On the genericity theorem for essential dimension

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All fields will have characteristic 0. Let k be a base field, Field_k the category of extensions of k. Take $k = \mathbb{Q}$ or $k = \mathbb{C}$. Let $F\colon \mathrm{Field}_k \to \mathrm{Sets}$ be a functor. If ξ is an object of some $F(K)$, a *field of definition* of ξ is an intermediate field $k \subseteq L \subseteq K$ such that ξ is in the image of $F(L) \to F(K)$.

Definition (Merkurjev)**.** The *essential dimension of* ξ, denoted by $\mathrm{ed}_k\,\xi$, is the least transcendence degree $\mathrm{tr\,deg}_k\,L$ of a field of definition L of ξ. The *essential dimension of* F, denoted by $\mathrm{ed}\,F$, is the supremum of the essential dimensions of all objects ξ of all $F(K)$.

The idea is that $F(K)$ represents isomorphism classes of some objects we care about. Thus the essential dimension of ξ can be interpreted as the minimal number of independent parameter that we need to write ξ.

It is easy to see that if F is represented by a scheme X locally of finite type over k, then $\operatorname{ed}_k F = \dim X$. Thus, for example, if g and d are natural numbers, and $F(K)$ is the set of smooth curves in $\mathbb{P}^n_K$ of genus g and degree d, the essential dimension of F is the dimension of the Hilbert scheme of smooth curves of genus g and degree d in $\mathbb{P}^n$. If, instead, $F(K)$ is the set of smooth curves in $\mathbb{P}^n_K$ of genus g and degree d, up to projective equivalence, the problem is much deeper.

The essential dimension $\operatorname{ed}_k \xi$ is finite, under weak hypotheses on F. But $\operatorname{ed}_k F$ could still be $+\infty$.

Merkurjev's definition generalizes the notion of *essential dimension of a group*, due to Buhler and Reichstein, who initiated the theory. The essential dimension of an algebraic group G over k is the essential dimension of the functor $\mathrm{H}^1(-, G)$ of isomorphism classes of G-torsors over $\operatorname{Spec} K$.

Let $\mathcal{X}$ be an algebraic stack over k. *The essential dimension of* $\mathcal{X}$ *over* k, denoted by $\operatorname{ed}_k \mathcal{X}$, is the essential dimension of the functor of isomorphism classes of objects of $\mathcal{X}$ defined over extensions of k. For example, take $\mathcal{X} = \mathcal{M}_g$. What can we say about $\operatorname{ed}_k \mathcal{M}_g$? This is a very natural question, which could have been asked a long time ago, but to our knowledge was not: how many independent parameters do we need to write a general smooth curve of genus g?

Theorem (P. Brosnan)**.** *Let* $\mathcal{X}$ *be an algebraic stack of finite type over a field. Assume that for each object* ξ *of* $\mathcal{X}(K)$*, where K is an algebraically closed field, the group scheme* $\operatorname{Aut}_K(\xi)$ *is affine. Then* $\operatorname{ed}_k \mathcal{X}$ *is finite.*

This follows easily from a result of Kresch, which ensures that such a stack is stratified by quotient stacks.

The condition of the theorem is satisfied for $g \neq 1$, hence $\operatorname{ed}_k \mathcal{M}_g < +\infty$ if $g \neq 1$.

If C is a smooth projective curve of genus 0 over an extension K of k, then C is isomorphic to a conic in $\mathbb{P}^2_K$. Such a conic can be written with an equation of the form $ax^2 + by^2 + z^2 = 0$, so is defined over $k(a, b)$. This implies that $\operatorname{ed}_k \mathcal{M}_0 \leq 2$. It follows by Tsen's theorem that $\operatorname{ed}_k \mathcal{M}_0 = 0$.

