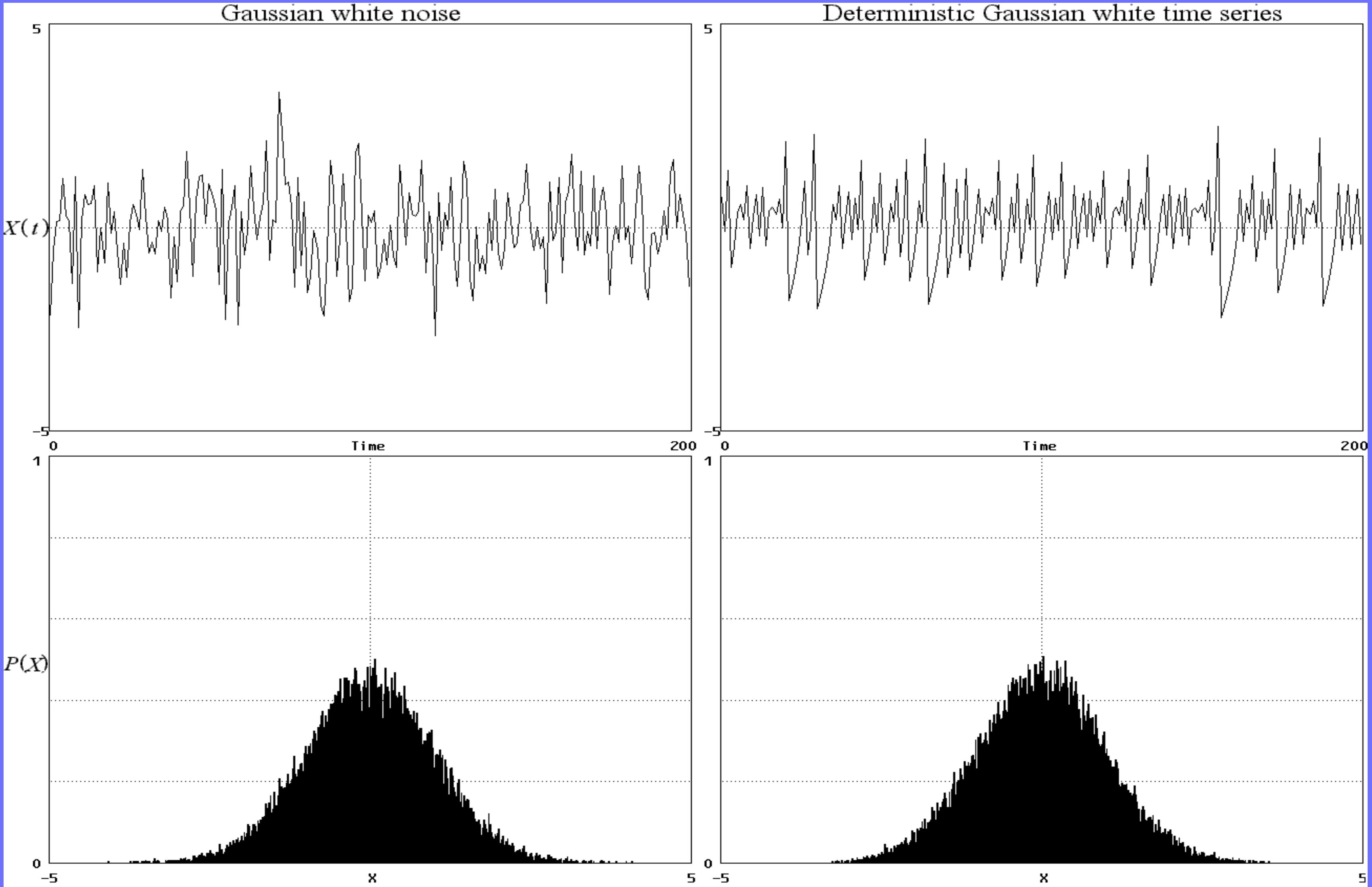
Examples of time-series in natural and social sciences

- Weather measurements (temperature, pressure, rain, wind speed, …) . If the series is very long …climate
- Earthquakes
- Lightcurves of variable stars
- Sunspots
- Macroeconomic historical time series (inflation, GDP, employment,…)
- Financial time series (stocks, futures, commodities, bonds, …)
- Populations census (humans or animals)
- Physiological signals (ECG, EEG, …)

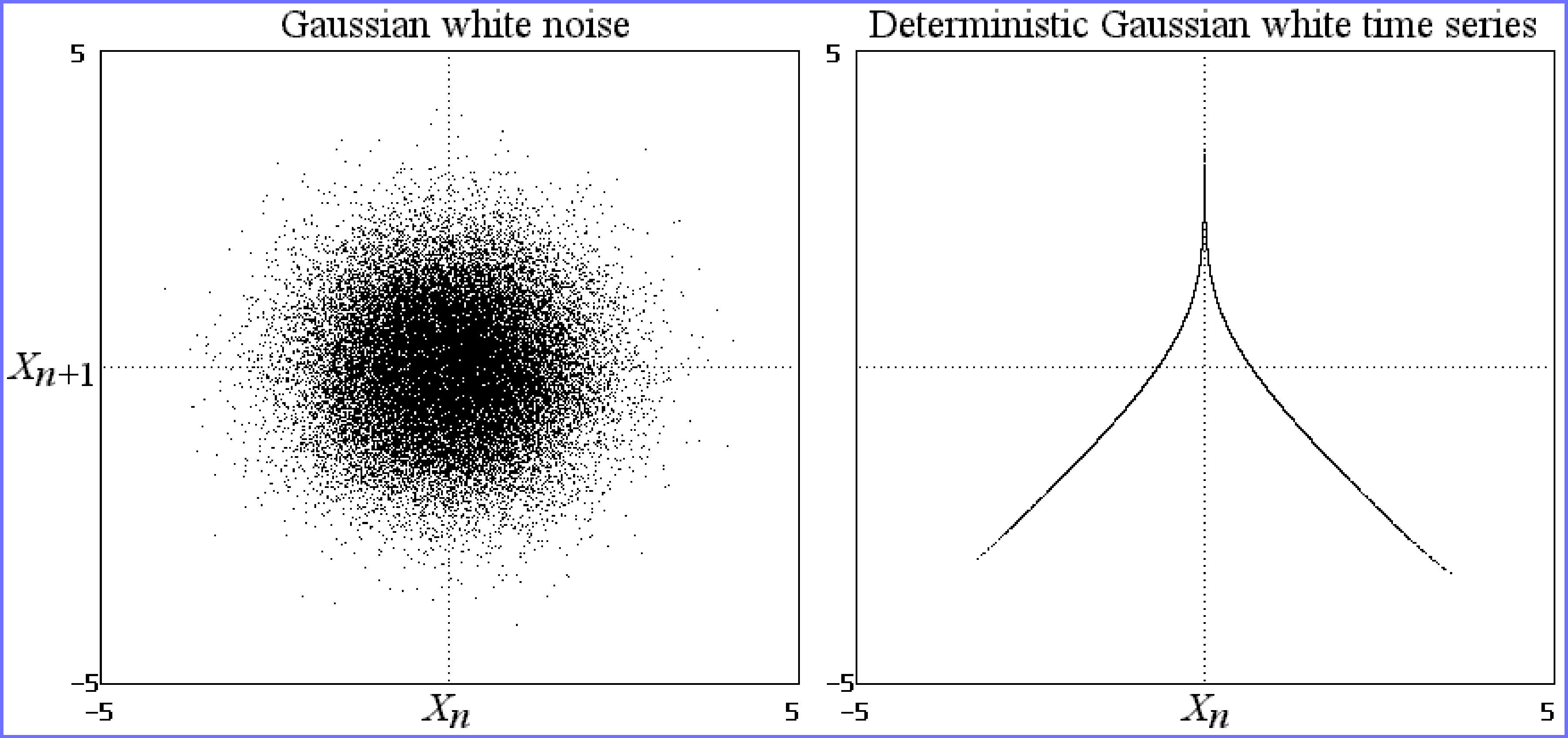
Stochastic or chaotic?

