Fields of moduli and the arithmetic of quotient singularities

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Let k be a field; for expository reasons, I will always assume $\operatorname{char} k = 0$. Consider a class $\mathscr{M}$ of objects of type (X, ξ), where X is a proper variety over an extension of k, and ξ is an additional structure, for example, a polarization on X.

We will be interested in *pointed varieties* (X, p), that is, varieties $X \to \operatorname{Spec} k$ with a fixed smooth rational point $p \in X(k)$.

We will require some conditions on $\mathscr{M}$, which will be satisfied in all the obvious examples: there should be a good notion of family of objects of $\mathscr{M}$, satisfying existence of pullbacks, a finite presentation condition, étale descent, and representability of isomorphism group schemes.

Let (X, ξ) be an object of $\mathscr{M}$ over the algebraic closure $\overline{k}$, $k \subseteq k' \subseteq \overline{k}$ an intermediate extension. We say that (X, ξ) *is defined over* k' if there exists an object (X', ξ') over k' such that $(X', \xi')_{\overline{k}} \simeq (X, \xi)$.

Set $G \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ and $G' \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k') \subseteq G$. By Galois descent, (X, ξ) is defined over k' if and only if there exists an action of G' on (X, ξ), compatible with the action of G' on $\operatorname{Spec} \overline{k}$.

If $s \in G$, denote by $(X, \xi)_s$ the pullback of (X, ξ) along $s \colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$. If H is the open subgroup of G consisting of $s \in G$ such that $(X, \xi)_s \simeq (X, \xi)$, the fixed subfield $\overline{k}^H$ is the *field of moduli* of (X, ξ), denoted by $\mathrm{M}(X, \xi)$. It is a finite extension of k, and is contained in every field of definition.

When $\mathscr{M}$ is a class of objects possessing a moduli space $M \to \operatorname{Spec} k$ (for example, smooth curves of genus at least 2, or polarized abelian varieties), an object (X, ξ) over $\overline{k}$ gives a morphism $\operatorname{Spec} \overline{k} \to M$; then if p is the image of $\operatorname{Spec} \overline{k} \to M$, then $\mathrm{M}(X, \xi)$ is the residue field $k(p)$.

The field of moduli was first defined for polarized varieties by T. Matsusaka in 1958; the definition was clarified and extended by G. Shimura and S. Koizumi. It has been intensively studied since then, particularly for curves and abelian varieties.

Here is the basic question: when is an object (X, ξ) over $\overline{k}$ defined over its field of moduli $\mathrm{M}(X, \xi)$?

If $\operatorname{Aut}(X, \xi)$ is trivial, then (X, ξ) is defined over its field of moduli.

Shimura gave examples of hyperelliptic curves not defined over their fields of definition.

Fix a genus $g \geq 2$. Let C be a conic over k with $C(k) = \emptyset$ and $D \subseteq C$ be a generic divisor of degree $2g + 2$. Then $D_{\overline{k}} \subseteq C_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$ is a smooth divisor of degree $2g + 2$, and there exists a unique hyperelliptic curve $X \to C_{\overline{k}}$ of genus g ramified along $D_{\overline{k}}$; if $s \in \mathrm{Gal}(\overline{k}/k)$, then $X_s \simeq X$, so $\mathrm{M}(X) = k$.

If X comes from a curve Y over k, by the uniqueness of the g^1_2 and the fact that $\mathrm{Aut}(C_{\overline{k}}, D_{\overline{k}})$ is trivial, this would be a double cover $Y \to C$ ramified along D; by the standard description of double covers this would correspond to a line bundle L on C with $L^{\otimes 2} \simeq \mathscr{O}_C(D)$. Then L would have degree $g + 1$, and if g is even there is no such L. Hence X is not defined over k.

Some known results.

(1) It is classical that every elliptic curve is defined over its field of moduli.

