Essential dimension of homogeneous polynomials

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The theory of *essential dimension* was born in 1997 with the publication of "On the essential dimension of a finite group", by Joe Buhler and Zinovy Reichstein. It has since attracted a lot of attention.

The basic question is: how complicated is it to write down an algebraic or geometric object in a certain class? How many independent parameters do we need?

Let us start with the very general definition, due to Merkurjev.

We will fix a base field k of characteristic 0. Can take $k = \mathbb{Q}$ or $k = \mathbb{C}$.

Let Fields_k the category of extensions of k. Let $F\colon \mathrm{Fields}_k \to \mathrm{Sets}$ be a functor. We should think of each $F(K)$ as the set of isomorphism classes of some class of objects we are interested in. If ξ is an object of some $F(K)$, a *field of definition* of ξ is an intermediate field $k \subseteq L \subseteq K$ such that ξ is in the image of $F(L) \to F(K)$.

Definition (Merkurjev)**.** The *essential dimension of* ξ, denoted by $\mathrm{ed}_k\, \xi$, is the least transcendence degree $\operatorname{tr\,deg}_k L$ of a field of definition L of ξ.

The *essential dimension of* F, denoted by $\mathrm{ed}_k\, F$, is the supremum of the essential dimensions of all objects ξ of all $F(K)$.

The essential dimension $\mathrm{ed}_k\, \xi$ is finite, under weak hypothesis on F. But $\mathrm{ed}_k\, F$ could still be $+\infty$.

It is easy to see that if F is represented by a scheme X of finite type over k, then $\mathrm{ed}_k F = \dim X$. Thus, for example, if g and d are natural numbers, and $F(K)$ is the set of smooth curves in $\mathbb{P}^n_K$ of genus g and degree d, the essential dimension of F is the dimension of the Hilbert scheme of smooth curves of genus g and degree d in $\mathbb{P}^n$. But if we ask for the essential dimension of the functor of smooth curves of genus g and degree d, up to projective equivalence, the question may be very hard.

Suppose that we have an action of GL_n on some scheme X which is of finite type over k. The we can define the *functor of orbits* $F\colon \mathrm{Fields}_k \to \mathrm{Sets}$ that sends each extension K of k into the set $X(K)/\mathrm{GL}_n(K)$ of orbits for the action of $\mathrm{GL}_n(K)$ on the set of K-rational points $X(K)$. The *essential dimension of the action* is the essential dimension of this functor. Clearly $\mathrm{ed}_k F \leq \dim X$.

Here are some interesting examples.

(1) Let $X_{n,d}$ be the affine space of dimension $\binom{d+n-1}{n-1}$ of forms of degree d in n variables, with the natural action of GL_n by base change. The functor of orbits is the functor $\mathbf{F}_{n,d}$ of forms of degree d in n variables, up to change of coordinates.

(2) The functor $F_{\mathcal{M}_g}$ be the functor that associates with each extension $k \subseteq K$ the set of isomorphism classes of smooth projective curves of genus g is isomorphic to a functor of orbits for $g \neq 1$.

(3) If $G \subseteq \mathrm{GL}_n$ is a closed subgroup, the functor of orbits for the action of GL_n on GL_n/G is isomorphic to the functor of isomorphism classes of G-torsors.

The essential dimension of the functor of isomorphism classes of G-torsors is known as the essential dimension of G. Buhler and Reichstein introduced this concept for finite groups, with a rather different geometric definition. This case has been studied a lot, but many important questions are still open. For example, the essential dimension of PGL_n is very interesting, because PGL_n-torsors correspond to Brauer–Severi varieties, and also to central simple algebras.

Assume that k contains enough roots of 1. It is know that $\mathrm{ed}_k \mathrm{PGL}_2 = \mathrm{ed}_k \mathrm{PGL}_3 = 2$; this follows from the fact that central simple algebras of degree 2 and 3 are *cyclic*. This is easy for degree 2; in degree 3 it is a theorem of Albert. A cyclic algebra of degree n over K has a presentation of the type $x^n = a$, $y^n = b$ and $yx = \omega xy$, where $a, b \in K^*$ and ω is a primitive n^{th} root of 1. Hence a cyclic algebra is defined over a field of the type $k(a, b)$, and has essential dimension at most 2.

When n is a prime larger than 3, it is only known (due to Lorenz, Reichstein, Rowen and Saltman) that

$$2 \leq \operatorname{ed}_k \mathrm{PGL}_n \leq \frac{(n-1)(n-2)}{2}.$$

Computing $\operatorname{ed}_k \mathrm{PGL}_n$ when n is a prime is an extremely important question, linked with the problem of cyclicity of simple algebras of prime degree. If every simple algebra of prime degree is cyclic, then $\operatorname{ed}_k \mathrm{PGL}_n = 2$. Most experts think that a generic simple algebra of prime degree larger than 3 should not be cyclic. One way to show this would be to prove that $\operatorname{ed}_k \mathrm{PGL}_n > 2$ when n is a prime larger than 3.