Every elliptic curve can be written in the form $y^2 z = x^3 + axz^2 + bz^3$, hence $\operatorname{ed}_k \mathcal{M}_{1,1} \leq 2$. In fact $\operatorname{ed}_k \mathcal{M}_{1,1} = 2$. However, $\mathcal{M}_1 \neq \mathcal{M}_{1,1}$! A smooth curve of genus 1 without a rational point can not be written in this form.

Here is a geometric interpretation of $\mathrm{ed}_k \mathcal{M}_g$, closer to the original definition of Buhler and Reichstein. Let $C \to S$ be a family in $\mathcal{M}_g$ where S is integral of finite type over k. A *compression* of $C \to S$ consists of a non-empty open subset $U \subseteq S$ and a cartesian diagram

$$\begin{array}{ccc} C_U & \longrightarrow & D \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

where $D \to T$ is in $\mathcal{M}_g$.

If K is the function field of S, the essential dimension of the generic fiber $C_K \to \operatorname{Spec} K$ is the minimal dimension of T, taken over all compressions of $C \to S$. The essential dimension of $\mathcal{M}_g$ is the supremum over all essential dimension of all families $C \to S$.

Suppose that $g \geq 2$, and let M_g be the moduli space of curves, which has dimension $3g - 3$. Suppose that the $C \to S$ has maximal variation in moduli, i.e., the morphism $S \to \mathrm{M}_g$ given by $C \to S$ is dominant; then any compression of $C \to S$ also has maximal variation in moduli; hence $\operatorname{ed}_k \mathcal{M}_g \geq 3g - 3$ if $g \geq 2$.

Theorem (P. Brosnan, Z. Reichstein, —)**.**

$$\operatorname{ed}_k \mathcal{M}_g = \begin{cases} 2 & \textit{if } g = 0 \\ +\infty & \textit{if } g = 1 \\ 5 & \textit{if } g = 2 \\ 3g - 3 & \textit{if } g \geq 3. \end{cases}$$

Genericity theorem (P. Brosnan, Z. Reichstein, A. Vistoli). *Let $\mathcal{X}$ be a smooth connected Deligne–Mumford stack of finite type over a field k, $\mathcal{U}$ a non-empty open substack. Then $\operatorname{ed}_k \mathcal{X} = \operatorname{ed}_k \mathcal{U}$.*

In particular, if the stabilizer of a generic point of $\mathcal{X}$ is trivial, then $\operatorname{ed}_k \mathcal{X} = \dim \mathcal{X}$.

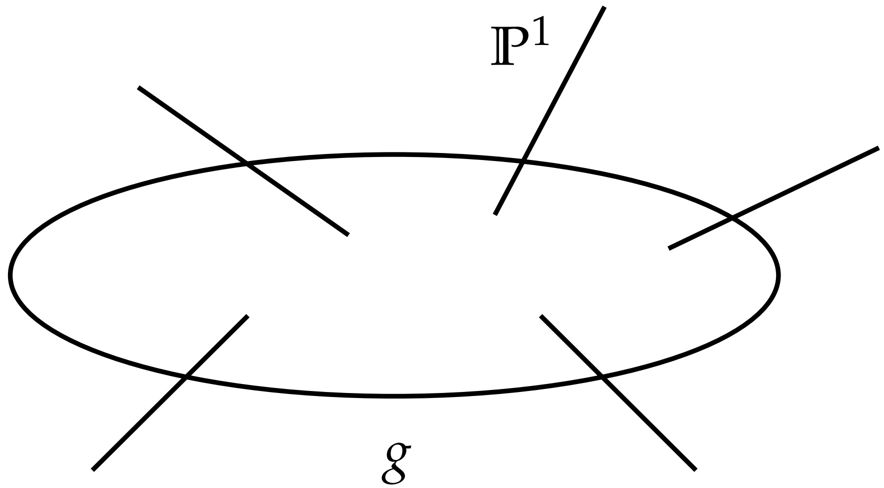
This takes care of the case $g \geq 3$. For more general cases we need a more precise form of the theorem.