(2) Let A_g be the moduli space of principally polarized abelian varieties over $\mathbb{C}$, k its field of rational functions, A the corresponding abelian variety over $\overline{k}$. Then the field of moduli of A is k. In 1971 Shimura proved that A is defined over k if and only if g is odd.

(3) If X is a hyperelliptic curve such that $\mathrm{Aut}(X)/\langle\tau\rangle$ is not cyclic, where $\tau\colon X \to X$ is the hyperelliptic involution, then X is defined over its field of moduli (B. Huggins, 2005).

(4) A smooth projective curve of genus 2 whose automorphism group has order larger than 2 is defined over its field of moduli (G. Cardona and J. Quer, 2005).

(5) Consider the pairs (X, D), where X is a smooth conic, $D \subseteq X$ a smooth divisor of degree n. If n is odd, then (X, D) is defined over its field of moduli (A. Marinatto, 2011).

Assume that $\mathrm{Aut}(X, \xi)$ is finite. If (X, ξ) is an object of $\mathscr{M}$ over $\overline{k}$, $s \in \mathrm{Gal}(\overline{k}/k)$, and $X_s \simeq X$ is an isomorphism over the isomorphism $s\colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$, this descends to a well defined isomorphism $\bigl(X/\mathrm{Aut}(X, \xi)\bigr)_s \simeq X/\mathrm{Aut}(X, \xi)$. This gives descent data for $X/\mathrm{Aut}(X, \xi)$ to $\mathrm{M}(X, \xi)$, and yields a model $\mathrm{Q}(X, \xi)$ of $X/\mathrm{Aut}(X)$ over $\mathrm{M}(X, \xi)$, the *compression* of (X, ξ).

Theorem [P. Dèbes – M. Emsalem (1998)]. Let X be a smooth curve, and assume that $\mathrm{Aut}(X, \xi) \subseteq \mathrm{Aut}(X)$. If $\mathrm{Q}(X, \xi)$ has a point over $\mathrm{M}(X, \xi)$, then (X, ξ) is defined over its field of moduli.

Dèbes and Emsalem only state this for naked smooth curves, and 1-pointed curves, but their proof (probably) yields this more general statement.

One case in which $\mathrm{Q}(X, \xi)$ has always a point over $\mathrm{M}(X, \xi)$ is when the data ξ include a rational point. Thus we get that n-pointed smooth curves of genus g, with $n \geq 1$ and $g \geq 1$, are defined over their field of moduli. Here is another way of stating this: if $\mathrm{M}_{g,n}$ is the moduli space of n-pointed smooth curves of genus g over a field of characteristic 0 and $p \in \mathrm{M}_{g,n}$, the corresponding curve over $\overline{k(p)}$ is defined over $k(p)$. This generalizes the classical case $(g, n) = (1, 1)$.

But how about the case $\dim X > 1$?

Smooth pointed surfaces are not necessarily defined over their fields of moduli (for example, Shimura showed that the generic principally polarized abelian surface over $\mathbb{C}$ is not defined over its field of moduli).

Theorem [Bresciani – V.]. Let (X, ξ) be an object of $\mathscr{M}$ over $\overline{k}$. Assume the following conditions.

(1) $\mathrm{Aut}(X, \xi)$ is finite, and $\mathrm{Aut}(X, \xi) \subseteq \mathrm{Aut}(X)$.

(2) X is irreducible.

(3) There exists a dominant rational map $Z \dashrightarrow \mathrm{Q}(X, \xi)$, where Z is an integral scheme of finite type over $\mathrm{M}(X, \xi)$, and a rational smooth point $p \in Z\big(\mathrm{M}(X, \xi)\big)$.

Then (X, ξ) is defined over $\mathrm{M}(X, \xi)$.

When X is a smooth curve we have that $\mathrm{Q}(X, \xi)$ is smooth, so our result gives that of Dèbes and Emsalem. In the general case it is not enough to assume that $\mathrm{Q}(X, \xi)$ has a point over $\mathrm{M}(X, \xi)$.