- An important goal of time-series analysis is to determine, given a times series (e.g. HRV) if the underlying dynamics (the heart) is:
  - Intrinsically random
  - Generated by a deterministic nonlinear chaotic system which generates a random output
  - A mix of the two (stochastic perturbations of deterministic dynamics)

Deterministic or random? Appearance can be misleading…



Time delay map



Logit and logistic

The logistic map $x \to L(x)=4x(1-x)$ preserves the probability measure $d\mu(x)=dx/(\pi\sqrt{x(1-x)})$

The transformation $h:[0,1] \to \mathbf{R}$, $h(x)=\ln x-\ln(1-x)$ conjugates L with a new map G

$$h \circ L = G \circ h$$

definined on **R.** The new invariant probability measure is $d\mu(x)=dx/[\pi(e^{x/2} + e^{-x/2})]$

Clearly G and L have the same dynamics (the differ only by a coordinates change)

Embedding method

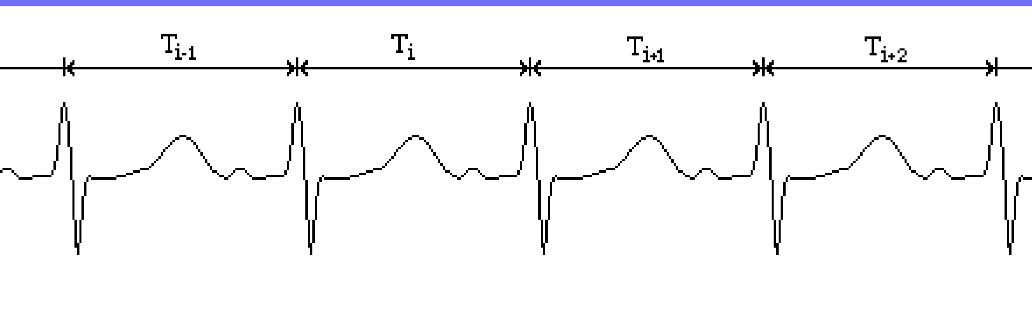
- Plot $x(t)$ vs. $x(t-\tau)$, $x(t-2\tau)$, $x(t-3\tau)$, …
- $x(t)$ can be any observable
- The embedding dimension is the # of delays
- The choice of τ and of the dimension are critical
- For a typical deterministic system, the orbit will be diffeomorphic to the attractor of the system (Takens theorem)

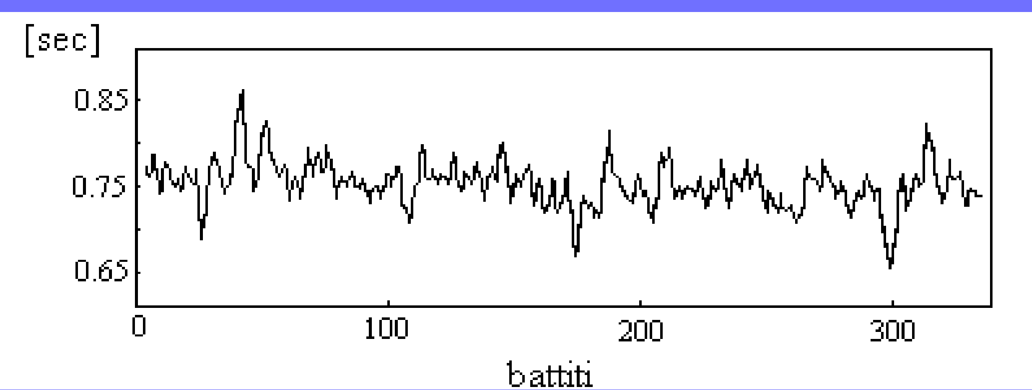
Time series analysis of physiological signals

Physiological signals are characterized by extreme variability both in healthy and pathological conditions. Complexity, erratic behaviour, chaoticity are typical terms used in the description of many physiological time series.

Quantifying these properties and turning the variability analysis from qualitative to quantitative are important goals of the analysis of time-series and could have relevant clinical impact.

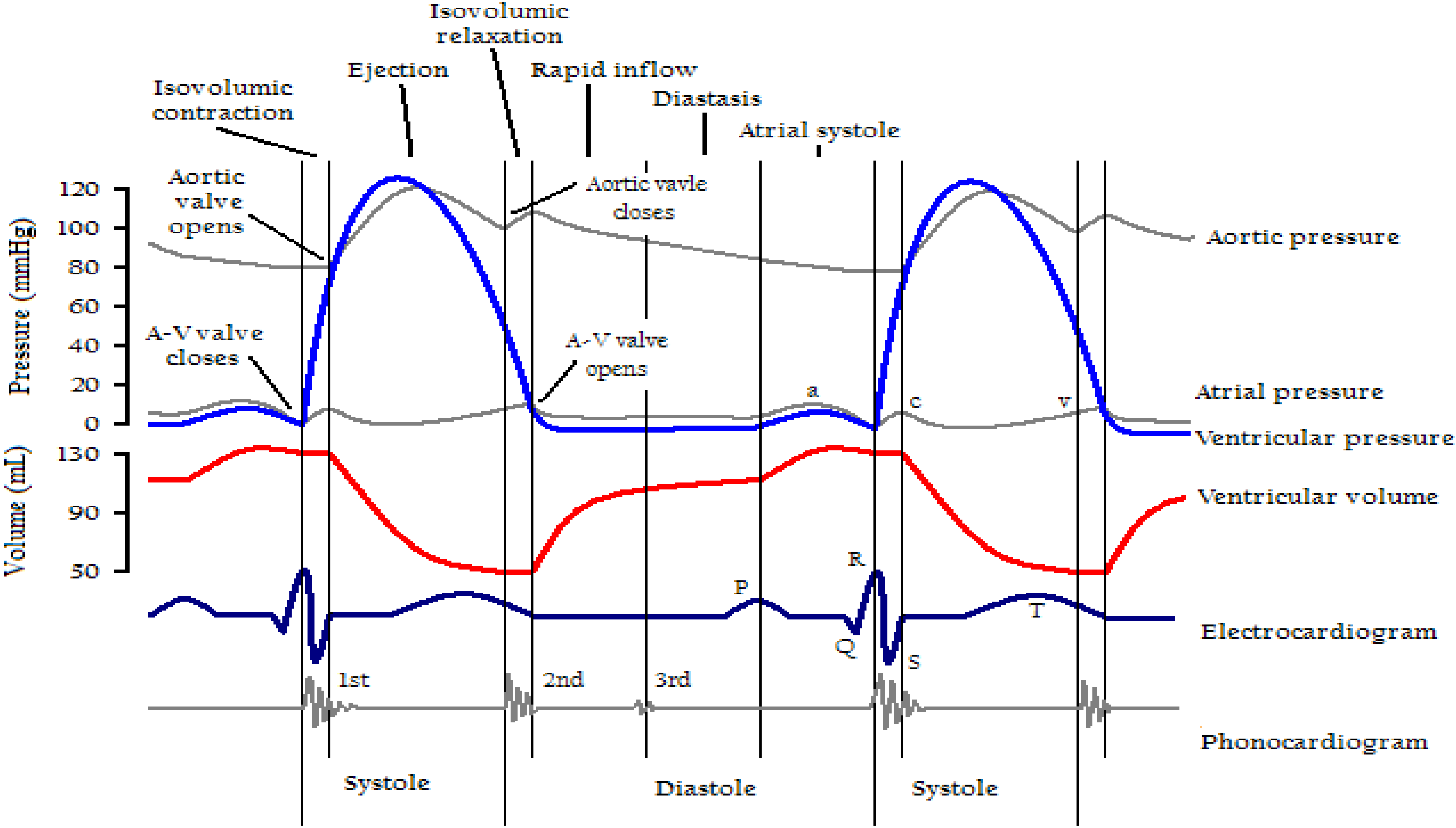
From ECG to heart rate variability time series



