The Nori Fundamental Group Scheme

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Grothendieck's theory of the fundamental group

Let X be a connected scheme. Recall that a *geometric point* of X is a morphism $x_0\colon \operatorname{Spec}\Omega \to X$, where Ω is a separably closed field. If $Y \to X$ is a morphism, the *geometric fiber* Y_{x_0} of Y over x_0 is the fiber product $\operatorname{Spec}\Omega \times_X Y$. If $Y \to X$ is étale, the fiber Y_{x_0} is a disjoint union of copies of $\operatorname{Spec}\Omega$; we think of it as a discrete set.

If we denote by (FÈt/X) the category of finite étale covers of X, sending $Y \to X$ to Y_{x_0} defines a *fiber functor* from (FÈt/X) $\to$ (FSet) to the category of finite sets.

The following very simple result is the basic one in the whole theory.

Fundamental Lemma. Let $Y \to X$ and $Y' \to X$ be finite étale covers with Y connected, and let $y_0 \colon \operatorname{Spec}\Omega \to Y$ be a geometric point of Y. Let f and g be morphisms of X-schemes $Y \to Y'$ such that $f(y_0) = g(y_0)$. Then $f = g$.

If G a finite group, G-cover of X consists of a finite étale map $\pi \colon Y \to X$, with an action of G on Y making π invariant, such that the induced action of G on a geometric fiber of $Y \to X$ is simply transitive. This condition does not depend on the geometric fiber.

If $Y \to X$ is a G-cover and Y is connected, then it follows from the fundamental Lemma that the natural group homomorphism $G \to \operatorname{Aut}_X Y$ is an isomorphism. Thus when Y is connected, the G-covering $Y \to X$ determines G.

Fix a geometric point $x_0\colon \operatorname{Spec}\Omega \to X$. Let $\{Y_i \to X\}_{i\in I}$ be a set of representatives for the isomorphism classes of connected Galois covers of X. Set $G_i = \operatorname{Aut}_X Y_i$, and for each i choose a geometric point $y_i\colon \operatorname{Spec}\Omega \to Y_i$ over x_0. There is a partial order on I: we define $i \leq\ j$ if there exists a (necessarily unique) morphism of X-schemes $f_{ij}\colon Y_j \to Y_i$ with $f_{ij}(y_j) = y_i$.

If $i \leq\ j$ and $g \in G_j$, there is a unique $h \in G_i$ such that $f_{ij}(gy_j) = hy_i$. This defines a group homomorphism $G_j \to G_i$. One shows that the partially ordered set I is directed.

Definition. The Grothendieck fundamental group $\pi_1^{\mathrm{alg}}(X, x_0)$ is the limit $\varprojlim_i G_i$, with its profinite topology.

Here are some important properties of $\pi_1^{\mathrm{alg}}(X, x_0)$.

(1) As an abstract group, $\pi_1^{\mathrm{alg}}(X, x_0)$ is the automorphism group of the fiber functor (FÈt/X) $\to$ (FSet).

(2) If G is a finite group, there is a natural correspondence between isomorphism classes of Galois G-covers $Y \to X$ with a fixed geometric point $y_0 \colon \operatorname{Spec}\Omega \to Y$ over x_0 and continuous homomorphisms $\pi_1^{\mathrm{alg}}(X, x_0) \to G$.

(3) There is a natural equivalence of categories between finite sets with a continuous action of $\pi_1^{\mathrm{alg}}(X, x_0)$ and finite étale covers of X.

An important generalization of Galois G-covers is given by *torsors* under a finite group scheme. It is natural question whether there is a theory similar to Grothendieck's, in which all torsors under finite group schemes appear, rather than only Galois G-covers. This is provided by Nori's theory.

Affine group schemes

From now on we will fix a base field k, over which all schemes will be defined.

Consider the scheme $\mathbb{G}_{\mathrm{m}} = \mathbb{A}^1 \smallsetminus \{0\} = \operatorname{Spec} k[x^{\pm 1}]$ over k. You want to think of $\mathbb{G}_{\mathrm{m}}$ as a group; however, as a set $\mathbb{G}_{\mathrm{m}}$ does not have a group structure. The scheme $\mathbb{G}_{\mathrm{m}}$ is an *affine group scheme*.

Denote by (Alg/k) the category of k-algebras, by (Aff/k) its dual, the category of affine schemes. Recall Grothendieck's functorial point of view: an affine scheme G can be identified with the functor $(\mathsf{Alg}/k) \to (\mathsf{Set})$ sending A to the set of homomorphisms of k-algebras $k[G] \to A$, or, dually, with the functor $\mathsf{h}_G \colon (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Set})$ sending an affine k-scheme T to the set of morphisms of k-schemes $T \to G$.

The first definition of a group scheme structure on G is a group structure on each $G(A)$, functorial in A.

For the second, a group scheme structure is given by a multiplication morphism $m\colon G \times G \to G$, an identity morphism $\operatorname{Spec} k \to G$, and and an inverse $G \to G$, which satisfies diagrammatic identities corresponding to the usual group axioms. For example, associativity can be expressed as the commutativity of the diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{id}_G} & G \times G \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times m} & & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G\,. \end{array}$$

The equivalence between these two point of view is proved with Yoneda's Lemma.

Dually (and this is the third definition), it is given by a structure of commutative Hopf algebra on $k[G]$.

A homomorphism of k-algebras $k[t^{\pm 1}] \to A$ corresponds to a unit $a \in A^*$; hence $\mathbb{G}_{\mathrm{m}}(A) = A^*$ has a natural group structure. The comultiplication $k[t^{\pm 1}] \to k[t^{\pm 1}] \otimes k[t^{\pm 1}]$ in the Hopf algebra structure is defined by $t \mapsto t \otimes t$.

If $H \subseteq G$ is a closed subscheme of an affine group scheme G, we say that H is a *subgroup scheme* if $H(A) \subseteq G(A)$ is a subgroup for any k-algebra A.

For example, if n is a positive integer, consider the subscheme $\boldsymbol{\mu}_n = \operatorname{Spec} k[t]/(t^n - 1) \subseteq \mathbb{G}_{\mathrm{m}}$; then $\boldsymbol{\mu}_n(A) = \{a \in A \mid a^n = 1\}$ is a subgroup of A^*, so $\boldsymbol{\mu}_n$ is a subgroup scheme of $\mathbb{G}_{\mathrm{m}}$.

As another example, GL_n is an open affine subscheme of the affine space $\mathrm{M}_n = \operatorname{Spec} k[x_{ij}]$ of $n \times n$ matrices; $\mathrm{GL}_n = \operatorname{Spec} k[x_{ij}]_{\det}$. If A is a k-algebra, $\mathrm{GL}_n(A)$ is the set of invertible $n \times n$ matrices with entries in A, which has a natural group structure. Clearly $\mathbb{G}_{\mathrm{m}} = \mathrm{GL}_1$.

