# Infinite root stacks of logarithmic schemes

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Let $X$ be a smooth projective connected curve over $\mathbb{C}$. Narasimhan and Seshadri have shown that the vector bundles arising from unitary finite-dimensional representations of $\pi_1(X)$ are exactly the polystable bundles of degree 0. Let $p_1$, ..., $p_d$ be distinct points on $X$, and set $D \stackrel{\text{def}}{=} p_1 + \cdots + p_d$ ($D$ plays the role of the boundary of the open Riemann surface $X \smallsetminus \{p_1, \ldots, p_d\}$). How about unitary representations of $\pi_1(X \smallsetminus D)$? Mehta and Seshadri discovered that they give rise to polystable *parabolic bundles* on $X$.

Let $\mathbf{w} = (w_1, \ldots, w_r)$ be a sequence of real numbers with $-1 < w_1 < \cdots < w_r < 0$ (the *weights*). A *parabolic bundle* on $(X, D)$ with weights $\mathbf{w}$ consists of a sequence of vector bundles with inclusions

$$E(-D) \subsetneqq E_{w_1} \subsetneqq \cdots \subsetneqq E_{w_r} \subsetneqq E.$$

Mehta and Seshadri define degrees of parabolic bundles and develop a theory of stability that parallels the classical theory for vector bundles, showing that the parabolic bundles arising from unitary representations of $\pi_1(X \smallsetminus D)$ are exactly the polystable parabolic bundles.

Is there some compact object $\mathscr{X}$ associated with $(X, D)$, such that vector bundles on $\mathscr{X}$ correspond to parabolic bundles? More generally, is there such an $\mathscr{X}$ that describes the geometry of $X \smallsetminus D$? For example, one could require that the fundamental group of $\mathscr{X}$ be $\pi_1(X \smallsetminus D)$. I know three possible approaches to the construction of such an $\mathscr{X}$, each of which is the prototype of a construction in logarithmic geometry.

The first approach is to take a real oriented blowup of $X$ at $D$. In other words, we replace each $p_i$ with a copy of $\mathbb{S}^1$. I will not discuss this, except to say that it is the model for the Kato–Nakayama construction in logarithmic geometry.

With the next two approaches we do not account for all parabolic bundles, but only those with rational weights. One could argue that parabolic bundles are intrinsically non-algebraic objects. For example, bundles arising from representations of the algebraic fundamental group $\widehat{\pi}_1(X \smallsetminus D)$ always have rational weights.

From now on we will only consider parabolic bundles with rational weights.

The second approach is to define the small étale site $(X, D)_{\text{ét}}$, in which the objects are maps $f\colon Y \to X$, where $Y$ is a smooth curve, and $f$ is étale on $Y \smallsetminus f^{-1}(D)$. This is the model for Kato's Kummer étale site.

The third is to consider the orbifold $\sqrt[n]{(X, D)}$, with a chart given in local coordinates by $z \mapsto z^n$ around each point $p_i$. In other words, we are replacing each $p_i$ with a copy of the classifying stack $\mathscr{B}\boldsymbol{\mu}_n$.

**Folklore theorem.** There is an equivalence of categories between vector bundles on $\sqrt[n]{(X, D)}$ and parabolic bundles with weights in $\frac{1}{n}\mathbb{Z}$.

Is there an object of this nature that accounts for all parabolic bundles on $(X, D)$? Yes.

If $m \mid n$ there is a map $\sqrt[n]{(X, D)} \to \sqrt[m]{(X, D)}$. Define the *infinite root stack*

$$\sqrt[\infty]{(X, D)} \stackrel{\text{def}}{=} \varprojlim_{n} \sqrt[n]{(X, D)}\,.$$

It is a proalgebraic stack with a map $\sqrt[\infty]{(X, D)} \to X$. It can be considered as an algebraic version of the real oriented blowup. We have replaced each $p_i$ with $\mathscr{B}\boldsymbol{\mu}_\infty$, where $\boldsymbol{\mu}_\infty \stackrel{\text{def}}{=} \varprojlim_n \boldsymbol{\mu}_n \simeq \widehat{\mathbb{Z}}$, and $\mathscr{B}\,\widehat{\mathbb{Z}}$ is an algebraic approximation of $\mathscr{B}\,\mathbb{Z} \simeq \mathbb{S}^1$.