Let $\mathcal{X}$ be a smooth connected separated Deligne–Mumford stack of finite type over a field k with moduli space $\mathcal{X} \to \mathbf{X}$. Let K be the function field of $\mathbf{X}$. Let $\mathcal{X}_K \stackrel{\text{def}}{=} \mathcal{X} \times_{\mathbf{X}} \operatorname{Spec} K$ be the *generic gerbe* of $\mathcal{X}$. If $\mathcal{X}$ is not separated, it will not have a moduli space, but the generic gerbe is still defined.

Theorem. $\operatorname{ed}_k \mathcal{X} = \dim \mathcal{X} + \operatorname{ed}_K(\mathcal{X}_K)$.

The genericity theorem applies to more general curves than smooth (or stable) curves. Suppose that $g \geq 2$. Let $\mathfrak{M}_g$ be the stack of all reduced connected local complete intersection curves of arithmetic genus g. This is a smooth irreducible Artin stack locally of finite type over k, containing $\mathcal{M}_g$ as an open substack. Let $\mathcal{M}_g^{\text{fin}}$ be the open substack of curves with finite automorphism groups. Then $\mathfrak{M}_g^{\text{fin}}$ is a smooth integral Deligne–Mumford stack locally of finite type over k; hence the genericity theorem applies, and we can conclude that $\operatorname{ed}_k \mathfrak{M}_g^{\text{fin}} = \operatorname{ed}_k \mathcal{M}_g$. In other words: the smallest number of independent parameters needed to write every geometrically reduced and geometrically connected local complete intersection curve of arithmetic genus g with finite automorphism group is $3g - 3$ if $g \geq 3$, and 5 if $g = 2$.

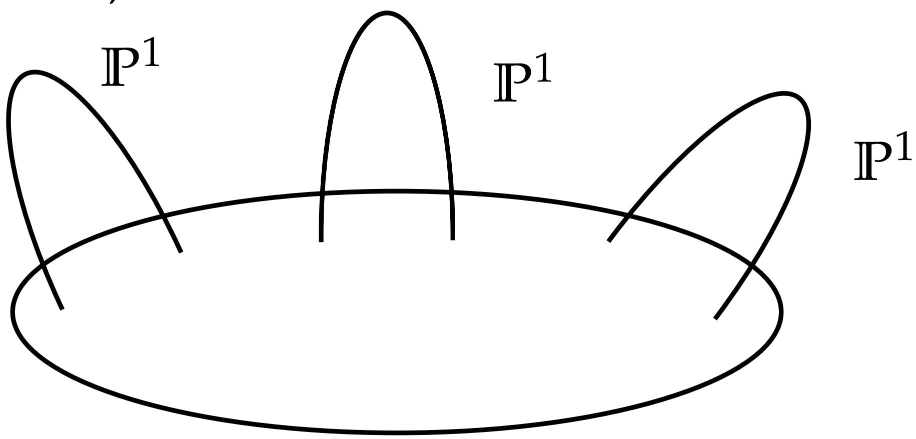
On the other hand, the essential dimension of $\mathfrak{M}_g$ is infinite. For example, let C be a curve over a field K, that over the algebraic closure $\overline{K}$ looks like a smooth curve of genus g, with n smooth rational tails attached.



Then it is easy to see that the "generic" such C has essential dimension $3g - 3 + n$. Thus the genericity theorem does not hold for $\mathfrak{M}_g$. Recall that an Artin stack is Deligne–Mumford when the automorphism group of one of its object is finite; so $\mathfrak{M}_g$ is not Deligne–Mumford.