More generally, if V is an n-dimensional vector space on k, we have the affine group scheme $\mathrm{GL}(V)$ with $\mathrm{GL}(V)(A) = \operatorname{Aut}_A(V \otimes_k A)$; this is isomorphic to GL_n.

Classical groups are defined by polynomial equations, so they have group scheme versions.

Assume that $\sigma\colon (k^n)^{\otimes r} \to (k^n)^{\otimes s}$ is a tensor on an n-dimensional k-vector space k^n; then we can consider the subgroup scheme $G \subseteq \mathrm{GL}_n$ of $n \times n$ invertible matrices preserving σ; this is defined by the system of polynomial equations $\sigma \circ X^{\otimes r} = X^{\otimes s} \circ \sigma$.

For example, SL_n is defined as the subgroup of GL_n preserving the determinant $\det\colon \bigwedge^n k^n \to k$; the set $\mathrm{SL}_n(A)$ consists of $n \times n$ matrices with entries in A and determinant equal to 1.

The orthogonal group O_n is defined as the subgroup scheme of matrices preserving the standard symmetric bilinear form $k^n \otimes k^n \to k$, $x \otimes y \mapsto \sum_i x_i y_i$. The corresponding system of equations is $X \cdot X^{\mathrm{t}} = \mathrm{I}_n$. The set $\mathrm{O}_n(A)$ consists of orthogonal $n \times n$ matrices with entries in A.

PGL_n is usually defined as a quotient $\mathrm{GL}_n/\mathbb{G}_{\mathrm{m}}$; but it can also be defined as the group scheme of automorphisms of the matrix algebra $\mathrm{M}_n \simeq k^{n^2}$, that is, as the group scheme of invertible $n^2 \times n^2$ matrices preserving the matrix multiplication $\mathrm{M}_n \otimes \mathrm{M}_n \to \mathrm{M}_n$.

Group schemes form a category, in which the arrows are morphisms of k-schemes that preserve the product, in the obvious sense.

If G is a finite group, then we can associate with it a group scheme $\coprod_{g \in G} \operatorname{Spec} k$, which we still denote by G. Its algebra $k[G]$ is the algebra of functions $G \to k$, with pointwise product and the comultiplication $k[G] \to k[G] \otimes k[G] = k[G \times G]$ induced by the product $G \times G \to G$. This is dual to the usual Hopf group algebra kG.

This defines a fully faithful embedding of the category of finite groups into the category of finite group schemes.

Notice that when k has characteristic prime to n, and contains all the n^{th} roots of 1 in $\overline{k}$, then $t^n - 1$ splits as a product of distinct linear factors, and $\boldsymbol{\mu}_n$ is the group scheme associated with the finite group $\boldsymbol{\mu}_n(k)$; but if k does not contain all the n^{th} roots of 1 then $\boldsymbol{\mu}_n$ is not a disjoint union of copies of $\operatorname{Spec} k$. When $\operatorname{char} k \mid n$, the polynomial $t^n - 1$ has 0 derivative, and $\boldsymbol{\mu}_n$ is not even smooth over k.

Theorem (Pierre Cartier). If $\operatorname{char} k = 0$, a group scheme of finite type over k is smooth.

If k is algebraically closed of characteristic 0, every finite group scheme over k comes from a finite group. If $\operatorname{char} k = 0$, every finite group scheme is a "twisted form" of a finite group.

If G is an affine group scheme, a *subgroup scheme* of G is a closed subscheme $H \subseteq G$ such that for any k-algebra A, the subset $H(A) \subseteq G(A)$ is a subgroup of $G(A)$. A subgroup scheme $H \subseteq G$ is *normal* if $H(A)$ is normal in $G(A)$ for all A. Equivalently, we can define a subgroup scheme H of G as a homomorphism of affine group schemes $H \to G$ such that the induced homomorphism of Hopf algebras $k[G] \to k[H]$ is surjective.

If $\phi\colon G \to H$ is a homomorphism of affine group schemes, the *kernel* $\ker \phi \subseteq G$ is the scheme-theoretic inverse image of the identity $\operatorname{Spec} k \subseteq H$. We have $(\ker \phi)(A) = \ker\big(G(A) \xrightarrow{\phi(A)} H(A)\big)$ for any k-algebra A; hence $\ker \phi$ is a normal subgroup scheme of G.

A homomorphism of affine group schemes $G \to H$ is a *quotient* if the induced homomorphism of Hopf algebra $k[H] \to k[G]$ is injective; this is equivalent to saying that it flat and surjective.

If $H \subseteq G$ is a normal subgroup, there exists a quotient $\pi \colon G \to G/H$ with $\ker \pi = H$; this gives an equivalence between quotients of G and normal subgroups of G.

If $\phi \colon G \to H$ is a homomorphism of affine group schemes, this factors uniquely as $G \to G/\ker\phi \to H$, where $G/\ker\phi \to H$ is a closed embedding.

The category of affine group schemes on k is closed under projective limits. These correspond to inductive limits of commutative Hopf algebras.

Affine group schemes have a fundamental finiteness property. If $G = \varprojlim_i G_i$ is a projective limit in the category of affine group schemes and H is an affine group scheme of finite type, then every homomorphism $G \to H$ factors through some G_i. More precisely, the induced function $\varinjlim_i \mathrm{Hom}(G_i, H) \to \mathrm{Hom}(G, H)$ is a bijection.

Furthermore, every affine group scheme is a projective limit of its quotient of finite type. This implies that the category of affine group schemes is equivalent to the pro-category of affine group schemes of finite type. Therefore, the embedding of the category of finite groups in the category of affine group schemes over k extends to a fully faithful embedding of the category of profinite groups into that of affine group schemes over k.

An affine group scheme is *profinite* if it is a projective limit of finite group schemes, or, equivalently, if everyone of its quotients of finite type is finite.

Over an algebraically closed field of characteristic 0, every profinite group scheme comes from a profinite group. If k has characteristic 0, profinite group are twisted forms of profinite groups, and correspond to profinite groups with a continuous action of the Galois group of k. This is completely false in positive characteristic, because of the existence of non-smooth finite group schemes.

For example, consider the projective limit $\boldsymbol{\mu}_\infty = \varprojlim \boldsymbol{\mu}_n$. The projective system is taken along the directed system of positive integers ordered by divisibility; if $m \mid n$, the homomorphism $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_m$ is defined by $x \mapsto x^{n/m}$. In characteristic 0, this corresponds to the profinite group $\prod_\ell \mathbb{Z}_\ell(1)$.