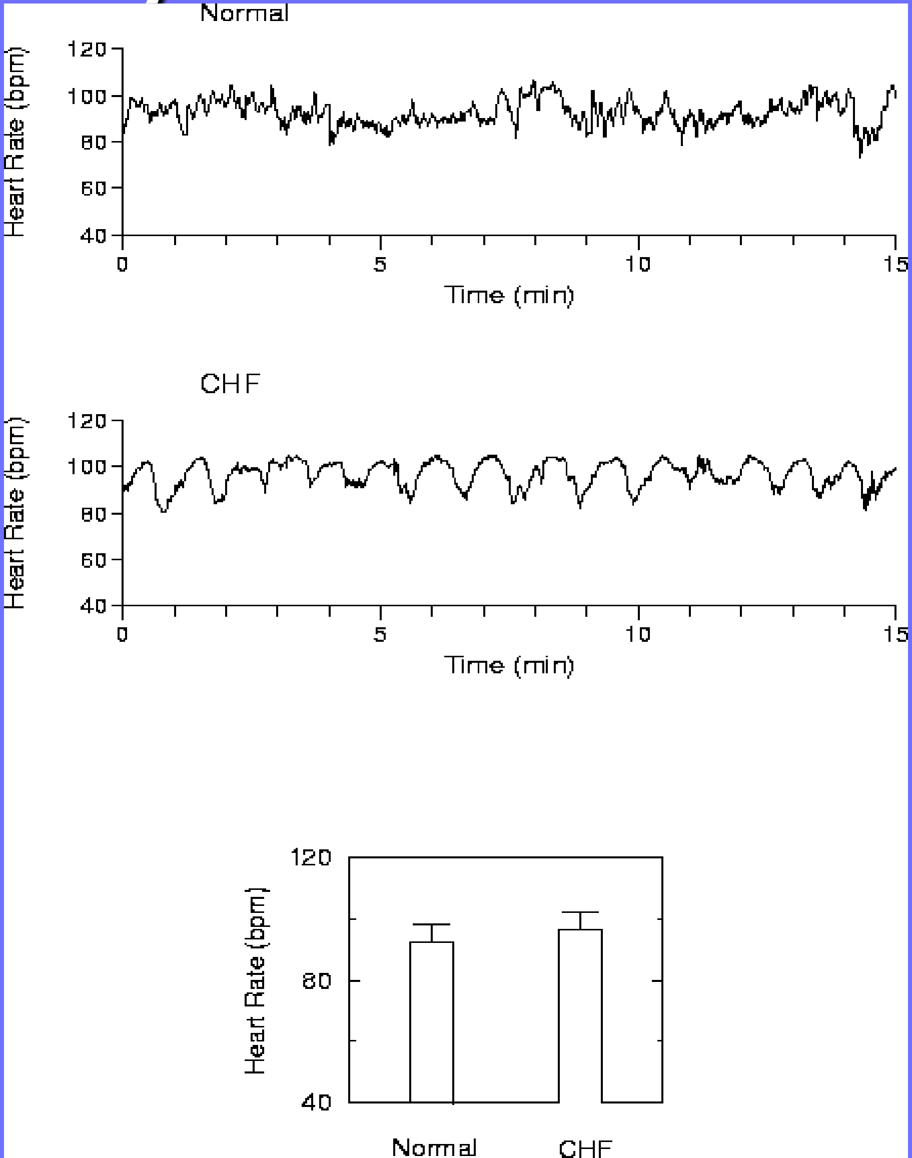


- Example of ECG signal
- The time interval between two consecutive R-wave peaks (R-R interval) varies in time
- The time series given by the sequence of the durations of the R-R intervals is called heart rate variability (HRV)

The heart cycle and ECG

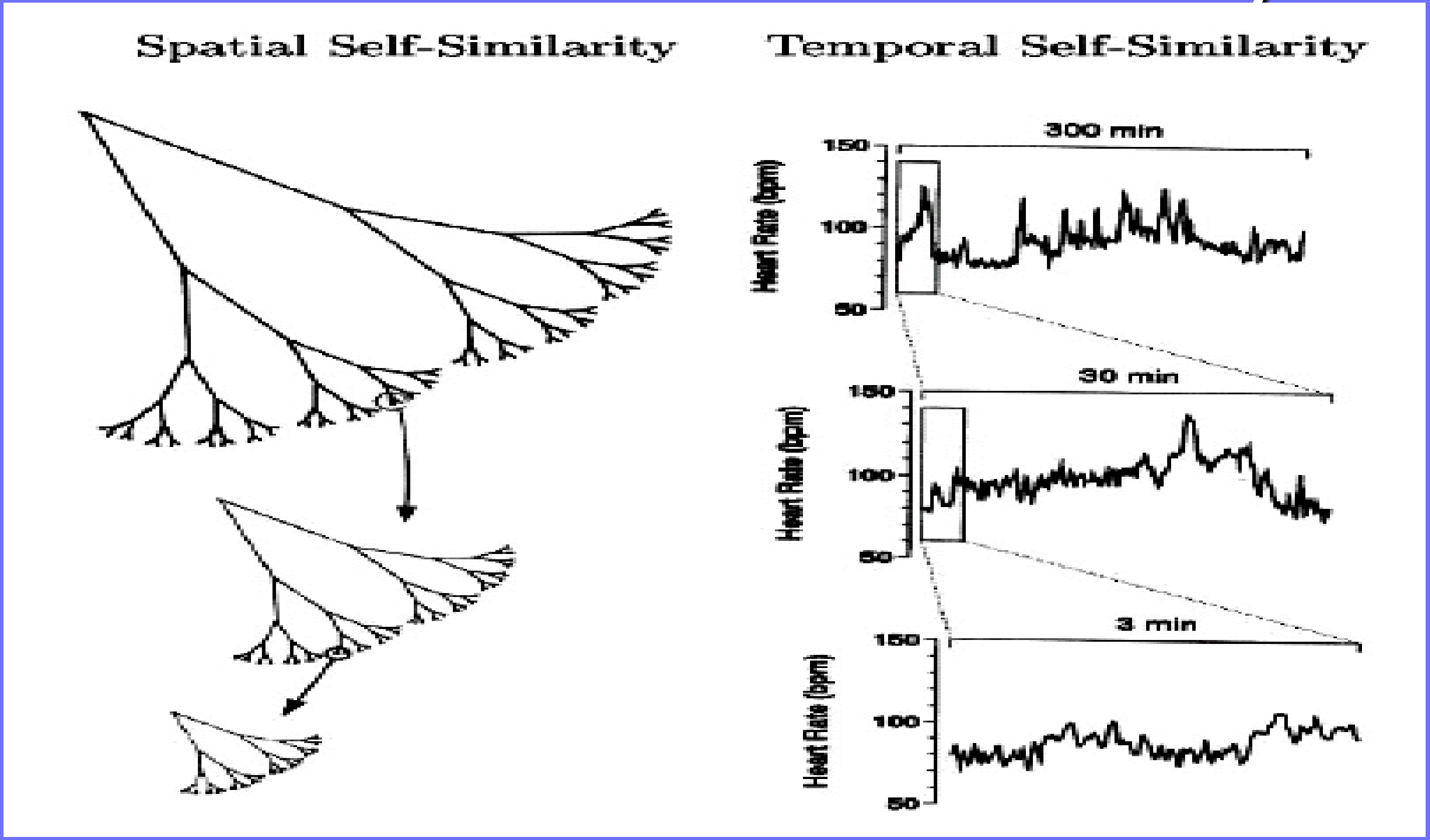


Healthy? Statistical vs. dynamical tools for diagnosis



- The HRV plots of an healthy patient show a very different dynamics from those of a sick patient but the traditional statistical measures (mean and variance) are almost the same.
- www.physionet.org