Consider the functor $\mathbf{F}_{n,2}$, associating with an extension K the set of isometry classes of quadratic forms. Of course, every quadratic form can be diagonalized, i.e., written in the form $\sum_{i=1}^{n} a_i x_i^2$; this implies that its orbit is defined on an extension $k(a_1, \ldots, a_n)$ of transcendence degree at most n. So $\operatorname{ed}_k \mathbf{F}_{n,2} \leq n$. Can one do better? It was proved by Z. Reichstein in 2000 that $\operatorname{ed}_k \mathbf{F}_{n,2} = n$.

In this examples, as in most cases, getting upper bounds is much easier than getting lower bounds.

In 2003, Grégory Berhuy and Giordano Favi proved that $\operatorname{ed}_k \mathbf{F}_{3,3} = 4$ (more or less).

In 2005 Berhuy and Reichstein proved the following result. Assume that $n \geq 4$ and $d \geq 3$, or $n = 3$ and $d \geq 4$, or $n = 2$ and $d \geq 5$ (these conditions mean that the generic hypersurface of degree d in n variables has no non-trivial projective automorphisms). Let $\Phi_{n,d}(x)$ be the generic n-form of degree d; in other words, the form all of whose coefficients are independent indeterminates; or the form corresponding to the generic point of $X_{n,d}$. The essential dimension $\operatorname{ed}_k \Phi_{n,d}(x)$ is the essential dimension of the orbit of $\Phi(x)$. There is an obvious lower bound for $\operatorname{ed}_k \Phi_{n,d}(x)$, which is $\binom{d+n-1}{n-1} - n^2$ (the dimension of the moduli space $M_{n,d}$ of n-forms of degree d). The point is that there is a dominant invariant rational map $X_{n,d} \dashrightarrow M_{n,d}$, so a field of definition of a form in the orbit of $\Phi_{n,d}(x)$ must always contain $k(M_{n,d})$.

Theorem (Berhuy, Reichstein)**.**

(a) *If* $\gcd(n,d) = 1$*, then*

$$\mathrm{ed}_k\,\Phi_{n,d}(x) = \binom{d+n-1}{n-1} - n^2 + 1\,.$$

(b) *Suppose that* $\gcd(n,d) = p^i$*, where* p *is a prime and* $i > 0$*. Call* p^j *the largest power of* p *dividing* d*. Then*

$$\mathrm{ed}_k\,\Phi_{n,d}(x) = \binom{d+n-1}{n-1} - n^2 + p^j\,.$$

But is $\mathrm{ed}_k\,\mathbf{F}_{n,d}$ equal to $\mathrm{ed}_k\,\Phi_{n,d}(x)$? In other words, could it happen that there are special forms that are more complicated than the generic one?

Suppose that X is an integral scheme of finite type over k with an action of GL_n, and call K its field of fraction. Let F be its orbit functor. We define the *generic essential dimension of F*, denoted by $\operatorname{g\,ed}_k F$, as the essential dimension of the orbit of the generic point $\operatorname{Spec} K \to X$. This turns out to depend only on F, and not on the specific group action. The result of Berhuy and Reichstein is about the generic essential dimension of $\mathbf{F}_{n,d}$. Obviously, $\operatorname{ed}_k F \geq \operatorname{g\,ed}_k F$.

In order to determine the essential dimension of F, we split the work into two parts.

(a) We compute $\operatorname{g\,ed}_k F$.

(b) We show that $\operatorname{ed}_k F = \operatorname{g\,ed}_k F$.

The techniques involved are very different.

Let us see an example in which $\operatorname{ed}_k F > \operatorname{g\,ed}_k F$. Let M_n be the affine space of $n \times n$ matrices, and let GL_n act on it by left multiplication. Let F_n be the orbit functor. The generic $n \times n$ matrix is invertible, so it has the identity matrix in its orbit, therefore $\operatorname{g\,ed}_k F_n = 0$. On the other hand, two matrices A in B in $\mathrm{M}_n(K)$ are in the same orbit if and only if $\ker A = \ker B$; so $F_n(K)$ can also be described as the set of linear subspaces of K^n. So $F_n(K)$ is the set of K-points of the disjoint union of Grassmannians $\coprod_{i=0}^{n} \mathbb{G}(i,n)(K)$; hence $\operatorname{ed}_k F_n$ equals the dimension of $\coprod_{i=0}^{n} \mathbb{G}(i,n)$, which is positive if $n \geq 2$.