We can also link $\sqrt[\infty]{(X,D)}$ with $(X,D)_{\text{ét}}$. If $Y \to X$ is a map in $(X,D)$, such that $f^{-1}(D) = \{q\}$ and the ramification index of $f$ at $q$ is $n$, then this lifts to an étale map $Y \to \sqrt[n]{(X,D)}$; thus we obtain an étale representable map

$$Y \times_{\sqrt[n]{(X,D)}} \sqrt[\infty]{(X,D)} \longrightarrow \sqrt[\infty]{(X,D)}.$$

Define the small étale site $\sqrt[\infty]{(X,D)}_{\text{ét}}$ as the site whose objects are étale representable maps $\mathscr{A} \to \sqrt[\infty]{(X,D)}$. One can show that in this way one gets an equivalence between $(X,D)_{\text{ét}}$ and $\sqrt[\infty]{(X,D)}_{\text{ét}}$.

With a little work one proves the following.

**Theorem.** There are equivalences between

(a) parabolic bundles on $(X, D)$,

(b) vector bundles on $\sqrt[\infty]{(X, D)}$, and

(c) vector bundles on $(X, D)_{\text{ét}}$.

Talpo and I generalize this result to arbitrary fine saturated logarithmic schemes, and arbitrary quasi-coherent sheaves, building on previous results of Niels Borne and myself.

Let us review the notion of logarithmic structure, in an unorthodox version that is due to Niels Borne and myself.

Recall that a symmetric monoidal category is a category $\mathscr{A}$ with a functor $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$, $(A, B) \mapsto A \otimes B$, which is associative, commutative, and has an identity, in an appropriate sense. The discrete symmetric monoidal categories are precisely the commutative monoids.

With a scheme $X$ we can associate the monoid $\operatorname{Div} X$ of effective Cartier divisors on $X$. It has a major drawback: it is not functorial. To remedy this, we extend it to a symmetric monoidal category $\mathfrak{Div}\, X$, the category of pairs $(L, s)$ where $L$ is a line bundle on $X$ and $s \in L(X)$. The monoidal structure is given by tensor product $(L, s) \otimes (L', s') = (L \otimes L, s \otimes s')$. The arrows are given by isomorphisms of line bundles preserving the sections. The identity in $\mathfrak{Div}\, X$ is $(\mathscr{O}, 1)$, and the only invertible objects are those isomorphic to $(\mathscr{O}, 1)$, that is, those pairs $(L, s)$ in which $s$ never vanishes.

If $A$ is a commutative monoid, we consider symmetric monoidal functors $L\colon A \to \mathfrak{Div}\, X$. This means that for each element $a \in A$ we have an object $L(a)$ of $\mathfrak{Div}\, X$. We are also given an isomorphism of $L(0)$ with $(\mathscr{O}_X, 1)$, and for $a$, $b \in A$ an isomorphism $L(a+b) \simeq L(a) \otimes L(b)$. These are required to satisfy various compatibility conditions.

**Definition.** A logarithmic structure $(A, L)$ on $X$ consist of the following data.

(a) A sheaf of commutative monoids $A$ on $X_{\text{ét}}$.

(b) For each étale map $U \to X$, a symmetric monoidal functor $L_U\colon A(U) \to \mathfrak{Div}\, U$, that is functorial in $U$.

We require that whenever $a \in A(U)$ and $L(a)$ is invertible, then $a = 0$.

Suppose that $P$ is a monoid and $\phi\colon P \to \mathfrak{Div}(X)$ is a symmetric monoidal functor. Then there exists a unique logarithmic structure $(A, L)$ on $X$, together with a homomorphism of monoids $P \to A(X)$, such that the composite $P \to A(X) \xrightarrow{L} \mathfrak{Div}(X)$ is isomorphic to $\phi$, and the image of $P$ in $A(X)$ generates $A$ as a sheaf.

The functor $\phi$ is called a *chart* for $A$. A logarithmic structure is called *fine saturated* if étale locally admits charts $\phi\colon P \to \mathfrak{Div}\, X$ with $P$ fine saturated. A fine saturated monoid is the monoid of integer points in a rational polyhedral cone in $\mathbb{R}^n$. We will only consider fine saturated logarithmic structures.

**Examples.**

(1) If $\Lambda = (L, s)$ is an object of $\mathfrak{Div}(X)$, we have a chart $\mathbb{N} \to \mathfrak{Div}\, X$ sending $n$ into $\Lambda^{\otimes n}$. In this case $A$ is the constant sheaf $\mathbb{N}$ supported on the zero scheme of $s$. This is *the logarithmic structure generated by* $(L, s)$.