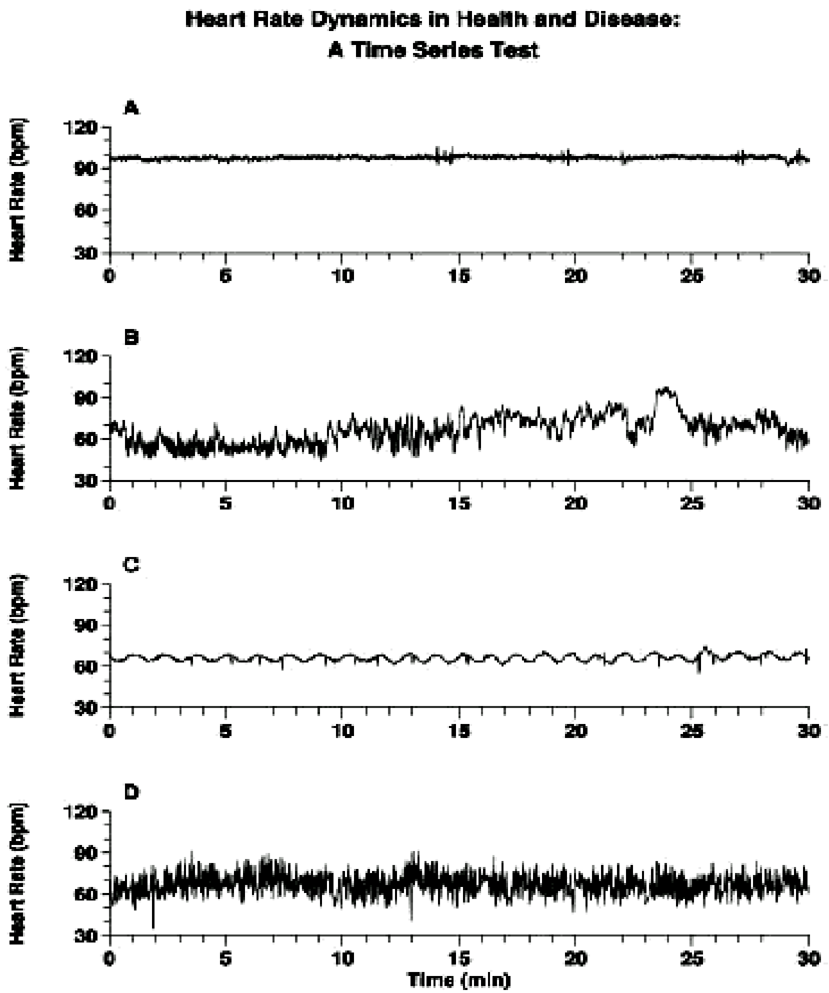
Time series and self-similarity

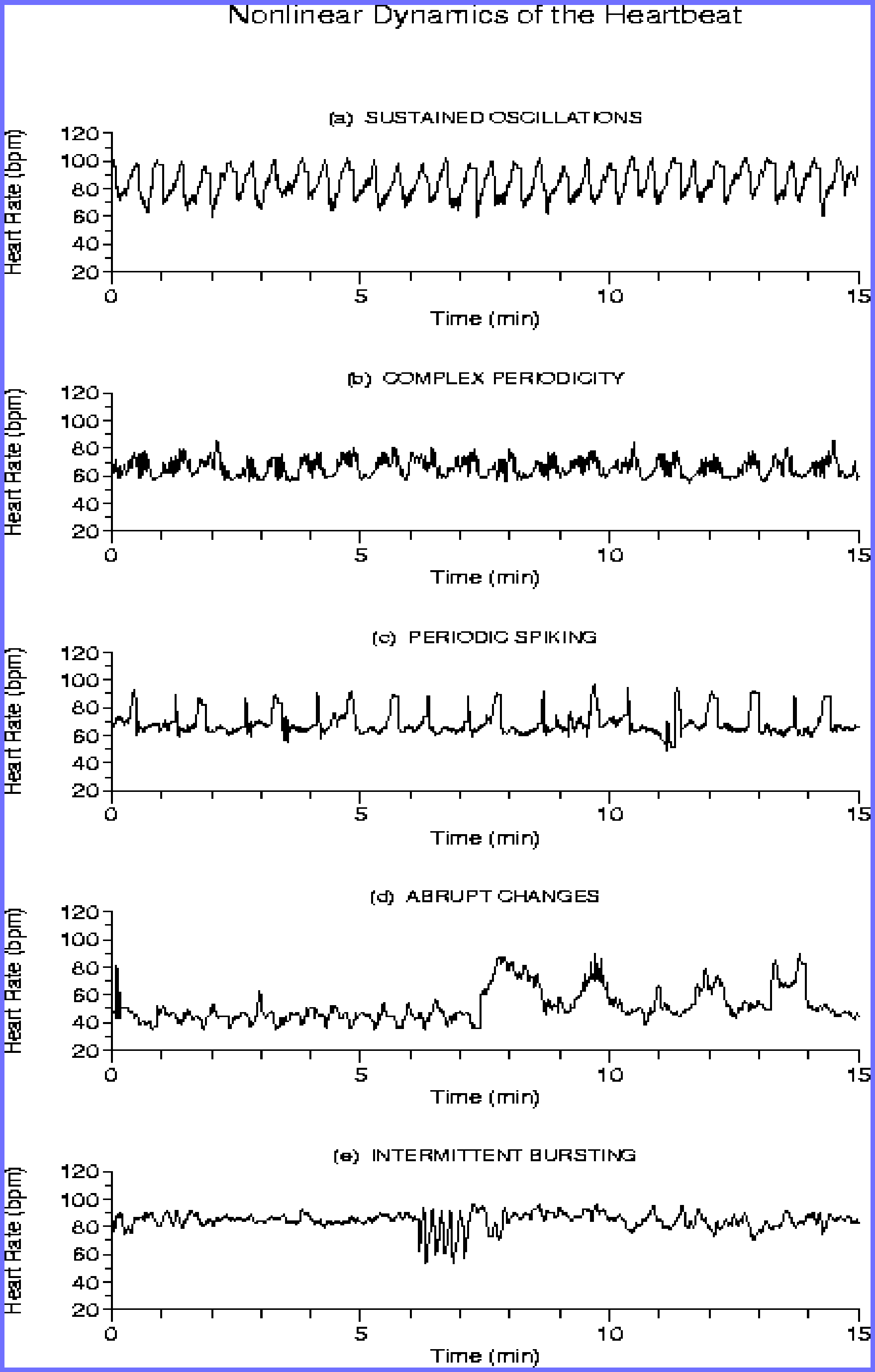


www.physionet.org

Fig. 2. Schematic representations of self-similar structures and self-similar fluctuations. The tree-like, spatial fractal (*Left*) has self-similar branchings, such that the small-scale structure resembles the large-scale form. A fractal temporal process, such as healthy heart rate regulation (*Right*), may generate fluctuations on different time scales that are statistically self-similar. Adapted from ref. 13.

Healthy ?





