Parabolic sheaves on logarithmic schemes

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Let X be an algebraic variety over $\mathbb{C}$, $x_0 \in X$. What can be recovered of the fundamental group $\pi_1(X, x_0)$ from algebro-geometric data? There are good reasons why the full fundamental group can not be reconstructed.

Grothendieck showed how to recover the *profinite completion* $\widehat{\pi}_1(X, x_0)$ that is, the projective limit of the finite quotients of $\pi_1(X, x_0)$. The essential point is that the finite quotients of $\pi_1(X, x_0)$ are the Galois groups of finite topological covers of X, and these are all algebraic. This completion carries strictly less information than $\pi_1(X, x_0)$; for example, it may happen that $\widehat{\pi}_1(X, x_0) = 0$ but $\pi_1(X, x_0) \neq 0$.

The profinite completion is defined using covers; but covers are hard to construct. For example, there is no algebraic know proof that if S is a finite subset of $\mathbb{P}^1$, then $\widehat{\pi}_1(\mathbb{P}^1 \smallsetminus S)$ is a finite profinite group over $|S| - 1$ generators, even when $|S| = 3$.

Suppose that X is reduced and projective. Then a theorem of Nori shows how to reconstruct $\widehat{\pi}_1(X, x_0)$ from the category $\mathrm{Vect}(X)$ of vector bundles on X, as a $\mathbb{C}$-linear tensor category, with a fiber functor $\mathrm{Vect}(X) \to \mathrm{Vect}_{\mathbb{C}}$. Consider the category $\mathrm{Rep}\big(\widehat{\pi}_1(X, x_0)\big)$ of continuous finite dimensional representations $\widehat{\pi}_1(X, x_0) \to \mathrm{GL}(V)$ of $\widehat{\pi}_1(X, x_0)$. Any such representation has a factorization $\widehat{\pi}_1(X, x_0) \to G \to \mathrm{GL}(V)$ through a finite quotient G of $\widehat{\pi}_1(X, x_0)$. With this representation we associate a vector bundle on X, as follows. The quotient G of $\widehat{\pi}_1(X, x_0)$ corresponds to a Galois G-cover $Y \to X$ with a marked point $y_0 \in Y$ over x_0. The quotient $(Y \times V)/G \to Y/G = X$ by the diagonal action is a vector bundle on X. This gives a functor $\mathrm{Rep}\big(\widehat{\pi}_1(X, x_0)\big) \to \mathrm{Vect}(X)$. It was observed by Weil that the vector bundles that one obtains in this way are very special: they are *finite*.

Vector bundles can be added (via direct sum) and multiplied (via tensor product); they can not be subtracted. Suppose that $f(t) \in \mathbb{N}[t]$ is a polynomial with non negative integer coefficients. If E is a vector bundle on X, we can define $f(E)$. The vector bundle E is called *finite* if there exist $f(t)$ and $g(t) \in \mathbb{N}[t]$, $f(t) \neq g(t)$, such that $f(E) \simeq g(E)$. Finite vector bundles form a tensor subcategory $\mathrm{Fin}(X)$ of $\mathrm{Vect}(X)$.

Theorem (Nori)**.** *The construction above gives an equivalence of $\mathbb{C}$-linear tensor categories* $\mathrm{Rep}\big(\widehat{\pi}_1(X, x_0)\big) \simeq \mathrm{Fin}(X)$*.*

It is known that one can recover $\widehat{\pi}_1(X, x_0)$ from $\mathrm{Rep}\big(\widehat{\pi}_1(X, x_0)\big)$ as the group of automorphism of the fiber functor. This is known as *Tannaka duality.*

What happens when X is not projective? Vector bundles on X are inadequate: for example, vector bundles on $\mathbb{P}^1 \smallsetminus S$ are all trivial. They must be replaced with *parabolic bundles*.

(Real) parabolic bundles were introduced by Mehta and Seshadri in dimension 1. Let us fast-forward, skipping a lot of important work of Maruyama–Yokogawa, Biswas, Simpson, Iyer–Simpson, and go to work of Niels Borne. Suppose that X is a smooth variety over an algebraically closed field of characteristic 0, D is a divisor with simple normal crossing on X. This means that D is a union of smooth hypersurfaces $D_1, \ldots, D_r$ intersecting transversally at each point. Fix a sequence $\mathbf{d} = (d_1, \ldots, d_r)$ of r positive integers. Consider the set $\frac{1}{\mathbf{d}}\mathbb{Z}^r \overset{\text{def}}{=} \frac{1}{d_1}\mathbb{Z} \times \cdots \times \frac{1}{d_r}\mathbb{Z}$ as a poset: $(w_1, \ldots, w_r) \leq (w'_1, \ldots, w'_r)$ if $w_i \leq w'_i$ for all i. This makes $\frac{1}{\mathbf{d}}\mathbb{Z}^r$ into a category: if $w \leq w'$ then there is a unique arrow from w to w', otherwise there is none.