Torsors

Torsors are algebraic-geometric versions of principal fiber bundles.

Let G be an affine group scheme on k and $P \to \operatorname{Spec} k$ a k-scheme. A *right action* of G on P consists of a morphism $P \times G \to P$ that satisfies the diagrammatic identities for a right action. Equivalently, it consists of a right action of $G(A)$ on $P(A)$, which is functorial in A. A morphism $P \to S$ is *G-invariant* when $P(A) \to S(A)$ is $G(A)$-invariant for all A.

For example, suppose that R is a k-algebra, and $r \in R$. Set $P \stackrel{\text{def}}{=} \operatorname{Spec} R[x]/(x^n - r)$. Then an element (ϕ, a) of $P(A)$ consists of a morphisms of k-algebras $\phi\colon R \to A$, together with an element $a \in A$ (the image of x) such that $a^n = \phi(r)$. Then $\boldsymbol{\mu}_n(A)$ acts on $P(A)$ by the rule $(\phi, a)u = (\phi, au)$. The morphism $P \to \operatorname{Spec} R$ is $\boldsymbol{\mu}_n$-invariant.

Let $P \to S$ be a morphism, together with an action of G on P leaving $P \to S$ invariant.

We say that $P \to S$ is a *trivial G-torsor* if there exists a G-equivariant isomorphism of S-schemes $P \simeq S \times G$, where G acts on $S \times G$ by right multiplication.

We say that $P \to S$ is a *G-torsor* if there exists an affine flat surjective morphism $S' \to S$, such that the pullback $S' \times_S P \to S'$ with the induced action of G is a trivial G-torsor.

Torsors over S form a category, the arrows being G-equivariant morphisms of S-schemes. This category is a groupoid, that is, all arrows are invertible. A torsor $P \to S$ is trivial if and only if it has a section $S \to P$.

If G is (the group scheme associated with) a finite group, then G-torsors are (not necessarily connected) étale Galois G-covers.

Assume that R is a k-algebra and $r \in R^*$, and set $S \stackrel{\text{def}}{=} \operatorname{Spec} R$. Consider the action of $\boldsymbol{\mu}_n$ on $\operatorname{Spec} R[x]/(x^n - r)$ defined above.

If $r = s^n$, then $P \to \operatorname{Spec} R$ is a trivial torsor: the isomorphism $P \simeq S \times \boldsymbol{\mu}_n = \operatorname{Spec} R[t]/(t^n - 1)$ sends x into st.

In general $P \to S$ is a $\boldsymbol{\mu}_n$-torsor: we can take $S' = P$. This torsor $P \to S$ is trivial if and only if it has a section $S \to P$, that is, if there is a homomorphism of R-algebras $R[x]/(x^n - r) \to R$, or, finally, if and only if r is a n^{th} power in R.

Every $\boldsymbol{\mu}_n$-torsor is Zariski-locally of this form. This is a version of Kummer theory.

Let $E \to X$ be a vector bundle on a scheme S of rank n. The *bundle of frames* $\operatorname{Fr} E$ is the open subscheme of the fibered product $E_X^n \stackrel{\mathrm{def}}{=} \underbrace{E \times_X E \times_X \cdots \times_X E}_{n \text{ factors}}$, consisting of n-tuples of vector that are linearly independent. The scheme $\operatorname{Fr} E$ represents the functor of isomorphisms $\mathscr{O}^n \simeq E$, where $\mathscr{O}^n$ is the trivial vector bundle or rank n. The natural right action of GL_n on $\operatorname{Fr} E$ makes $\operatorname{Fr} E$ into a GL_n-torsor.

By descent theory, this construction gives an equivalence between the groupoid of vector bundles of rank n on S and that of GL_n-torsors.

If $G \subseteq \mathrm{GL}_n$ is the subgroup scheme of matrices preserving a tensor $\sigma\colon (k^n)^{\otimes r} \to (k^n)^{\otimes s}$, then G-torsors over a k-scheme S correspond to *twisted forms* of σ, that is, rank n vector bundles $E \to S$ with a tensor $\tau\colon E^{\otimes r} \to E^{\otimes s}$ that becomes isomorphic to $(\mathscr{O}_{S'}^n, \sigma)$ after a faithfully flat affine base change $S' \to S$.

If (E, τ) is a vector bundle of rank n with a tensor τ as above, the corresponding G-torsor is the subscheme of $\operatorname{Fr} E$ consisting of bases in which σ is exactly equal to τ.

Taking $\sigma = \det\colon \bigwedge k^n \to k$, we see that SL_n-torsors correspond to vector bundles $E \to S$ with an isomorphism $\omega\colon \bigwedge^n E \simeq \mathscr{O}_S$. The torsor is the subscheme of $\operatorname{Fr} E$ defined by the equation $\omega(v_1 \wedge \cdots \wedge v_n) = 1$.

If σ is the standard quadratic form, we see that O_n torsors correspond to vector bundles with non-degenerate quadratic form $E \otimes E \to E$. The torsor is the subscheme of orthonormal frames in E.

We can think of $\boldsymbol{\mu}_n$ as the subgroup of $\mathbb{G}_{\mathrm{m}}$ consisting of invertible 1×1 matrices preserving the tensor $\sigma_n \colon k^{\otimes n} \to k^{\otimes 0} = k$ that sends $x_1 \otimes \cdots \otimes x_n$ into $x_1 \dots x_n$. As a consequence, $\boldsymbol{\mu}_n$-torsors on S correspond to line bundles L on S with an isomorphism $L^{\otimes n} \simeq \mathscr{O}_S$. If L is a line bundle, then $\operatorname{Fr} L$ is the complement of the 0-section in L; an isomorphism $L^{\otimes n} \simeq \mathscr{O}$ corresponds to a nowhere vanishing section s of $L^{\otimes n}$. Hence the corresponding $\boldsymbol{\mu}_n$-torsor is the subscheme of non-zero sections x of L with $x^{\otimes n} = s$.

When $L = \mathscr{O}$ and $X = \operatorname{Spec} R$, then $L^{\otimes n} = \mathscr{O}$, and $s \in R^*$. In this case the torsor is exactly $\operatorname{Spec} R[x]/(x^n - s)$. Since every invertible sheaf is locally trivial, every $\boldsymbol{\mu}_n$-torsor is of this form.

This proves a generalized version of Kummer's Theorem: every $\boldsymbol{\mu}_n$-torsor is Zariski-locally of the form $\operatorname{Spec} R[x]/(x^n - r)$ for some $r \in R$.