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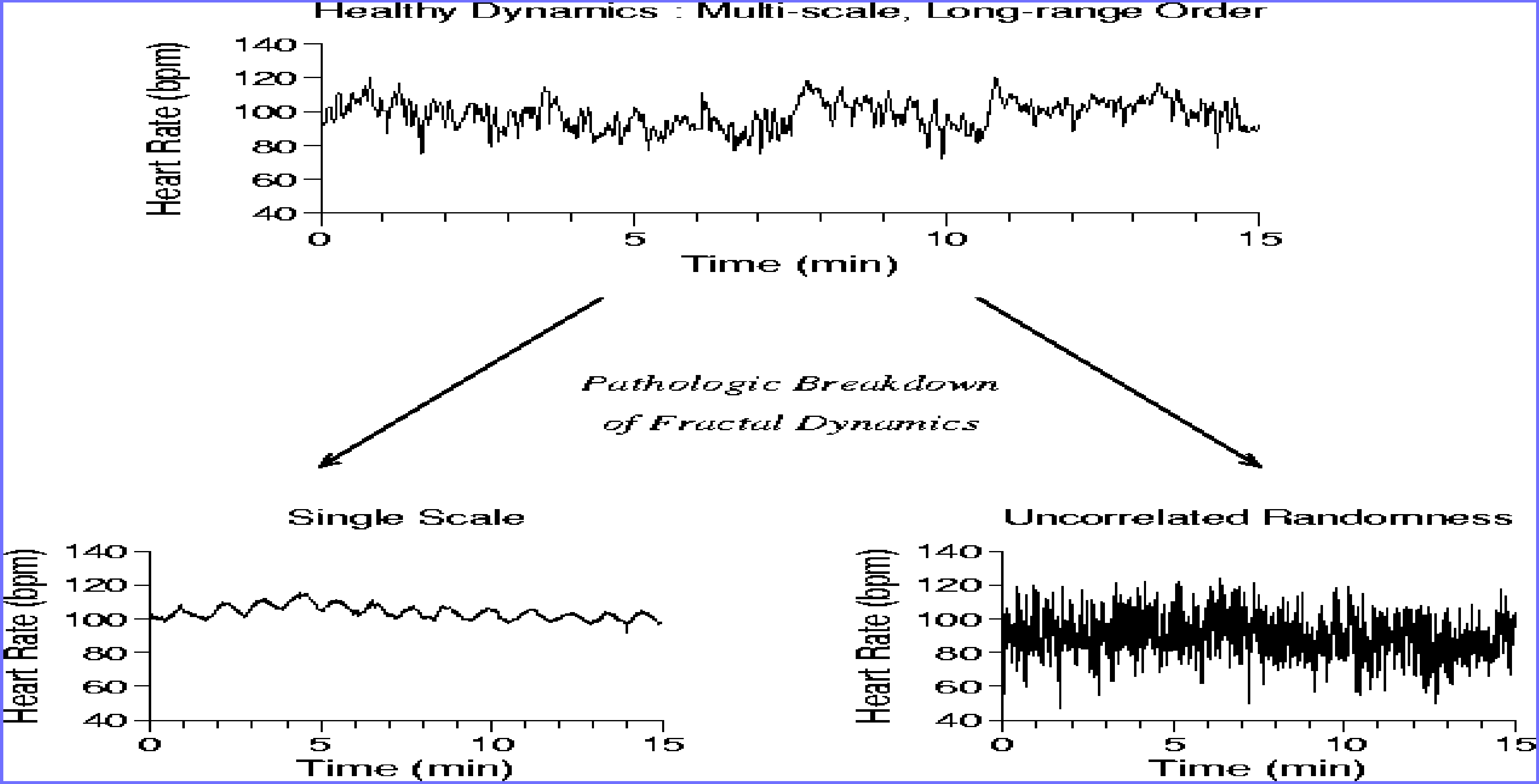
Healthy or not?



(Adapted from Goldberger AL. Non-linear dynamics for clinicians:
chaos theory, fractals, and complexity at the bedside. *Lancet* 1996;347:1312-1314.)

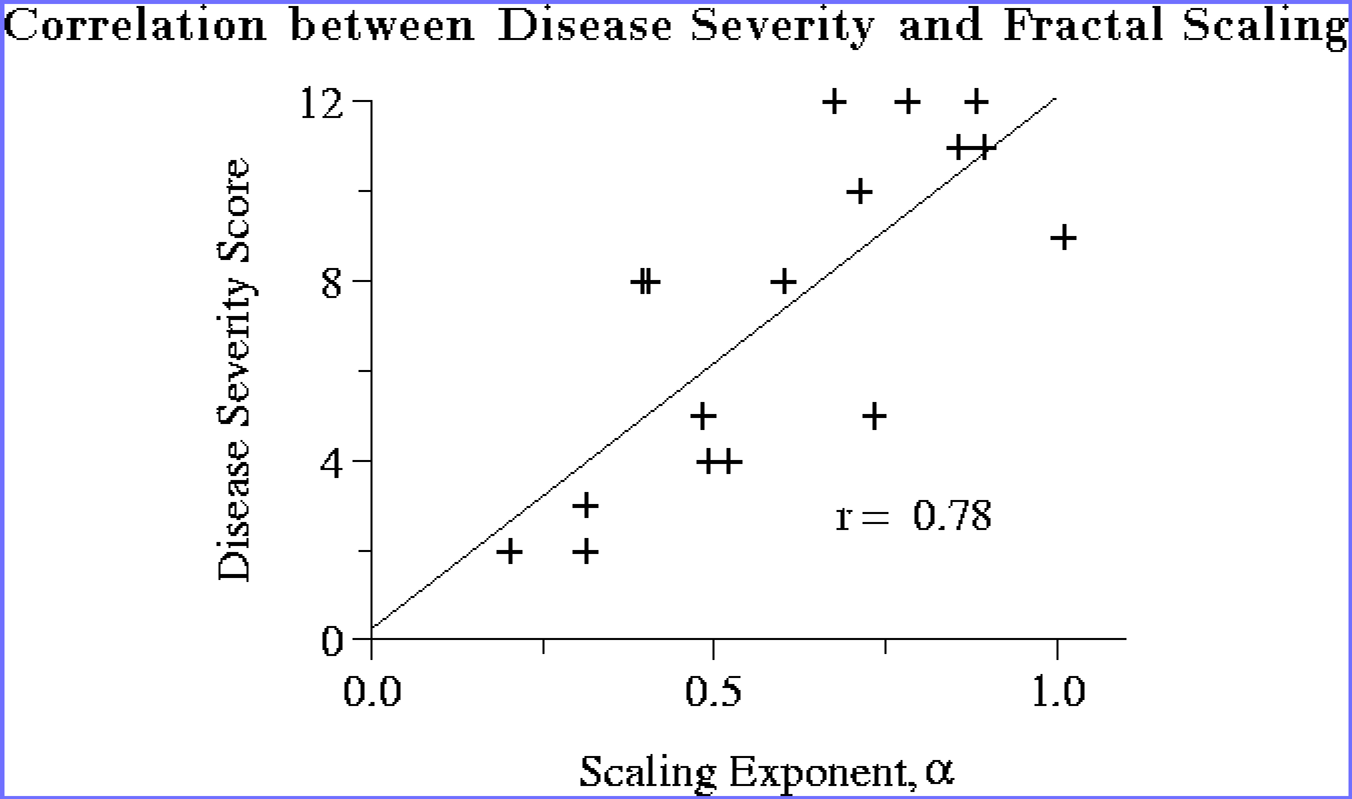
Healthy or not?