Definition (Simpson, Borne). A parabolic bundle on (X, D) with weights in $\frac{1}{\mathbf{d}}\mathbb{Z}^r$ is a functor $\frac{1}{\mathbf{d}}\mathbb{Z}^r \to \mathrm{Vect}(X)$, satisfying the following conditions.

(a) if $w \leq w'$, then $E_w \to E_{w'}$ is injective.

(b) if $\mathrm{e}_i \in \mathbb{Z}^r \subseteq \frac{1}{\mathbf{d}}\mathbb{Z}^r$ is the i^{th} element of the canonical basis, then the embedding $E_{w-\mathrm{e}_i} \to E_w$ identifies $E_{w-\mathrm{e}_i}$ with $E_w(-D_i)$.

Parabolic bundles on (X, D) with weights in $\frac{1}{\mathbf{d}}\mathbb{Z}^r$ form a category, denoted by $\mathrm{ParV}_{1/\mathbf{d}}(X)$. If $\mathbf{d}$ divides $\mathbf{d}'$, there is an embedding $\mathrm{ParV}_{1/\mathbf{d}}(X) \subseteq \mathrm{ParV}_{1/\mathbf{d}'}(X)$. The union of these categories is the category of parabolic vector bundles on (X, D) denoted by $\mathrm{ParV}(X, D)$. The category $\mathrm{ParV}(X)$ has a (highly non-obvious) tensor product, allowing to talk about the category of finite parabolic bundles $\mathrm{Fin}(X, D)$.

Theorem (Nori, Borne). *There is an equivalence of tensor categories* $\operatorname{Rep}\big(\widehat{\pi}_1(X \smallsetminus D, x_0)\big) \simeq \operatorname{Fin}(X, D)$.

It is not at all clear how to associate a parabolic bundle with a representation of $\widehat{\pi}_1(X \smallsetminus D, x_0)$. Borne does this by interpreting parabolic bundles as bundles on certain orbifolds.

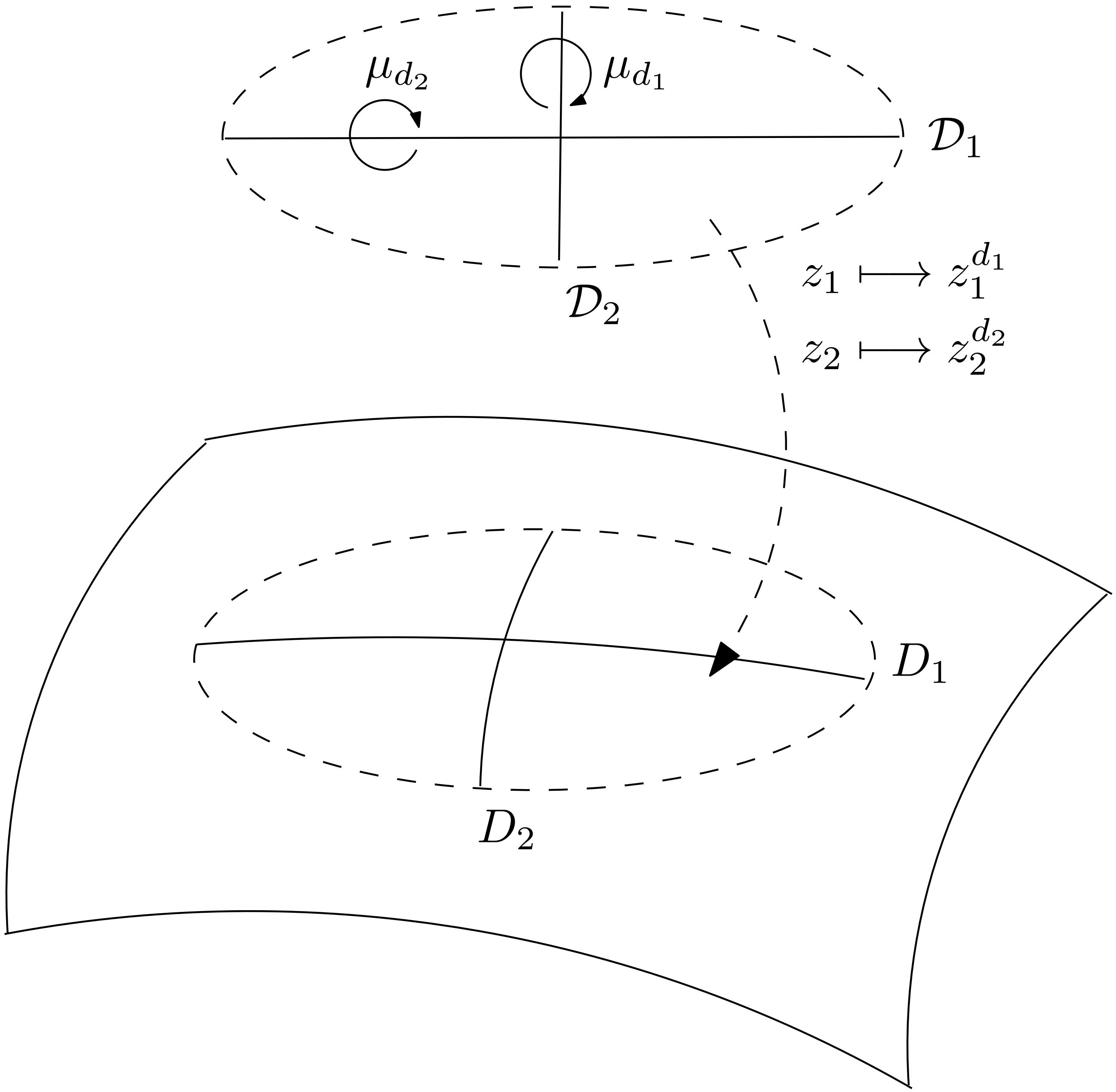
We embed the monoid of effective Cartier divisors on a variety X in the symmetric monoidal category $\mathfrak{Div}\, X$ of pairs (L, s), where L is an invertible sheaf and $s \in L(X)$. An arrow from (L, s) to (L', s') is an isomorphism $L \simeq L'$ carrying s into s'. The monoidal structure is given by tensor product. This category is more complicated than the monoid of divisors, but invertible sheaves with sections have the advantage of admitting pullbacks under arbitrary morphisms.