If $\phi\colon G \to H$ is a homomorphism of affine group schemes and P is G-torsor on a scheme X, there exists an H-torsor $P \times^G H$, with a ϕ-equivariant morphism of X-schemes $P \to P \times^G H$. This is unique, up to a unique isomorphism; it can be constructed as usual as the quotient $(P \times H)/G$, where the action of G is defined by $(p, h) \cdot g \stackrel{\text{def}}{=} \bigl(pg, \phi(g)^{-1}h\bigr)$. The action of H is defined by right multiplication on H.

If $\phi\colon G \to H$ is a closed embedding and $Q \to X$ is an H-torsor, a *reduction of structure group* of Q to G consists of a subscheme $P \subseteq Q$ that is G-invariant and a G-torsor on X. Equivalently, it consists of a G-torsor $P \to X$ and an isomorphism of H-torsors $P \times^G H \simeq Q$. A reduction of structure group $P \subseteq Q$ gives a section $X = P/G \to Q/G$; conversely, gives a section $X \to Q/G$, its inverse image in Q is a reduction of structure group to G.

For example, an O_n-torsor consists of a GL_n torsor with a reduction of structure group to O_n. If E is a vector bundle on X, the quotient $\mathsf{Fr}(E)/\mathrm{O}_n$ is the bundle of non-degenerate quadratic forms on E. This reproves that O_n-torsors correspond to vector bundles with a non-degenerate quadratic form.

As another example, a $\boldsymbol{\mu}_n$-torsor corresponds to a $\mathbb{G}_\mathrm{m}$-torsor with a reduction of structure group to $\boldsymbol{\mu}_n$. If L is a line bundle on X, then $\mathsf{Fr}(L)/\boldsymbol{\mu}_n$ is the $\mathbb{G}_\mathrm{m}$-torsor $\mathsf{Fr}(L^{\otimes n})$; hence $\boldsymbol{\mu}_n$-torsors correspond to line bundle L with isomorphisms $L^{\otimes n} \simeq \mathscr{O}_X$.

Nori's fundamental group scheme

Let X be a connected and geometrically reduced scheme over k with a fixed rational point $x_0 \in X(k)$.

Theorem (Madhav Nori). There exists a profinite group scheme $\pi_1^{\mathrm{N}}(X, x_0)$, called the *fundamental group scheme of* (X, x_0), such that for any finite group scheme G over k there a functorial correspondence between homomorphisms $\pi_1^{\mathrm{N}}(X, x_0) \to G$ and isomorphism classes of G-torsors $P \to X$, with a fixed rational point $p_0 \in P(k)$ over x_0.

So, in contrast with Grothendieck's, Nori's Galois theory is relative to the base field. If $X = \operatorname{Spec} k$, then $\pi_1^{\mathrm{N}}(X, x_0)$ is trivial, Grothendieck's fundamental group is the Galois group of k.

The proof of the theorem is not particularly hard; $\pi_1^{\mathrm{N}}(X, x_0)$ is constructed as an explicit projective limit.

If k is algebraically closed of characteristic 0, the group $\pi_1^{\mathrm{N}}(X, x_0)$ is the Grothendieck fundamental group $\pi_1^{\mathrm{alg}}(X, x_0)$.

In general, in characteristic 0 the group scheme $\pi_1^{\mathrm{N}}(X, x_0)$ can be reconstructed as follows.

Theorem (Nori). Let k' be an algebraic separable extension of k; set $X' = \operatorname{Spec} k' \times_{\operatorname{Spec} k} X$, and call x_0' the k'-rational point of X' corresponding to x_0. Then

$$\pi_1^{\mathrm{N}}(X', x_0') = \operatorname{Spec} k' \times_{\operatorname{Spec} k} \pi_1^{\mathrm{N}}(X, x_0)\,.$$

Nori conjectured this to be true for arbitrary algebraic extension, but this turned out to be false; the first counterexample was found by V. B. Mehta and S. Subramanian.

Assume that k has characteristic 0, call $\overline{k}$ the algebraic closure of k, and $\mathscr{G}$ the Galois group of $\overline{k}/k$. Set $\overline{X} = \operatorname{Spec}\overline{k} \times_{\operatorname{Spec} k} X$, and call $\overline{x}_0\colon \operatorname{Spec}\overline{k} \to X$ the point of X corresponding to x_0. By Nori's theorem above,

$$\operatorname{Spec}\overline{k} \times_{\operatorname{Spec} k} \pi_1^{\mathrm{N}}(X, x_0) = \pi_1^{\mathrm{N}}(\overline{X}, \overline{x}_0) = \pi_1^{\mathrm{alg}}(\overline{X}, \overline{x}_0)\,;$$

the corresponding action of $\mathscr{G}$ comes from the action of $\mathscr{G}$ on $(\overline{X}, \overline{x}_0)$.

This is false in positive characteristic, even for $k = \overline{k}$, as soon as there are non-smooth finite groups G over k and non-trivial G-torsors on X.

Suppose that X is an abelian variety with origin $x_0 \in X(k)$.

If n is a positive integer, call $X[n]$ the kernel of the morphism $X \xrightarrow{x \mapsto x^n} X$. If $m \mid n$, there is a homomorphism $X[n] \to X[m]$ defined by $x \mapsto x^{n/m}$.

Theorem (Nori). $\pi_1^{\mathrm{N}}(X, x_0) = \varprojlim_n X[n]$.

If the characteristic of k divides n, then $X[n]$ is not smooth; hence if $\operatorname{char} k > 0$, the affine group scheme $\pi_1^{\mathrm{N}}(X, x_0)$ is a profinite group scheme that does not come from a profinite group.

Tannakian categories

In Grothendieck's theory, the main point is the equivalence between finite sets with a continuous action of $\pi_1^{\mathrm{alg}}(X, x_0)$ and the category (FÈt/X) of finite étale covers of X.

There is no satisfactory analogue of this correspondence for $\pi_1^{\mathrm{N}}(X, x_0)$: Marco Antei and Michel Emsalem gave an interesting definition of *essentially finite cover*, but its formal aspects are still far from clear.

Instead, with some properness hypothesis on X there is an extremely important interpretation of the category of representations of $\pi_1^{\mathrm{N}}(X, x_0)$. This is the main point of Nori's theory.

Let us sketch the main results of Tannaka duality, in the algebraic version due to Grothendieck, Saavedra-Rivano and Deligne.

If G is an affine group scheme, a (finite-dimensional) representation ρ of G consists of a finite-dimensional vector space V on k, together with a homomorphism of groups schemes $\rho\colon G \to \mathrm{GL}(V)$. Equivalently, a representation can be described as a comodule over the Hopf algebra $k[G]$.

Given two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\sigma\colon G \to \mathrm{GL}(W)$, A morphism of representation $\phi\colon V \to W$ is a k-linear function, with the property that for any k-algebra A and any $g \in G(A)$, we have

$$\sigma(g) \circ \phi_A = \phi_A \circ \rho(g)\,.$$

where $\phi_A \stackrel{\mathrm{def}}{=} \mathrm{id}_A \otimes \phi\colon A \otimes_k V \to A \otimes_k W$.