(2) Suppose $D$ is a subset of pure codimension 1 of a regular scheme $X$. Define a sheaf $A$ on $X_{\text{ét}}$, whose sections over an étale map $U \to X$ are the effective Cartier divisors supported on the inverse image of $D$; the symmetric monoidal functor $L_U \colon A(U) \to \mathfrak{Div}(U)$ is the tautological functor. If $p$ is a geometric point of $X$, then the stalk $A_p$ is the free commutative monoid $\mathbb{N}^t$ generated by the branches of $D$ through $p$.

If $D$ is a Cartier divisor on a regular scheme $X$, the two logarithmic structures defined above coincide when $D$ is reduced and unibranch (for example, when $D$ is a smooth divisor on a curve).

Every toric scheme carries a canonical logarithmic structure. If $P$ is a fine saturated monoid and $X_P \stackrel{\mathrm{def}}{=} \operatorname{Spec} \mathbb{Z}[P]$, we get a chart $P \to \mathfrak{Div}\, X_P$ by sending $p \in P$ into $(\mathscr{O}, p)$. Every logarithmic structure on $X$ comes, étale locally, from a map from $X$ into a toric scheme.

Let $X$ be a logarithmic scheme and $n$ a positive integer. We will construct an algebraic stack $\sqrt[n]{X} \to X$; this is due to Borne and myself, is inspired by constructions of M. Olsson in many particular cases.

Let us assume for simplicity that the logarithmic structure comes from a chart $L\colon P \to \mathfrak{Div}\, X$. The algebraic stack $\sqrt[n]{X} \to X$ sends each morphism $f\colon T \to X$ into the category formed by pairs $(M, \phi)$, where $M\colon \frac{1}{n}P \to \mathfrak{Div}\, T$ is a symmetric monoidal functor, and $\phi$ is an isomorphism of the restriction of $M$ to $P$ with $f^* \circ L\colon P \to \mathfrak{Div}\, T$. If $p \in P$, then $M(p/n) \in \mathfrak{Div}\, T$ is such that $M(p/n)^{\otimes n} \simeq f^* L(p)$. We can think of $(M, \phi)$ as a refinement of $f^* \circ L$ obtained by adding $n^{\text{th}}$ roots for of all the $f^* L(p)$.

If $(A, L)$ is the logarithmic structure generated by a Cartier divisor $D$ of $X$, then $\sqrt[n]{X}$ is the $n^{\text{th}}$ root stack of $D$, as defined by Abramovich–Graber–V. and Cadman. In particular, if $X$ is a smooth curve with the logarithmic structure defined by a smooth divisor $D$, then $\sqrt[n]{X}$ is precisely the orbifold $\sqrt[n]{(X, D)}$ defined earlier.

We defined the *infinite root stack* $\sqrt[\infty]{X}$ as $\varprojlim \sqrt[n]{X}$. This is a proalgebraic stack. It is functorial: a morphism of logarithmic schemes $f\colon Y \to X$ induces a morphism $\sqrt[\infty]{f}\colon \sqrt[\infty]{Y} \to \sqrt[\infty]{X}$.

We claim that $\sqrt[\infty]{X}$ captures completely the geometry of $X$.

**Theorem 1** (Talpo–V.)**.** If $X$ and $Y$ are logarithmic schemes, an isomorphism of stacks $\sqrt[\infty]{Y} \simeq \sqrt[\infty]{X}$ comes from a unique isomorphism of logarithmic schemes $Y \simeq X$.

The category of logarithmic schemes has two natural topologies, the Kummer-étale topology, mostly used to study $l$-adic phenomena, and the Kummer-flat topology. Both were defined by Kato. A homomorphism of commutative monoids $P \to Q$ is *Kummer* if it is injective, and for each $q \in Q$ there exists $n > 0$ such that $nq \in P$. A morphism of logarithmic schemes is Kummer-flat if fppf-locally looks like a morphism of toric schemes $X_Q \to X_P$ induced by a Kummer homomorphism $P \to Q$.

**Theorem 2** (Talpo–V.)**.** A morphism of logarithmic schemes $Y \to X$ is Kummer-flat if and only if the induced morphism $\sqrt[\infty]{Y} \to \sqrt[\infty]{X}$ is representable, flat and locally finitely presented.