Let D be an effective Cartier divisor on a scheme S and d be a positive integer. We can construct the *stack of roots* $S[\frac{1}{d}D]$. A morphism $T \to S[\frac{1}{d}D]$ consists of a morphism $f\colon T \to S$, together with an object $\mathcal{D}$ of $\mathfrak{Div}\, T$ and an isomorphism $\mathcal{D}^{\otimes d} \simeq f^*D$. In the previous context, consider

$$X[\tfrac{1}{\mathbf{d}}D] \stackrel{\text{def}}{=} X[\tfrac{1}{d_1}D_1] \times_X \cdots \times_X X[\tfrac{1}{d_r}D_r].$$

Morphisms $T \to X[\frac{1}{\mathbf{d}}D]$ correspond to morphisms $f\colon T \to X$, together with objects $\mathcal{D}_{1,T}, \ldots, \mathcal{D}_{r,T}$ of $\mathfrak{Div}\, T$, and an isomorphism $\mathcal{D}_{i,T}^{\otimes d_i} \simeq f^*D_i$ for each i. The stack $X[\frac{1}{\mathbf{d}}D]$ is algebraic, and a Deligne–Mumford stack in characteristic 0. Its moduli space is X. In $\mathfrak{Div}\, X[\frac{1}{\mathbf{d}}D]$ there are tautological divisors $\mathcal{D}_1, \ldots, \mathcal{D}_r$. Call $\pi\colon X[\frac{1}{\mathbf{d}}D] \to X$ the natural morphism.

Here is a local picture of π in the case $r = 2$.


μ_{d_2}
μ_{d_1}
$\mathcal{D}_1$
$\mathcal{D}_2$
$z_1 \longmapsto z_1^{d_1}$
$z_2 \longmapsto z_2^{d_2}$
D_1
D_2

If E is a vector bundle on $X[\frac{1}{\mathbf{d}}D]$, we can associate with it a parabolic bundle in $\mathrm{ParV}_{1/d}(X, D)$: for each $\frac{\mathbf{i}}{\mathbf{d}} = (\frac{i_1}{d_1}, \dots, \frac{i_r}{d_r}) \in \frac{1}{\mathbf{d}}\mathbb{Z}^r$, set

$$E_{\mathbf{i}/\mathbf{d}} \overset{\text{def}}{=} \pi_* E(i_1\mathcal{D}_1 + \cdots + i_r\mathcal{D}_r).$$

Theorem (Borne)**.** *There is an equivalence of tensor categories* $\mathrm{Vect}(X[\frac{1}{\mathbf{d}}D])$ *with* $\mathrm{ParV}_{1/\mathbf{d}}(X)$.