Representations of G form, in an obvious way, a k-linear abelian category $\operatorname{Rep} G$.

If $\rho\colon G \to \mathrm{GL}(V)$ and $\sigma\colon G \to \mathrm{GL}(W)$ are representations, we can form the tensor product $\rho \otimes \sigma\colon G \to \mathrm{GL}(V \otimes W)$.

One also defines, in the obvious way, the *dual representation*.

This gives $\operatorname{Rep} G$ the structure of a *neutral tannakian category*.

A *tensor category* $\mathscr{C}$ over k is a k-linear abelian category, with finite dimensional Hom's, together with a symmetric monoidal bilinear functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$, denoted by $(X, W) \mapsto V \otimes W$, called the *tensor product*. This means that there are given isomorphisms $V \otimes W \simeq W \otimes V$ and $(V \otimes W) \otimes Z \simeq V \otimes (W \otimes Z)$; furthermore there is a neutral object $\mathbf{1}$, with isomorphisms $\mathbf{1} \otimes V \simeq V$. These are required to satisfy a number of complicated compatibility conditions.

A tensor category $\mathscr{C}$ is *rigid* when we have fixed a k-linear functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, denoted by $V \mapsto V^{\vee}$, with functorial isomorphisms

$$\mathrm{Hom}(V \otimes W, Z) \simeq \mathrm{Hom}(W, V^{\vee} \otimes Z),$$

and the k-linear maps

$$\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(V', W') \longrightarrow \mathrm{Hom}(V \otimes V', W \otimes W')$$

defined by the tensor product are isomorphisms.

Given a k-algebra A, denote by $\operatorname{Vect} A$ the category of finitely generated projective A-modules, or vector bundles on $\operatorname{Spec} A$. Given a rigid tensor category $\mathscr{C}$, a *fiber functor* $\Phi\colon \mathscr{C} \to \operatorname{Vect} A$ is a k-linear exact functor that preserves tensor products, in the sense that we are given isomorphisms $\Phi(V \otimes W) \simeq \Phi(V) \otimes_A \Phi(W)$, compatible with the various isomorphisms.

Definition. A *tannakian category* $\mathscr{C}$ is a rigid tensor category over k, such that

(a) $\operatorname{Hom}(\mathbf{1}, \mathbf{1}) = k$, and

(b) there exists an extension K of k, and a fiber functor $\mathscr{C} \to \operatorname{Vect} K$.

A *neutral tannakian category* is a pair $(\mathscr{C}, \Phi)$, where $\mathscr{C}$ is a tannakian category, and $\Phi\colon \mathscr{C} \to \operatorname{Vect} k$ is a fiber functor.

The category Rep G is a neutral tannakian category. The monoidal structure is given by tensor product, the functor $V \to V^\vee$ is given by dual, and the neutral element $\mathbf{1}$ is k with the trivial action of G. The fiber functor $\operatorname{Rep} G \to \operatorname{Vect} k$ is the forgetful functor.

Conversely, give a neutral tannakian category $(\mathscr{C}, \Phi)$, we defined a functor $G\colon (\mathrm{Alg}/k) \to (\mathrm{Set})$ sending each k-algebra A into the set of automorphisms (as a tensor functor) of the fiber functor $\Phi_A\colon \mathscr{C} \to \operatorname{Vect} A$ obtained by composing Φ with the obvious functor $\operatorname{Vect} k \to \operatorname{Vect} A$ obtained by tensoring with A. This turns out to be an affine group scheme.

Theorem (Grothendieck, Saavedra-Rivano, Deligne). These constructions give an equivalence between the category of affine group schemes on k and the category of neutral tannakian categories.

Nori's correspondence

When X satisfies some properness hypothesis, the category of representations of $\pi_1^{\mathrm{N}}(X, x_0)$ turns out to have a very interesting interpretation in terms of vector bundles on X. This is the main point of Nori's theory. I am going to present a version of this, due to Niels Borne and myself, which works in greater generality than Nori's original version.

The idea of associating a vector bundle to a representation of the fundamental group goes back at least to Weil. Let $\pi_1^{\mathrm{N}}(X, x_0) \to \mathrm{GL}(V)$ be a representation; this has a factorization $\pi_1^{\mathrm{N}}(X, x_0) \to G \to \mathrm{GL}(V)$, where G is a finite quotient of $\pi_1^{\mathrm{N}}(X, x_0)$. The quotient $\pi_1^{\mathrm{N}}(X, x_0) \to G$ corresponds to a G-torsor $P \to X$; we associate with these data a vector bundle $(P \times V)/G \to P/G = X$.

This yields a k-linear functor

$$\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0) \longrightarrow \operatorname{Vect} X\,,$$

where $\operatorname{Vect} X$ is the category of vector bundles on X. This functor is exact, and preserves tensor products.

In order for this to be well-behaved, we need some properness hypothesis on X.

Definition. A scheme over k is *pseudo-proper* if it is quasi-compact and quasi-separated, and for any two vector bundles E and F on X we have $\dim_k \operatorname{Hom}(E, F) < \infty$.

Obviously any proper scheme is pseudo-proper, but the converse is not true. For example, if $\overline{X}$ is proper and $S \subseteq \overline{X}$ is a closed subset of codimension at least 2, then $\overline{X} \smallsetminus S$ is pseudo-proper.

Let $\mathscr{A} = \operatorname{Vect} X$, where X is a pseudo-proper scheme, or $\mathscr{A} = \operatorname{Rep} G$, where G is an affine group scheme.

By an old result of Atiyah, the Krull–Schmidt theorem holds in $\mathscr{A}$, that is, every object decomposes as a direct sum of indecomposable objects, and this decomposition is unique up to isomorphisms.

In the category $\mathscr{A}$ we have a neutral object $\mathbf{1}$, direct sums and tensor products; hence if $f \in \mathbb{N}[x]$ is a polynomial with natural number coefficients and E is an object of $\mathscr{A}$, we can evaluate f at E, interpreting sums as direct sums and products as tensor products. For example, if $f(x) = 1 + 2x^3$ we have

$$f(E) = \mathbf{1} \oplus E^{\otimes 3} \oplus E^{\otimes 3}$$

Definition. An object E of $\mathscr{A}$ is *finite* if there exist f and g in $\mathbb{N}[x]$ with $f \neq g$ and $f(E) \simeq g(E)$.

The object E is *essentially finite* if it is the kernel of a map between finite objects.

Nori's definition of an essentially finite bundle is different from ours, and only works when X is proper.