FRACTAL DYNAMICS OF HEART RATE AND GAIT

	FRACTAL HEART DYNAMICS	**FRACTAL GAIT DYNAMICS**
Features	Extends over thousands of beats	Extends over thousands of steps
In Health	Persists during different activities (asleep or awake)	Persists regardless of gait speed (slow, normal or fast)
Potential	Altered with advanced age	Altered with advanced age
Diagnostic & Prognostic	Altered with cardiovascular disease (e.g. Heart Failure)	Altered with nervous system disease (e.g. Parkinson's D.)
Utility	Helps predict survival	May predict falls among elderly

Source: *htpp://www.physionet.org*

Correlation between disease severity and fractal scaling exponent



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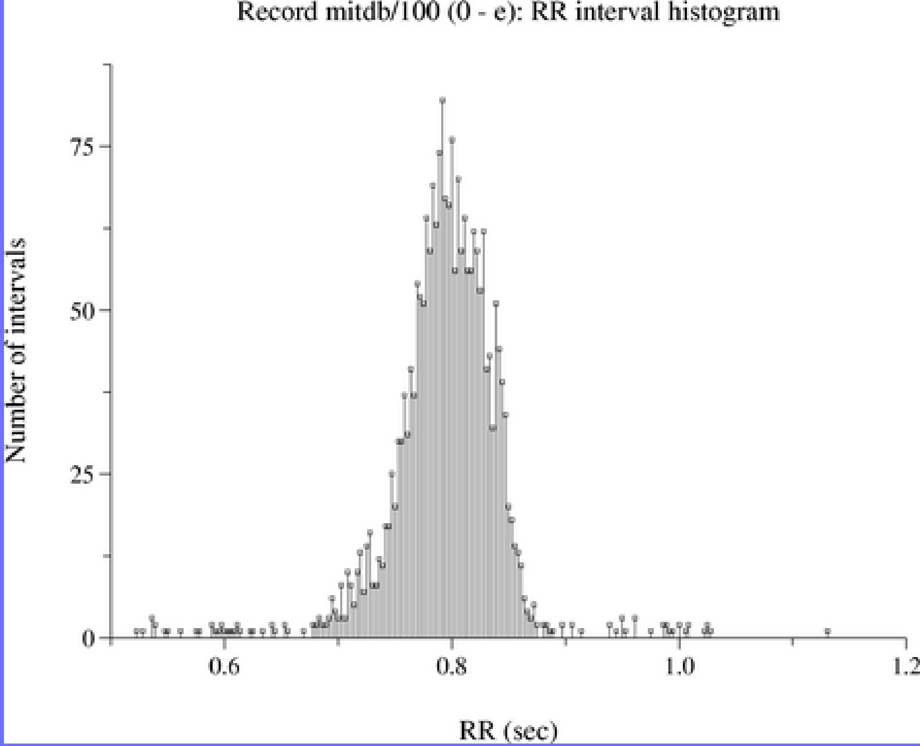
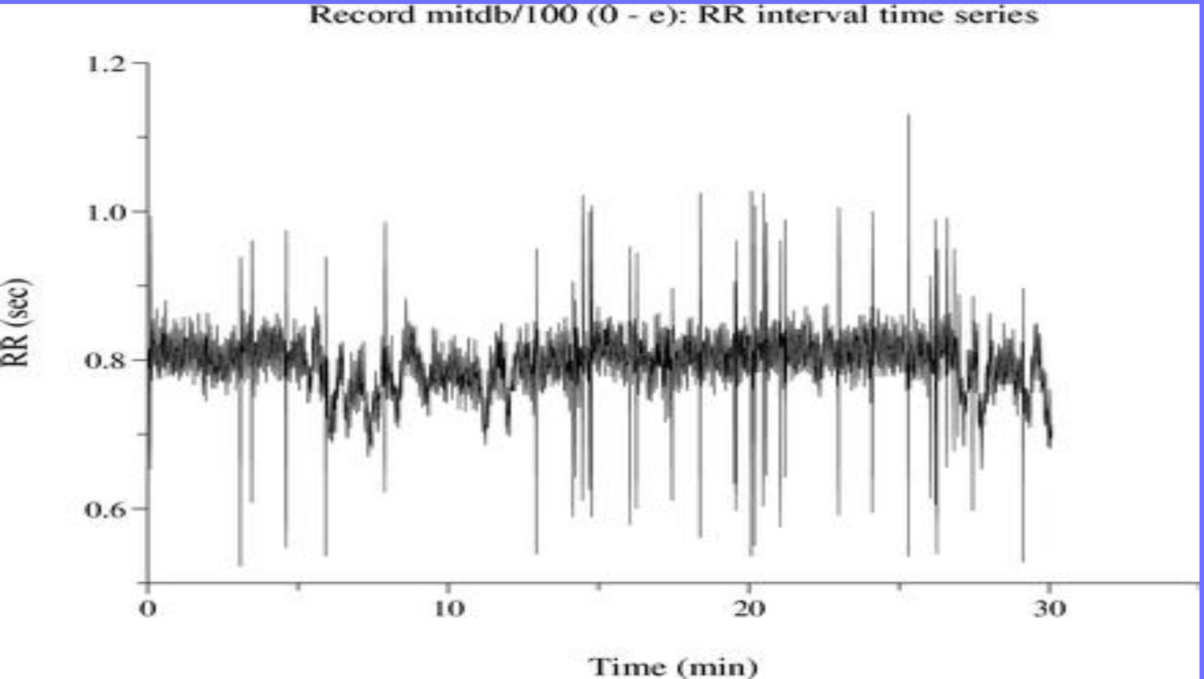
Distribution of R-R intervals





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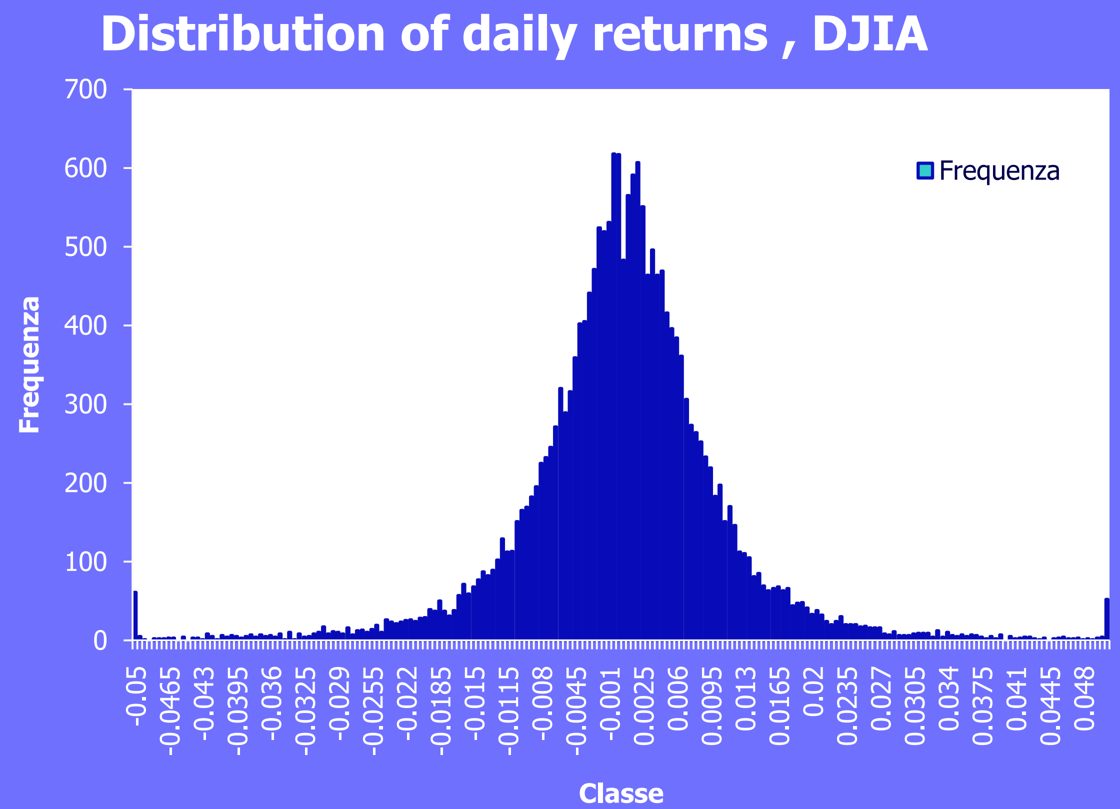
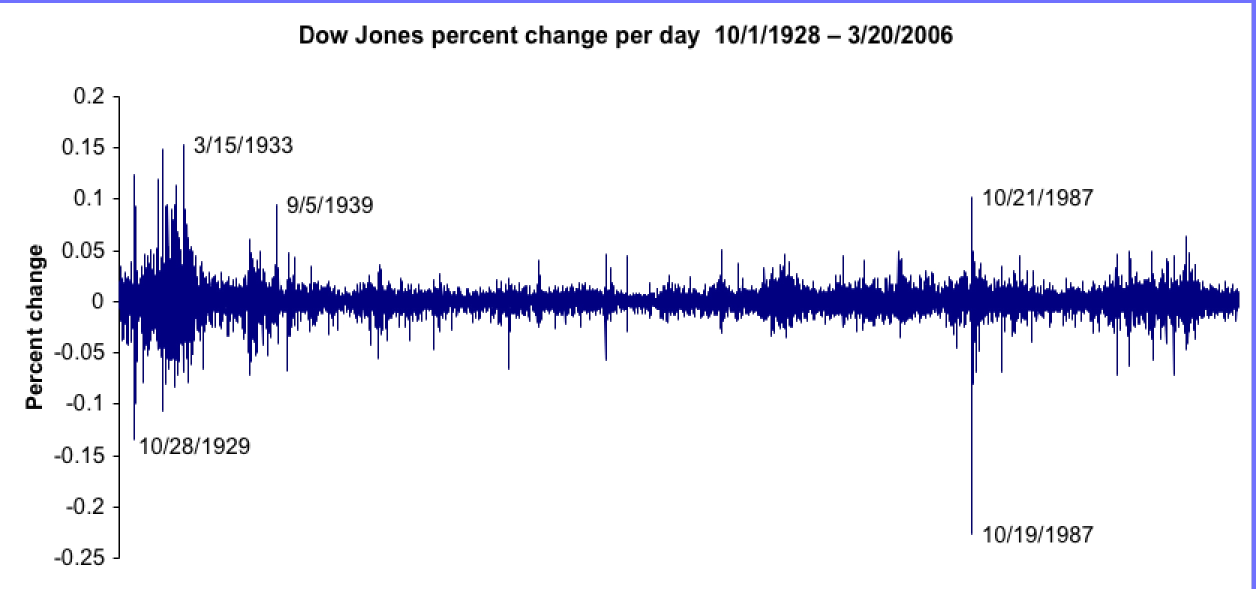
Distribution of daily returns, Dow Jones 1928-2007