Abhyankar's lemma says that $\widehat{\pi}_1(X \smallsetminus D)$ is the limit of the $\widehat{\pi}_1(X[\frac{1}{\mathbf{d}}D])$. Hence every continuous representation of $\widehat{\pi}_1(X \smallsetminus D)$ comes from a representation of $\widehat{\pi}_1(X[\frac{1}{\mathbf{d}}D])$ for some $\mathbf{d}$. With this you associate a finite vector bundle on $X[\frac{1}{\mathbf{d}}D]$, hence a parabolic bundle. The Nori–Borne theorem says that this gives an equivalence.

Borne's work left two main questions open.

1) In what generality can one define parabolic bundles? For example, how about divisors with non-simple normal crossing? It seems clear that one must use *sheaves* of weights.

2) How about parabolic coherent sheaves?

These problems are best seen in the context of logarithmic geometry.

Borne and I define a notion of parabolic (quasi-)coherent sheaf for logarithmic varieties with given weights. We also prove a general version of Borne's correspondence in this context.

Let us review the notion of logarithmic structure in a somewhat unorthodox language, using *Deligne–Faltings structures*. The fact that a Deligne–Faltings structure defines a logarithmic structure is somehow implicit in Kato's construction of the logarithmic structure associated with a homomorphism of monoids $P \to \mathcal{O}(X)$; going in the other direction, the construction is contained in a paper of Lorenzon.

All sheaves on a scheme X will be defined on the small étale site $X_{\text{ét}}$. All monoids will be commutative.

Recall that for any scheme X we denote by $\mathfrak{Div}\, X$ the symmetric monoidal category of invertible sheaves with a specified section. The arrows in $\mathfrak{Div}\, X$ are given by isomorphisms. The identity is $(\mathcal{O}_X, 1)$. The invertible objects are the pairs (L, s) in which s does not vanish anywhere.

If A is a monoid, we will think of A as a discrete symmetric monoidal category. We will consider symmetric monoidal functors $L\colon A \to \mathfrak{Div}\, X$. This means that for each element $a \in A$ we have an object $L(a) = (L_a, s_a)$ of $\mathfrak{Div}\, X$. We are also given an isomorphism of $L(0)$ with $(\mathcal{O}_X, 1)$, and for $a, b \in A$ an isomorphism $L(a+b) \simeq L(a) \otimes L(b)$. These are required to satisfy various compatibility conditions.

Definition. A Deligne–Faltings (DF) structure (A, L) on a scheme X consists of a sheaf of monoids A and a symmetric monoidal functor $L_U\colon A(U) \to \mathfrak{Div}\, U$ for each étale map $U \to X$, such that

(a) the L_U are compatible with pullbacks via étale maps $V \to U$ (this is best expressed with the language of fibered categories); and

(b) if $L(a)$ is invertible for some $a \in A(U)$, then $a = 0$.

The category of DF structures on a scheme X is equivalent to the category of quasi-integral logarithmic structures on X. Recall that a logarithmic structure $M \to \mathcal{O}_X$ is *quasi-integral* if the action of $\mathcal{O}_X^*$ on M is free. If (X, M) is quasi-integral, the natural projection $\pi\colon M \to \overline{M}$ makes M into a $\mathcal{O}_X^*$-torsor over $\overline{M}$; then we obtain a DF structure by setting $A \overset{\text{def}}{=} \overline{M}$, and sending a section $a \in A(U)$ into the sheaf L_a of $\mathcal{O}_U^*$-equivariant morphisms $\pi^{-1}(a) \to \mathcal{O}_U$, which is an invertible sheaf. The section s_a is obtained from the given homomorphism $M \to \mathcal{O}$.

A particularly interesting feature of this approach is the treatment of charts.

Let A be a sheaf of monoids on $X_{\text{ét}}$. A *chart* for A consists of a finitely generated monoid P, with a homomorphism of monoids $P \to A(X)$, such that the induced homomorphism of sheaves of monoids $P_X \to A$ (where P_X is the constant sheaf corresponding to P) is a cokernel. This means that if K is the kernel of $P_X \to A$, the induced homomorphism $P_X/K \to A$ is an isomorphism. This is much stronger than asking that $P_X \to A$ be surjective.