An object E is finite if and only if the set of isomorphism classes of the indecomposable components of all the powers $E^{\otimes n}$ is finite.

Hence:

(1) $E \oplus F$ is finite if and only if E and F are both finite.

(2) If E and F are finite, then $E \otimes F$ is finite.

(3) A line bundle, or a representation of degree 1, is finite if and only if it is torsion.

For example, essentially finite bundles on $\mathbb{P}^n$ are trivial. It is enough to prove that finite bundles are trivial; this follows from the structure theorem for vector bundles on $\mathbb{P}^1$, and the fact that a bundle on $\mathbb{P}^n$ that is trivial on each line is in fact trivial.

Lemma. If G is a profinite group scheme, every representation of G is essentially finite. If $\operatorname{char} k = 0$, then every representation of G is finite.

Sketch of proof. Every representation $G \to \mathrm{GL}(V)$ factors through a finite quotient of G; hence we can assume that G is finite. Then we have the regular representation $k[G]$; as in the case of finite groups, one proves that for every representation V of G we have an isomorphism $V \otimes k[G] \simeq k[G]^{\oplus \operatorname{rk} V}$. In particular $k[G]^{\otimes 2} \simeq k[G]^{\oplus \operatorname{rk} k[G]}$, so $k[G]$ is finite. Also, every representation V of G is a subrepresentation of a direct sum $k[G]^{\oplus n}$; by embedding $k[G]^{\oplus n}/V$ into some $k[G]^{\oplus m}$, we write V as the kernel of a homomorphism $k[G]^{\oplus n} \to k[G]^{\oplus m}$.

If $\operatorname{char} k = 0$, one proves the analogue of Maschke's Theorem, that representations of G are sums of irreducible representations. Hence indecomposable representations are irreducible, and since every irreducible representation appears in $k[G]$, there can be only be finitely many isomorphism classes of irreducible representations.

Assume that X is pseudo-proper; denote by $\operatorname{Fin} X$ and $\operatorname{EFin} X$ the subcategories of $\operatorname{Vect} X$ consisting respectively of finite and essentially finite bundles. The functor $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0) \to \operatorname{Vect} X$ is exact and preserves tensor products; since every representation of $\pi_1^{\mathrm{N}}(X, x_0)$ is essentially finite, its essential image is contained in $\operatorname{EFin} X$. If $\operatorname{char} k = 0$, then the essential image is contained in $\operatorname{Fin} X$.

There is also a fiber functor $\operatorname{EFin} X \to \operatorname{Vect} k$, sending a vector bundle E on X to its fiber over x_0.

Theorem (Nori, Borne–V.). Assume that X is pseudo-proper, connected and geometrically reduced. Then the functor $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0) \to \operatorname{EFin} X$ is an equivalence of neutral tannakian categories.

Furthermore, when $\operatorname{char} k = 0$ we have $\operatorname{EFin} X = \operatorname{Fin} X$.

Nori assumes that X is proper, and has a different and more complicated notion of essentially finite bundle.

This is quite remarkable: it says that the category $\operatorname{Vect} X$ determines $\pi_1^{\mathrm{N}}(X, x_0)$ with its monoidal structure and its fiber functor. The category $\operatorname{EFin} X$ is defined purely in terms of $\operatorname{Vect} X$ with tensor product; this is tannakian category, neutralized by the fiber functor $\operatorname{EFin} X \to \operatorname{Vect} k$ given by restriction to x_0. (With his definition, Nori is able to prove this directly, but we can not.) The corresponding affine group scheme is $\pi_1^{\mathrm{N}}(X, x_0)$.

A vector bundle E is in the image of $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0)$ if and only if it has a reduction of structure group to a finite group scheme; when $k = \mathbb{C}$ and X is proper and smooth, this happens if and only if E has a flat holomorphic connection with finite monodromy. So we deduce that a vector bundle on a smooth proper algebraic complex variety has a flat holomorphic connection with finite monodromy if and only if it is finite.

We have seen that every essentially finite bundle on $\mathbb{P}^n_k$ is trivial; this implies that the fiber functor $\operatorname{EFin} X \to \operatorname{Vect} k$ is an equivalences. Hence $\pi_1^{\mathrm{N}}(\mathbb{P}^n_k)$ is trivial. This can be expressed by saying that if $P \to \mathbb{P}^n_k$ is a torsor under a finite group scheme that is trivial on a rational point, it is trivial. I don't know a direct proof of this fact in positive characteristic.

Nori clarified the dependence of $\pi_1^{\mathrm{N}}(X, x_0)$. If $x_1 \in X(k)$ is another rational point, $\pi_1^{\mathrm{N}}(X, x_1)$ is not necessarily isomorphic to $\pi_1^{\mathrm{N}}(X, x_0)$, but it an *inner form* of it. This implies that they become isomorphic after an extension of k; furthermore, if $\pi_1^{\mathrm{N}}(X, x_0)$ is abelian, then they are canonically isomorphic.

This is clarified considerably by developing the theory without base points, as Borne and I do. In this case the fundamental group scheme must be replaced with a *gerbe*.

Affine gerbes

In Grothendieck's functorial point of view, a scheme X over k is identified with the functor $\mathsf{h}_X \stackrel{\text{def}}{=} \mathsf{Hom}(-, X)\colon (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Set})$. We need to extend the formalism to (pseudo)-functors $(\mathsf{Alg}/k) = (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Groupoid})$ to the category of groupoids (categories in which all arrows are isomorphisms).

A key example: if $G \to \operatorname{Spec} k$ is an algebraic group, we have the "classifying stack" $\mathscr{B}_k G\colon (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Groupoid})$, sending each affine k-scheme S into the category $\mathscr{B}_k G(S)$ of G-torsors over S, which is a groupoid.

The 2-categorical version of Yoneda's lemma says that if $\Gamma\colon (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Groupoid})$ is a pseudo-functor and X is an affine k-scheme, natural transformations ("morphisms") $X = \mathsf{h}_X \to \Gamma$ form a category equivalent to $\Gamma(X)$. Thus, for example, morphisms $X \to \mathscr{B}_k G$ correspond to G-torsors over X.

We are interested in *affine gerbes* over k. These are pseudo-functors $\Gamma\colon (\mathsf{Alg}/k) = (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Groupoid})$ such that:

(1) They are stacks in the fpqc topology.

(2) There exists some field extension k'/k such that $\Gamma(k') \neq \emptyset$.

(3) Any two objects are fpqc-locally isomorphic, that is, given two objects ξ and η in $\Gamma(S)$, where S is an affine k-scheme, there exists a faithfully flat morphism $f\colon T \to S$ with T affine, such that $f^*\xi \simeq f^*\eta$.