Kato defined the Kummer-flat $X_{\mathrm{Kfl}}$ site of a logarithmic scheme $X$, whose objects are Kummer-flat maps $Y \to X$; coverings are surjective Kummer-flat maps.

We define the fppf site $\sqrt[\infty]{X}_{\mathrm{fppf}}$, in which objects are representable, flat and locally finitely presented maps $\mathscr{A} \to \sqrt[\infty]{X}$.

**Theorem 3** (Talpo–V.)**.** The functor sending a Kummer-flat morphism $Y \to X$ into $\sqrt[\infty]{Y} \to \sqrt[\infty]{X}$ induces an equivalence of topoi $\mathsf{Sh}(X_{\mathrm{Kfl}}) \simeq \mathsf{Sh}(\sqrt[\infty]{X}_{\mathrm{fppf}})$.

We wish to argue that quasi-coherent sheaves on a logarithmic scheme $X$ should be defined as quasi-coherent sheaves on $\sqrt[\infty]{X}$. K. Hagihara and W. Nizioł have studied the K-theory of coherent and locally free sheaves on $X_{\mathrm{Kfl}}$.

**Corollary** (Talpo–V.)**.** There is an equivalence between finitely presented sheaves of $\mathscr{O}$-modules on $\sqrt[\infty]{X}$ and on $X_{\mathrm{Kfl}}$.

This allows to approach the study of the K-theory of $X_{\mathrm{Kfl}}$ using $\sqrt[\infty]{X}$, with tools such as the Toën–Riemann–Roch theorem.

Finally, quasi-coherent sheaves on $\sqrt[\infty]{X}$ have a parabolic interpretation.

Suppose for simplicity that $X$ has a global chart $L\colon P \to \mathfrak{Div}\, X$. For each $p \in P$ we set $L(p) = (L_p, s_p)$.

Denote by $P^{\mathrm{gr}}$ the group generated by $P$. Set $P^{\mathrm{gr}}_{\mathbb{Q}} \overset{\mathrm{def}}{=} P^{\mathrm{gr}} \otimes \mathbb{Q}$, and denote by $P_{\mathbb{Q}}$ the rational polyhedral cone in $P^{\mathrm{gr}}_{\mathbb{Q}}$ generated by $P$.

The *weight space* $P^{\mathrm{wt}}_{\mathbb{Q}}$ is $P^{\mathrm{gr}}_{\mathbb{Q}}$, with the partial ordering defined by $x \leq\ y$ if $y - x \in P_{\mathbb{Q}}$. We think of $P^{\mathrm{wt}}_{\mathbb{Q}}$ as a category in the usual way.

Finally, $\mathfrak{QCoh}\, X$ is the category of quasi-coherent sheaves on $X$.

The following is an immediate extension of the definition of parabolic sheaf due to Borne and myself.

**Definition.** A *parabolic sheaf* $E$ on $X$ consists of the following data.

(a) A functor $E\colon P_{\mathbb{Q}}^{\mathrm{wt}} \to \mathfrak{QCoh}\, X$, denoted by $a \mapsto E_a$.

(b) For each $a \in P_{\mathbb{Q}}^{\mathrm{wt}}$ and $p \in P$, an isomorphism $E_{a+p} \simeq E_a \otimes L_p$.

We require that the composite $E_a \to E_{a+p} \simeq E_a \otimes L_p$ of the arrow $E_a \to E_{a+p}$ coming from the inequality $a \leq a + p$ with the given isomorphism $E_{a+p} \simeq E_a \otimes L_p$ be given by $e \mapsto e \otimes s_p$, and other compatibility conditions.

The essential difference between this and the classical definition of parabolic bundle is that we don't require the arrows $E_a \to E_b$ defined when $a \leq b$ to be injective.

The following is a simple extension of a result of Borne and myself.

**Theorem.** We have an equivalence between the category of parabolic sheaves on $X$ and the category of quasi-coherent sheaves on $\sqrt[\infty]{X}$.

Talpo has also studied the moduli theory of parabolic bundles, when $X$ is a polarized variety over a field. Suppose for simplicity that $X$ is integral, the logarithmic structure is simplicial and generically trivial. Then Talpo gives notions of stability and semistability for torsion-free finitely presented parabolic bundles, and show that there a moduli space for them, which is a disjoint union of projective varieties.