Here is another example. Let M_n be the affine space of $n \times n$ matrices over k. Let GL_n act on M_n by left multiplication, and consider the quotient stack $[\mathrm{M}_n/\mathrm{GL}_n]$. The associated functor $\mathrm{Field}_k \to \mathrm{Sets}$ sends each extension K of k into the quotient set $\mathrm{M}_n(K)/\mathrm{GL}_n(K)$. But two matrices in $\mathrm{M}_n(K)$ are in the same orbit if and only if they have the same kernel; hence the essential dimension of $[\mathrm{M}_n/\mathrm{GL}_n]$ is the essential dimension of the functor $\mathrm{Field}_k \to \mathrm{Sets}$ that sends and extension K of k into the set of all the subspaces of K^n. This functor is also the functor associated with the disjoint union $\bigsqcup_{i=0}^{n} \mathbb{G}(i,n)$ of all the grassmanians of i-dimensional subspaces in K^n; so its essential dimension is the dimension of $\bigsqcup_{i=0}^{n} \mathbb{G}(i,n)$, which is positive. On the other hand, $[\mathrm{M}_n/\mathrm{GL}_n]$ contains the open substack $[\mathrm{GL}_n/\mathrm{GL}_n] = \operatorname{Spec} k$, so the essential dimension of its generic gerbe is 0.

What goes wrong? I tried to construct examples of curves with essential dimension larger than that of a generic smooth curve, but without rational tails, and could not do it.



The difference between this kind of curve and the previous ones, violating the genericity theorem, is that in this case the automorphism groups are of the type $\mathbb{G}_m$, while in the previous case where $\mathbb{G}_m \ltimes \mathbb{G}_a$; so in this case they are reductive, while in the previous case they are not.

This example, and many others, suggest a conjecture.

Conjecture (Z. Reichstein, —). Let $\mathcal{X}$ be a smooth connected Artin stack locally of finite type over k, which is generically Deligne–Mumford. Let $\mathcal{X}_K \to \operatorname{Spec} K$ be its generic gerbe. Let $\xi \in \mathcal{X}(L)$ be an object of $\mathcal{X}$ defined over some extension L of k; assume that $\operatorname{Aut}_L \xi$ is reductive. Then

$$\operatorname{ed}_k \xi \leq \dim \mathcal{X} + \operatorname{ed}_K \mathcal{X}_K.$$

In particular, if the automorphism group of any object of $\mathcal{X}$ is reductive, then

$$\operatorname{ed}_k \mathcal{X} = \dim \mathcal{X} + \operatorname{ed}_K \mathcal{X}_K.$$

This seems hard to prove. The condition that a point have reductive stabilizer is neither open nor closed; for example, in the example of the quotient $[\mathrm{M}_n/\mathrm{GL}_n]$ the only points that have reductive stabilizers are the generic point $\operatorname{Spec} k = [\mathrm{GL}_n/\mathrm{GL}_n]$ and the origin $\operatorname{Spec} k \to \mathrm{M}_n \to [\mathrm{M}_n/\mathrm{GL}_n]$; thus the points with reductive stabilizer form a constructible set which is neither open nor closed.

Definition. Let G be an affine algebraic group over a field. Then G is *extremely reductive* if it satisfies one of the following equivalent conditions.

1. All the algebraic subgroups of G are reductive.

2. G does not contain any non-trivial unipotent subgroup.

3. The connected component of the identity is a torus.

One can show that in an Artin stack locally of finite type over a field the points with extremely reductive stabilizers form an open substack.

Theorem. *Let $\mathcal{X}$ be a smooth integral Artin stack locally of finite type over a field. Suppose that $\mathcal{X}$ is generically Deligne–Mumford, and let $\mathcal{X}_K \to \operatorname{Spec} K$ be its generic gerbe. Suppose that the automorphism group of each object of $\mathcal{X}$ defined over a field is extremely reductive. Then*

$$\operatorname{ed}_k \mathcal{X} = \dim \mathcal{X} + \operatorname{ed}_K \mathcal{X}_K .$$