Our aim is the following: if (X, p) is a pointed variety, find conditions on $\mathrm{Aut}(X, p)$ implying that, if conditions (1) and (2) above are satisfied, then (3) is also satisfied.

A *singularity* over a field k is a pair (U, p), where U is a scheme of finite type over k, and $p \in U(k)$. Two singularities (U, p) and (U', p') are *equivalent* if $\widehat{\mathscr{O}}_{U,p} \simeq \widehat{\mathscr{O}}_{U',p'}$.

Let (U, p) be a singularity, $\pi \colon \widetilde{U} \to U$ a resolution. We say that (U, p) is *liftable* if $\widetilde{U}$ has a k-rational point over p. This condition is independent of the resolution; furthermore, a singularity equivalent to a liftable singularity is itself liftable.

When $k = \mathbb{R}$ liftable singularities are called *central points*, and play a role in real algebraic geometry.

If G is a finite group, a *G-singularity* is a singularity (U, p) over k such that $(U, p)_{\overline{k}}$ is equivalent to a singularity of type $(V/G, [q])$, where V is a smooth variety over $\overline{k}$ and G acts faithfully on V with $q \in V(\overline{k})$ as a fixed point. For example, a 2-dimensional $(\mathbb{Z}/2)$-singularity is a surface A_1-singularity.

Definition. Let d be a positive integer. A finite group G is R_d if every d-dimensional G-singularity is liftable.

From our main theorem we get the following.

Corollary. Let (X, p) be an irreducible d-dimensional pointed variety over $\overline{k}$. If $\mathrm{Aut}(X, p)$ is a finite R_d-group, then (X, p) is defined over its field of moduli.

So, for example, from Shimura's result that a generic abelian surface is not defined over its field of moduli we see that $\mathbb{Z}/2$ is not R_2. And in fact it is easy to produce examples of surface A_1-singularities (U, p) with a minimal resolution $\widetilde{U}$ in which the exceptional divisor is a conic without rational points.

To apply this we need to have classes of finite groups that are R_d. If G is not a subgroup of $\mathrm{GL}_d(\overline{k})$, then G is R_d, but in an uninteresting way: there exist no d-dimensional G-singularities.

Clearly, every finite group is an R_1 group. One can show that cyclic groups of even order are not R_2.

Giulio worked out a (very complicated) complete classification of R_2-groups. The following is a sample.

Proposition. The following classes of finite groups are R_2.

(1) Groups of odd order.

(2) Groups of type $(\mathbb{Z}/m)^2 \times (\mathbb{Z}/n)$, where n is odd.

(3) Dihedral groups.

(4) Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ that are not cyclic of even order.

(5) More generally, finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ that do not contain any pseudoreflexion and are not cyclic of even order.

For general d we have two main results.

Theorem. Any finite group of order prime to $d!$ is R_d.

Theorem. Let G be a group with trivial center, such that the quotient map $\operatorname{Aut} G \to \operatorname{Out} G$ from automorphisms to outer automorphisms is split. Assume that either G is perfect, or that every proper normal subgroup of G is perfect. Then G is R_d for all d.

Examples include the symmetric and alternating groups S_n and A_n for $n = 5$ or $n \geq 7$. Simple groups with this property are fairly common; they have been classified by A. Lucchini, F. Menegazzo and M. Morigi, and include all sporadic groups.

Another result that we can prove is the following. A group of order 2 is not R_d for any $d \geq 2$; however, the following holds.

Theorem. Let (X, p) be an irreducible d-dimensional pointed variety over $\overline{k}$. If $\mathrm{Aut}(X, p)$ has order 2, d is odd, and p is an isolated fixed point, then (X, p) is defined over its field of moduli.

This is a vast generalization of Shimura's result that a generic principally polarized abelian variety of odd dimension is defined over its field of moduli.

A sketch of proof of our main theorem.

Theorem. Let (X, ξ) be an object over $\overline{k}$. Assume the following conditions.