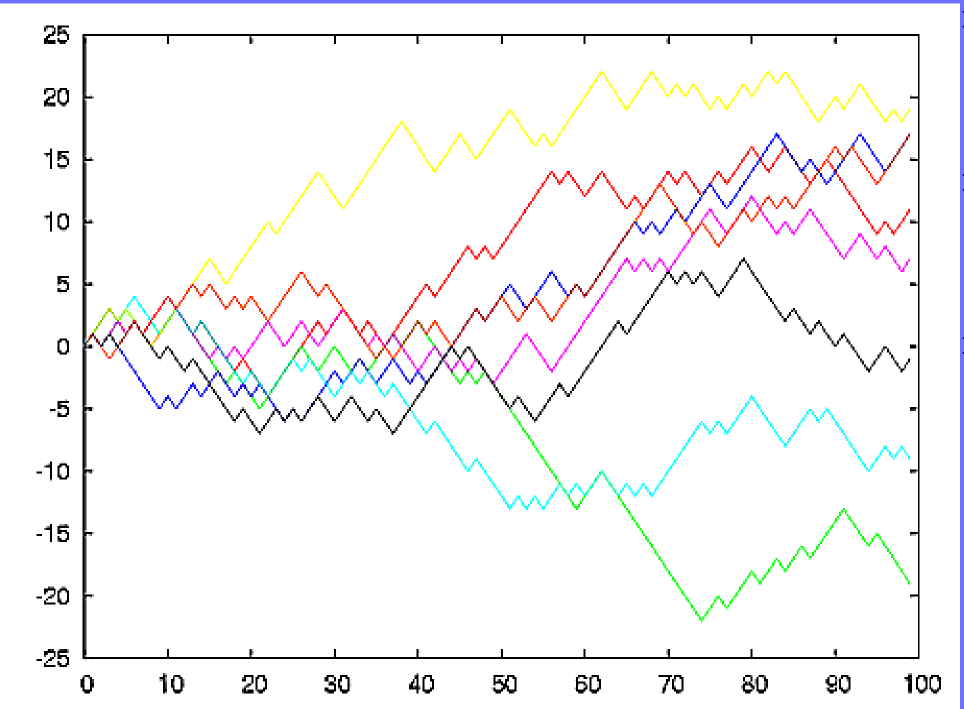


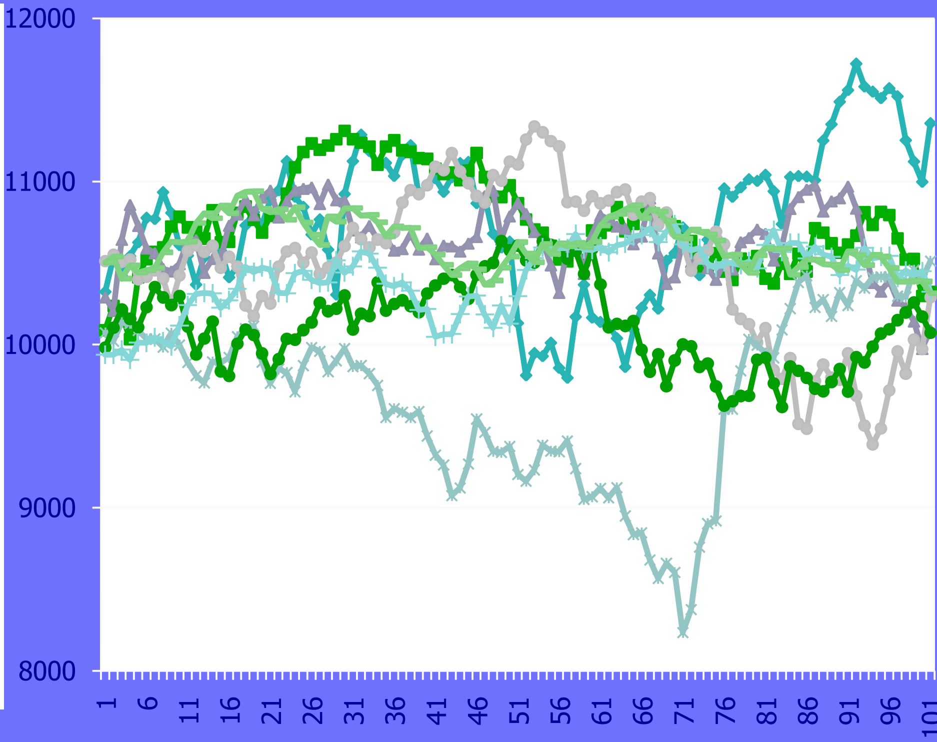
Geometric daily return at time t = (Price of index at time t / Price of index at time t-1)-1

Here the index is the Dow Jones Industrial Average, t is integer and counts only open market days