Corollary. *Let C be a geometrically reduced and geometrically connected curve of genus $g \geq 2$ defined over an extension K of k. Suppose that the smooth part of C does not contain a copy of $\mathbb{A}^1$. Then*

$$\operatorname{ed}_k C \leq \operatorname{ed}_k \mathcal{M}_g .$$

Let us give another application of the result. Fix two positive integers d and n. Let $\mathbf{F}_{n,d}\colon \text{Field}_k \to \text{Sets}$ be the functor that associates with each extension K of k the set of all homogeneous polynomials $f(x)$ of degree d in n variables, modulo base change, that is, modulo the natural action of $\mathrm{GL}_n(K)$ defined by the rule $A \cdot f(x) = f\bigl(A^{-1}x\bigr)$. What is the essential dimension of this functor? Let $\mathbb{A}_k^N$ be the affine space of all forms of degree d in n variables, where $N = \binom{d+n-1}{n-1}$. The functor $\mathbf{F}_{n,d}$ is the functor of isomorphism classes of the quotient stack $\mathcal{F}_{n,d} \overset{\text{def}}{=} [\mathbb{A}^N/\mathrm{GL}_n]$; so we are asking about the essential dimension of $\mathcal{F}_{n,d}$.

When $d \geq 3$ and $n \geq 2$, the automorphism group of a general form in $\mathbf{F}_{n,d}(K)$ is finite; in other words, $\mathcal{F}_{n,d}$ is generically Deligne–Mumford. In this range, there is a generic moduli space for $\mathcal{F}_{n,d}$, which has dimension $N - n^2$; hence $\operatorname{ed}_k \mathbf{F}_{n,d} \geq N - n^2$.

Of course $\operatorname{ed}_k \mathcal{F}_{n,1} = 0$. For $d = 2$, we know from basic linear algebra that any quadratic form can be written in the form $a_1 x_1^2 + \cdots + a_n x_n^2$; hence $\operatorname{ed}_k \mathcal{F}_{n,2} \leq n$. Reichstein proved that $\operatorname{ed}_k \mathcal{F}_{n,2} = n$.

Let us only consider from now on the case $n = 3$. Berhuy and Favi, in 2003, showed that $\operatorname{ed}_k \mathcal{F}_{3,3} = 3$. Suppose that $d \geq 4$. Let $\Phi(x)$ the generic form of degree d; in other words, the form all of whose coefficients are independent indeterminates; or the form corresponding to the generic point of $\mathbb{A}^N$. In 2005 Berhuy and Reichstein calculated the essential dimension of $\Phi(x)$.

Theorem (Berhuy and Reichstein)**.**

$$\operatorname{ed}_k \Phi(x) = \begin{cases} \binom{d+2}{2} - 8 & \text{if } 3 \nmid d \\ \binom{d+2}{2} - 6 & \text{if } 3 \mid d \end{cases}$$

Let $(\mathcal{F}_{3,d})_K$ be the generic gerbe of $\mathcal{F}_{3,d}$.

Theorem (Brosnan, Reichstein and —)**.**

$$\dim \mathcal{F}_{3,d} + \operatorname{ed}_K(\mathcal{F}_{3,d})_K = \begin{cases} \binom{d+2}{2} - 8 & \textit{if } 3 \nmid d \\ \binom{d+2}{2} - 6 & \textit{if } 3 \mid d \end{cases}$$

Thus one can try to apply the genericity theorem. The stack $\mathcal{F}_{3,d}$ satisfyies all the hypothesis of the new genericity theorem, except that the automorphism of all objects over points must be extremely reductive.

However

Theorem (Reichstein, —)**.**

$$\operatorname{ed}_k \mathbf{F}_{3,d} = \begin{cases} \binom{d+2}{2} - 8 & \textit{if } 3 \nmid d \\ \binom{d+2}{2} - 6 & \textit{if } 3 \mid d \end{cases}$$

The point is that the “bad” forms, those having a unipotent symmetry group, are very special, and can be classified (they factor in terms of degrees 1 and 2); this allows to estimate their essential dimension.Thus, one applies the genericity theorem to the complement of this bad locus; the bad forms do not change the result.