Is there a general case in which we can assert that $\operatorname{ed}_k F = \operatorname{g\,ed}_k F$?

Yes.

Genericity theorem (Brosnan, Reichstein, —). *Suppose that* GL_n *acts with finite stabilizers on a connected smooth variety* X *over* k. *Let* F *be the orbit functor. Then* $\operatorname{ed}_k F = \operatorname{g\,ed}_k F$.

This is a particular case of the general statement about Deligne–Mumford stacks.

This is definitely false, in general, when X is singular. It seems very hard to say something in the singular case.

Corollary. *Suppose that* GL_n *acts with finite stabilizers on a connected smooth variety* X *over* k, *with trivial generic stabilizer. Let* F *be the orbit functor. Then* $\operatorname{ed}_k F = \dim X - n^2$.

Here is an application. Recall that $F_{\mathcal{M}_g}$ is the functor that associates with each extension $k \subseteq K$ the set of isomorphism classes of smooth projective curves of genus g. What is $\operatorname{ed}_k F_{\mathcal{M}_g}$? In other words, how many independent variables do you need to write down a general curve of genus g?

Curves of genus 0 are conics, hence they can be written in the form $ax^2 + by^2 + z^2 = 0$, so $\operatorname{ed}_k F_{\mathcal{M}_0} \leq 2$. By Tsen's theorem, $\operatorname{ed}_k F_{\mathcal{M}_0} = 2$. An easy argument using moduli spaces of curves reveals that $\operatorname{ed}_k F_{\mathcal{M}_g} \geq 3g - 3$ for $g \geq 2$, and $\operatorname{ed}_k F_{\mathcal{M}_1} \geq 1$.

Theorem (Brosnan, Reichstein, —)**.**

$$\operatorname{ed} F_{\mathcal{M}_g} = \begin{cases} 2 & \textit{if } g = 0 \\ +\infty & \textit{if } g = 1 \\ 5 & \textit{if } g = 2 \\ 3g - 3 & \textit{if } g \geq 3. \end{cases}$$

What can one say when the stabilizers are not finite? Let us go back to our example of the action of GL_n by left multiplication on M_n. In this case the generic essential dimension is 0. If a matrix $A \in \mathrm{M}_n(K)$ has rank r, then the orbit of A in $\mathrm{M}_n(K)$ is in natural correspondence with the K-points of the Grassmannian $\mathbb{G}(r, n)$ of quotients of dimension r; so its essential dimension is 0 exactly when A is invertible or $A = 0$. The stabilizer of A is a parabolic subgroup of GL_n, and this is never reductive, unless A is invertible or A is 0.

Recall that a linear algebraic group G over k is *reductive* when one of the following equivalent condition is satisfied.

(a) G contains no non-trivial normal unipotent subgroups.

(b) G is *linearly reductive*, i.e., the linear representations of G are completely reducible.

This, and many other examples, let Reichstein and myself to conjecture the following result, which recently became a theorem.

Generalized genericity theorem (Reichstein, —)**.** *Let X be a smooth connected variety with an action of GL_n, and call F its orbit functor. Assume that the generic stabilizer is finite. Let K be an extension of k, and let $\xi \in X(K)$, such that the stabilizer of ξ is reductive. Then the essential dimension of the orbit of ξ is at most equal to the generic essential dimension of the orbit functor.*

In particular, if all stabilizers are reductive, then $\mathrm{ed}_k F = \mathrm{g\,ed}_k F$.

More generally, this can be stated for algebraic stacks.

This can be applied to the action of GL_n on the space of forms $X_{n,d}$. The forms whose stabilizer is not reductive are very special, and they live in a subvariety of $X_{n,d}$ of high codimension.

Theorem (Reichstein, —). *Assume that $n \geq 2$ and $d \geq 3$. Then*

$$\operatorname{ed}_k \mathbf{F}_{n,d} = \operatorname{g ed}_k \mathbf{F}_{n,d} \,.$$

Corollary. *Assume that $n \geq 4$ and $d \geq 3$, or $n = 3$ and $d \geq 4$, or $n = 2$ and $d \geq 5$.*

(a) *If $\gcd(n, d) = 1$, then*

$$\operatorname{ed}_k \mathbf{F}_{n,d} = \binom{d+n-1}{n-1} - n^2 + 1 \,.$$

(b) *Suppose that $\gcd(n, d) = p^i$, where p is a prime and $i > 0$. Call p^j the largest power of p dividing d. Then*

$$\operatorname{ed}_k \mathbf{F}_{n,d} = \binom{d+n-1}{n-1} - n^2 + p^j \,.$$