If (A, L) is a DF structure on X, a chart for (A, L) is a chart for A. From a chart $P \to A(X)$ one gets a symmetric monoidal functor $P \to \mathfrak{Div}\, X$. Given a scheme X, a finitely generated monoid P and a symmetric monoidal functor $P \to \mathfrak{Div}\, X$, there exists a DF structure (A, L), unique up to a unique isomorphism, with a chart $P \to A(X)$, with an isomorphism of the composite $P \to A(X) \xrightarrow{L_X} \mathfrak{Div}\, X$ with the given symmetric monoidal functor $P \to \mathfrak{Div}\, X$.

A sheaf of monoids A is *coherent* if it is sharp, and étale locally has charts. A DF structure (A, L) is coherent of A is coherent. We show that a DF structure is coherent if and only if the associated logarithmic structure is. Our charts exists on larger open sets than Kato charts. For example, if $D = (L, s)$ is an effective Cartier divisor on X, the DF structure (A, L) generated by D is defined by the symmetric monoidal functor $\mathbb{N} \to \mathfrak{Div}\, X$ sending $n \in \mathbb{N}$ into $D^{\otimes n}$, so it has a global chart; but it has a Kato chart only when D is principal.

Now, in order to define our correspondence between parabolic quasi-coherent sheaves and sheaves on root stacks we need to define parabolic sheaves and root stacks. For both, we need a DF structure (A, L) and a sheaf of denominators.

Let A be a coherent sheaf of monoids on X. The *weight system* A^{wt} of A is a category whose objects are elements of A^{gp}, and the arrows from x to y are elements $a \in A$ such that $x + a = y \in A$. For example, $(\mathbb{N}^r)^{\mathrm{wt}}$ is the category associated with the poset $\mathbb{Z}^r$, with the ordering defined by $(w_1, \ldots, w_r) \leq (w'_1, \ldots, w'_r)$ if $w_i \leq w'_i$ for all i.

This construction generalizes to sheaves of monoids, and associates a stack of categories A^{wt} with every sheaf of monoids A.

A *system of denominators* $A \to B$ is an injective homomorphism of sheaves of monoids, where B is coherent, such that every section of B locally has a multiple that comes from A.

Given a coherent DF structure (A, L) on a scheme X and a system of denominators $A \subseteq B$, let us define parabolic sheaves.

If X is a scheme, we denote by $\mathfrak{QCoh}\, X$ the category of quasi-coherent sheaves on X.

Let us write $L(a) = (L_a, s_a)$ for each $a \in A(U)$.

Definition. A parabolic sheaf E with weights in B is given by a functor : $B^{\mathrm{wt}}(U) \to \mathfrak{QCoh}\, U$, denoted by $b \mapsto E_b$, for each étale map $U \to X$, and a functorial isomorphism $E_{b+a} \simeq E_b \otimes L_a$ for each $b \in B^{\mathrm{wt}}(U)$ and each $a \in A(U)$, such that

(a) these data are compatible with pullbacks,

(b) The composite
$$E_b \longrightarrow E_{b+a} \simeq E_b \otimes L_a$$
is given by multiplication by s_a, and

(c) other compatibility conditions are satisfied.

Parabolic sheaves with weights in B form a category, denoted by $\mathrm{Par}_B(X, A, L)$.

Let (A, L) be a coherent DF structure over a scheme X and $A \subseteq B$ be a system of denominators. With this data we can associate a stack $X_{B/A} \to X$, which we call the *root stack*. Several particular cases of the construction that follows have been worked out by Martin Olsson, and the idea should be attibuted to him. It is a vast generalization of the notion of root stack for sections of invertible sheaves.

If $f\colon T \to X$, there is a pullback DF structure (f^*A, f^*L). The sheaf f^*A is simply the pullback sheaf, but f^*L is a little complicated to define. A morphism $T \to X_{B/A}$ corresponds to a morphism $f\colon T \to X$ and a DF structure Λ on T, with sheaf of monoids f^*B, together with an isomorphism of the restriction of of Λ to $f^*A \subseteq f^*B$ with the pullback of L to T. One can show that $X_{B/A}$ is a tame algebraic stack with moduli space X, and a Deligne–Mumford stack in characteristic 0.

Let (A, L) be a coherent DF structure over a scheme X and $A \subseteq B$ be a system of denominators.

Theorem (Borne, —). *The category of quasi-coherent sheaves on the root stack $X_{B/A}$ is naturally equivalent to the category of parabolic sheaves* $\mathrm{Par}_B(X, A, L)$.

To Do list:

1. Make this into a theory of quasi-coherent sheaves on a fine saturated logarithmic scheme, by going to the limit over all systems of denominators. This presents some non-trivial difficulties.

2. Develop a theory of parabolic sheaves with real weights.

3. Prove a version of Nori's theorem for fine saturated logarithmic schemes.