(4) If k'/k is a field extension and ξ is in $\Gamma(k')$, the functor $\underline{\mathrm{Aut}}_{k'}\,\xi\colon (\mathsf{Aff}/k')^{\mathrm{op}} \to (\mathsf{Grp})$ sending each affine k-scheme $f\colon S \to \operatorname{Spec} k'$ into the automorphism group of $f^*\xi$ is represented by an affine group scheme.

If G is an affine group scheme over k, then $\mathscr{B}_k G$ is an affine gerbe; the trivial torsor $G \to \operatorname{Spec} k$ gives a distinguished element of $\mathscr{B}_k G(k)$, or, more suggestively, a section $\operatorname{Spec} k \to \mathscr{B}_k G$ of the structure morphism $\mathscr{B}_k G \to \operatorname{Spec} k$.

Conversely, let Γ be an affine gerbe, and $\xi \in \Gamma(k)$, or $\xi\colon \operatorname{Spec} k \to \Gamma$. We obtain an affine group scheme $G \overset{\text{def}}{=} \underline{\mathrm{Aut}}_k\, \xi$; descent theory gives an isomorphism $\mathscr{B}_k G \simeq \Gamma$.

So, $\{$affine group schemes$\} = \{$affine gerbes with sections$\}$.

A pairs (Γ, ξ), where Γ is an affine gerbe and $\xi \in \Gamma(k)$, corresponds to the group scheme $\underline{\mathrm{Aut}}_k\, \xi$. Homomorphisms of group schemes correspond to pairs $(\phi, \alpha)\colon (\Gamma, \xi) \to (\Delta, \eta)$, where $\phi\colon \Gamma \to \Delta$ is a functor and $\alpha\colon \eta \simeq \phi(\xi)$ is an isomorphism in $\Delta(k)$.

There are gerbes with $\Gamma(k) = \emptyset$.

Also, different sections $\operatorname{Spec} k \to \Gamma$ can give rise to non-isomorphic groups; equivalently, there may be non-isomorphic affine group scheme G and H with $\mathscr{B}_k G \simeq \mathscr{B}_k H$.

Let us give an example of an affine gerbe Γ with $\Gamma(k) = \emptyset$. Let $X \to \operatorname{Spec} k$ be a Brauer–Severi variety of degree n, that is, a variety over k such that $X_{\overline{k}} \simeq \mathbb{P}^{n-1}_{\overline{k}}$. There are examples of Brauer–Severi varieties which are not projective spaces, for example, smooth conics in $\mathbb{P}^2_k$ without rational points.

If A is k-algebra, we define a groupoid $\Gamma_X(A)$ whose objects are line bundles L on X_A, which have degree 1 on the geometric fibers of the projection $X_A \to \operatorname{Spec} A$; the arrows are given by isomorphisms of line bundles. The groupoid $\Gamma_X(A)$ is functorial in A; one shows that Γ is an affine gerbe. Clearly, $\Gamma_X(k) \neq \emptyset$ if and only if $X \simeq \mathbb{P}^{n-1}_k$.

Here is a very general construction of affine gerbes. Let G be an affine group scheme over k and V a homogeneous space over G. This means that G acts on V, and that if $v_0 \in V(K)$ is a K-rational point over some extension K of k, the induced map $G_K \xrightarrow{g \mapsto v_0 g} V_K$ is flat and surjective. Define a pseudo-functor $\Gamma\colon (\mathrm{Alg}/k) \to (\mathrm{Groupoid})$ sending each k-algebra A to the category whose objects are pairs (P, ϕ), where $P \to \operatorname{Spec} A$ is a G-torsor, and $\phi\colon P \to V$ is a G-equivariant map. The arrows in $\Gamma(A)$ are morphisms of G-torsors, commuting with the map to V. Then Γ is an affine gerbe. (The gerbe Γ is the stack-theoretic quotient $[V/G]$.)

If $V = \operatorname{Spec} k$, then $\Gamma = \mathscr{B}_k G$.

If H is a quotient of G and $V \to \operatorname{Spec} k$ is an H torsor, then $\Gamma(k) \neq \emptyset$ if and only V comes from a G-torsor. In the previous example we have $G = \mathrm{GL}_n$ and $H = \mathrm{PGL}_n$ (there is an equivalence between Brauer–Severi varieties and PGL_n-torsors).

Here is an example of an affine gerbe giving rise to different group schemes. Assume that char $k \neq 2$, fix a positive integer n and let $\mathscr{Q}_n \colon (\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Groupoid})$ be the pseudo-functor such that $\mathscr{Q}_n(S)$ is the groupoid of vector bundles $E \to S$ of rank n with a non-degenerate quadratic form; the arrows are given by isometries. Any non-degenerate quadratic form can be put étale-locally in canonical form; hence $\mathscr{Q}_n$ is a gerbe.

A section $\operatorname{Spec} k \to \mathscr{Q}_n$ corresponds to a pair (V, q), where V is an n-dimensional vector space and q is a non-degenerate quadratic form on V. The group $\underline{\mathrm{Aut}}_k(V, q)$ is the orthogonal group $\mathrm{O}(V, q)$. Hence all gerbes of the form $\mathscr{B}_k\mathrm{O}(V, q)$ are equivalent to $\mathscr{Q}_n$.

Suppose that we have an affine gerbe Γ with two objects ξ, $\eta \in \Gamma(k)$, corresponding to group schemes $G \overset{\text{def}}{=} \underline{\mathrm{Aut}}_k\, \xi$ and $H \overset{\text{def}}{=} \underline{\mathrm{Aut}}_k\, \eta$. We have a functor

$$I \overset{\text{def}}{=} \underline{\mathrm{Isom}}_k(\xi, \eta) \colon (\mathsf{Alg}/k) \longrightarrow (\mathsf{Set})$$

sending each k-algebra A into the set of of isomorphisms of ξ_A and η_A in $\Gamma(A)$. Composition defines a right action of G and a left action of H, which commute. This makes I into a (G, H)-*bitorsor*, that is, a right G-torsor and a left H-torsor, inducing an isomorphism of H with the functor of automorphisms of I as an G-torsor. This is a twisted version of G, obtained by descent from I and the action of G on itself via conjugation. We say that H is an *inner form* of G. So, if G is abelian then $H = G$.

Conversely, a (G, H)-bitorsor induces an equivalence $\mathscr{B}_k G \simeq \mathscr{B}_k H$.

Representations of gerbes

If Γ is a gerbe, a vector bundle on Γ is a natural transformation from Γ to the pseudo-functor $\mathrm{Vect}\colon (\mathrm{Alg}/k) \to (\mathrm{Categories})$ sending a k-algebra A into the category $\mathrm{Vect}\, A$ of projective modules on A. This associates with every $\xi \in \Gamma(A)$ a projective module over A. These form a k-linear *abelian category* $\mathrm{Vect}\,\Gamma$.