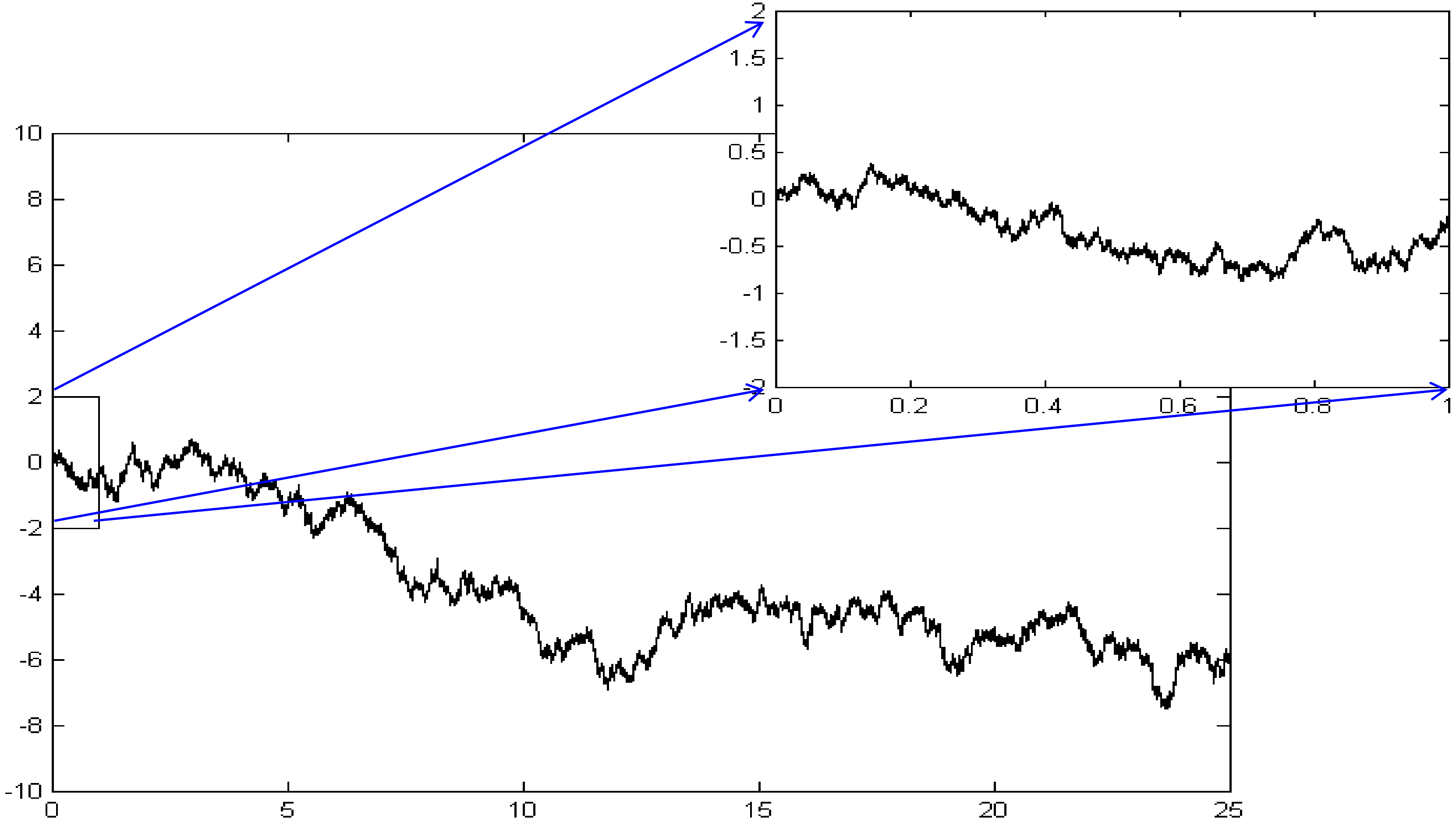
The value of the index is at the close

Random walks vs. Dow Jones





Selfsimilarity



Una passeggiata aleatoria?

- In un famosissimo articolo Paul Samuelson ("Proof that Properly Anticipated Prices Fluctuate Randomly", *Industrial Management Review,* 6:2, 41-49 (1965)) diede una dimostrazione matematica di questo fatto. Tutti gli argomenti rigorosi devono fondarsi su assiomi e definizioni e su impiegare una logica impeccabile, che certamente approssima ma non incarna l'esperienza quotidiana dei mercati. Scrive lo stesso Samuelson nelle conclusioni dell'articolo: "Non si dovrebbero dedurre troppe conseguenze dal teorema che ho appena dimostrato. In particolare non ne segue che i mercati competitivi reali funzionino bene."

The normal distribution

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R},$$

$$\varphi_{0,t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right),$$

$$\frac{\partial}{\partial t}\varphi_{0,t}(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi_{0,t}(x).$$

Do daily returns follow a normal distribution?

Class	*Observed Frequency*	Theoretical Frequency
x< -0.05	67	0.093902
-0.05<x<-0.045	19	0.567355
-0.045<x<-0.04	41	3.207188
--0.04<x<0.035	51	14.9652
-0.035<x<-0.03	78	57.64526
-0.03<x<-0.025	117	183.3153
-0.025<x<-0.02	247	481.2993
-0.02<x<-0.015	484	1043.367
-0.015<x<-0.01	1111	1867.6
-0.01<x<-05	2433	2760.391
-0.05<x<0	4879	3369.05
0<x<05	5119	3395.468
05<x<0.01	2881	2825.84
01<x<0.015	1219	1941.987
0.015<x<0.02	539	1102.011
0.02<x<0.025	241	516.3589
0.025<x<0.03	105	199.7674
0.03<x<0.035	77	63.8089
0.035<x<0.04	43	16.82651
0.04<x<0.045	27	3.662964
0.045<x<0.05	20	0.658208
x> 0.05	50	0.110887

Mean	00204
Median	00411
Moda	0
Standard deviation	0.011355
Varianza campionaria	00129
Kurtosis	26.84192
Asymmetry	-0.67021
Intervallo	0.399044
Minimum	-0.25632
Maximum	0.142729
Sum	4.058169
Number of observations	19848

Theoretical and observed frequency of outliers in the history of 15 stockmarkets

Exhibit 4: Outliers – Expected and Observed

This exhibit shows, for the indexes and sample periods in Exhibit 2, the expected (Exp) and observed (Obs) number of daily returns three standard deviations (SD) below and above the arithmetic mean return (AM); the ratio between the number of these observed and expected returns; and the total number of expected (TE) and observed (TO) returns more than three SDs away from the mean. 'Exp' figures are rounded to the nearest integer.

	Lower Tail				Upper Tail						
Market	AM–3-SD	Exp	Obs	Ratio	AM+3-SD	Exp	Obs	Ratio	TE	TO	Ratio
Australia	–2.46%	17	73	4.4	2.52%	17	53	3.2	33	126	3.8
Canada	–2.48%	11	73	6.9	2.55%	11	43	4.1	21	116	5.5
France	–3.11%	13	79	6.2	3.19%	13	61	4.8	25	140	5.5
Germany	–3.51%	16	85	5.3	3.57%	16	76	4.8	32	161	5.1
Hong Kong	–5.53%	12	77	6.2	5.67%	12	80	6.5	25	157	6.4
Italy	–3.82%	12	71	6.0	3.91%	12	48	4.0	24	119	5.0
Japan	–3.12%	19	132	6.8	3.19%	19	112	5.8	39	244	6.3
New Zealand	–2.51%	12	61	4.9	2.56%	12	57	4.6	25	118	4.7
Singapore	–3.12%	14	90	6.4	3.18%	14	86	6.1	28	176	6.3
Spain	–3.22%	11	52	4.8	3.31%	11	61	5.6	22	113	5.2
Switzerland	–2.74%	13	101	7.9	2.79%	13	62	4.8	26	163	6.4
Taiwan	–4.55%	15	103	6.8	4.65%	15	81	5.3	30	184	6.0
Thailand	–4.40%	10	62	6.0	4.48%	10	81	7.8	21	143	6.9
UK	–3.00%	13	69	5.3	3.07%	13	60	4.6	26	129	5.0
USA	–3.35%	28	180	6.4	3.40%	28	173	6.1	56	353	6.3
Average	**–3.39%**	**14**	**87**	**6.0**	**3.47%**	**14**	**76**	**5.2**	**29**	**163**	**5.6**

Estrada, Javier: Black Swans and Market Timing: How Not to Generate Alpha.
Available at SSRN: http://ssrn.com/abstract=1032962