(1) $\mathrm{Aut}(X, \xi)$ is finite, and $\mathrm{Aut}(X, \xi) \subseteq \mathrm{Aut}(X)$.

(2) X is irreducible.

(3) There exists a dominant rational map $Z \dashrightarrow \mathrm{Q}(X, \xi)$, where Z is an integral scheme of finite type over k, and a rational smooth point $p \in Z(k)$.

Then (X, ξ) is defined over $\mathrm{M}(X, \xi)$.

To prove this, we can base change from k to $\mathrm{M}(X, \xi)$, and assume $k = \mathrm{M}(X, \xi)$. We need consider the moduli problem of twisted forms of (X, ξ): these are families $(Y \to S, \eta)$ such that, when pulled back to $\overline{k}$, are étale locally products with (X, ξ).

These families over S correspond to morphisms $S \to \mathscr{G}$, where $\mathscr{G}$ is a Deligne–Mumford stack $\mathscr{G} \to \operatorname{Spec} k$. It is a *gerbe*; in other words, any two objects are étale-locally isomorphic. It is usually called the *residual gerbe* of (X, ξ). In particular, there is only one object in $\mathscr{G}(\overline{k})$, which is (X, ξ) itself. So, (X, ξ) is defined over $\mathrm{M}(X, \xi)$ if and only if $\mathscr{G}(k) \neq \emptyset$. Let us show that there exists a non-empty open subscheme $V \subseteq \mathrm{Q}(X, \xi)$ and a morphism $V \to \mathscr{G}$.

Set $Q = \mathrm{Q}(X, \xi)$, and ignore ξ, for simplicity of notation. Because X is irreducible, there exists a non-empty open subscheme $U \subseteq X$ on which $\mathrm{Aut}(X)$ acts freely. The largest such U is invariant under both $\mathrm{Gal}(\overline{k}/k)$-invariant and $\mathrm{Aut}(X)$, so $U/\mathrm{Aut}(X) \subseteq X/\mathrm{Aut}(X)$ descents to an open subscheme $V \subseteq Q$.

There is a family $U \times X \to U$ in $\mathscr{G}$ over U; since $\mathrm{Aut}(X)$ acts freely over U this descends to a family $(U \times X)/\mathrm{Aut}(X)$ over $U/\mathrm{Aut}(X)$. The action of $\mathrm{Gal}(\overline{k}/k)$ on this family is well defined, so it descends to a family over V. This gives a morphism $V \to \mathscr{G}$.

So we have a non-empty open subscheme $V \subseteq \mathrm{Q}(X, \xi)$, and a morphism $V \to \mathscr{G}$, so a rational function $Z \dashrightarrow \mathscr{G}$. The hypothesis is that Z has a rational smooth point, but the rational map might not be defined at this point.

This screams for a version of the Lang–Nishimura theorem!

Let us recall the statement of the Lang–Nishimura theorem.

Theorem. Let X and Y integral schemes of finite type over a field k, $f: X \dashrightarrow Y$ a rational map. Assume that

(1) Y is proper, and

(2) X has a k-rational smooth point.

Then $Y(k) \neq \emptyset$.

Theorem [Bresciani – V.]. The same statement holds when X and Y are Deligne–Mumford stacks (in characteristic 0).

The proof depends on a new version of the valuative criterion of properness for algebraic stacks, which in characteristic 0 is due to Giulio alone. This completes the proof of the main theorem.

We use our version of the Lang–Nishimura theorem and stack theoretic techniques even in the proof of our results on R_d-groups.

In positive characteristic the definition of field of moduli must be modified to keep inseparable extensions into account.

A version of our main theorem still holds. The automorphism group scheme $\underline{\mathrm{Aut}}_{\overline{k}}(X, \xi)$ is not required to be reduced, but it must be linearly reductive. The Lang–Nishimura theorem does not hold for Deligne–Mumford stacks, but for tame stacks, as defined by D. Abramovich, M. Olsson and myself.

The proof of the valuative criterion in positive characteristic is much harder.