If G is an affine group scheme over k, we have a natural functor $\mathrm{Rep}\, G \to \mathrm{Vect}\, \mathscr{B}_k G$; if V is a representation of G and $P \to S$ is a G-torsor, we have that $(V \times P)/G \to S$ is a vector bundle over S. This functor is an equivalence; so, $\mathrm{Rep}\, G$ only depends on $\mathscr{B}_k G$. If Γ is a gerbe, we set $\mathrm{Rep}\,\Gamma \overset{\mathrm{def}}{=} \mathrm{Vect}\,\Gamma$.

If G is an affine group scheme, the fiber functor $\mathrm{Rep}\,\mathscr{B}_k G = \mathrm{Rep}\, G \to \mathrm{Vect}\, k$ is the pullback along the section $\mathrm{Spec}\, k \to \mathscr{B}_k G$ given by the trivial torsor $G \to \mathrm{Spec}\, k$.

If Γ is an affine gerbe, the category $\operatorname{Rep}\Gamma$ is a tannakian category.

Conversely, if $\mathscr{C}$ is a tannakian category we can define an affine gerbe $\Gamma\colon (\mathrm{Alg}/k) \to (\mathrm{Groupoid})$ by sending a k-algebra A into the groupoid of fiber functors $\mathscr{C} \to \operatorname{Vect} A$.

Theorem (Grothendieck, Saavedra Rivano, Deligne). These constructions give an equivalence of the 2-category of affine gerbes over k with the the 2-category of tannakian categories.

Thus, affine gerbe correspond to tannakian categories, and sections of affine gerbes correspond to fiber functors.

The fundamental gerbe

An affine gerbe is (pro-)finite if for some (hence for all) ξ in $\Gamma(k')$, where k' is an extension of k, the group scheme $\underline{\mathrm{Aut}}_{k'}\,\xi$ is (pro-)finite. A profinite gerbe is a projective limit of finite gerbes.

Theorem 1 (Borne–V.). Suppose that X is a geometrically connected and geometrically reduced scheme over k. Then X has a *fundamental gerbe*, that is, a profinite gerbe $\Pi_{X/k}$ with a morphism $X \to \Pi_{X/k}$ such that any morphism $X \to \Gamma$, where Γ is a profinite gerbe, factors uniquely through $\Pi_{X/k}$, up to a unique isomorphism.

Our theorem also applies to some non-reduced schemes, and to much more general fibered categories.

If char $k = 0$, then $\Pi_{X/k}$ is the gerbe associated with Deligne's relative fundamental groupoid.

Our result also applies when $X(k) = \emptyset$. If $x_0 \in X(k)$, then the composite $\operatorname{Spec} k \xrightarrow{x_0} X \to \Pi_{X/k}$ determines a section $\xi_0\colon \operatorname{Spec} k \to \Pi_{X/k}$, hence a profinite group scheme $\underline{\mathrm{Aut}}_k\, \xi_0$. If $P \to X$ is a G-torsor, this corresponds to a morphism $X \to \mathscr{B}_k G$, which factors uniquely through $\Pi_{X/k}$. To obtain a homomorphism $\underline{\mathrm{Aut}}_k\, \xi_0 \to G$ we must specify an isomorphism of the image of ξ_0 in $\mathscr{B}_k G(k)$, corresponding to the restriction $P\mid_{x_0}$, with the trivial torsor. This is given by a rational point $p_0 \in P$ over x_0.

Hence $\underline{\mathrm{Aut}}_k\, \xi_0 = \pi_1^{\mathrm{N}}(X, x_0)$. So, the fundamental group schemes of X relative to different rational points correspond to different sections of the same gerbe $\Pi_{X/k}$.

As in the case of the fundamental group scheme, the most interesting part of the theory is the tannakian interpretation of $\Pi_{X/k}$.

Theorem 2 (Borne–V.). Assume that X is a geometrically connected, geometrically reduced scheme and pseudo-proper scheme over k. The pullback $\operatorname{Rep}\Pi_{X/k} \to \operatorname{Vect} X$ induces an equivalence of $\operatorname{Rep}\Pi_{X/k}$ with $\operatorname{EFin} X$.

Once again, this applies to much more general objects.

Sketch of proof. Denote by $\pi\colon X \to \Pi_{X/k}$ the canonical morphism.

Lemma. $\pi_*\mathscr{O}_X = \mathscr{O}_{\Pi_{X/k}}$.

Using the projection formula, this implies that the pullback $\pi^*\colon \operatorname{Rep}\Pi_{X/k} \to \operatorname{Vect} X$ is fully faithful.

Lemma. Every representation of a profinite gerbe is essentially finite.

This implies that π^* gives a fully faithful functor $\operatorname{Rep}\Pi_{X/k} \to \operatorname{EFin} X$. So we have to show that it essentially surjective.

Let E be a finite bundle on X of rank n. Choose two distinct polynomials f, $g \in \mathbb{N}[x]$ with an isomorphism $f(E) \simeq g(E)$. Let I be the scheme representing the vector spaces isomorphisms of $f(k^n)$ and $g(k^n)$; it is isomorphic to $\mathrm{GL}_{f(n)}$. There is a natural action of GL_n on I; the quotient stack $[I/\mathrm{GL}_n]$ is the stack of vector bundles E with isomorphisms $f(E) \simeq g(E)$. This gives a map $X \to [I/\mathrm{GL}_n]$; it is enough to show that this factors through $\Pi_{X/k}$.

Lemma. The action of GL_n on I has finite stabilizers.

By affine GIT, there exists a geometric quotient I/GL_n which is an affine variety. The composite $X \to [I/\mathrm{GL}_n] \to I/GL_n$ factors through a rational point on I/GL_n; let Ω be the corresponding orbit on I. Then $X \to [I/\mathrm{GL}_n]$ factors through $[\Omega/\mathrm{GL}_n]$. Since the action of GL_n on I is transitive with finite stabilizer, the quotient $[\Omega/\mathrm{GL}_n]$ is a finite gerbe. Hence $X \to [\Omega/\mathrm{GL}_n]$ factors through $\Pi_{X/k}$, and this concludes the proof.

Grothendieck's Section Conjecture

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Let X be a proper variety, geometrically connected and geometrically reduced on k, with a geometric point $\xi\colon \operatorname{Spec}\overline{k} \to X$. The natural morphism

$$\pi_1^{\mathrm{alg}}(X,\xi) \longrightarrow \pi_1^{\mathrm{alg}}(\operatorname{Spec} k, \operatorname{Spec}\overline{k}) = \mathrm{Gal}(\overline{k}/k)$$

is surjective, with kernel $\pi_1^{\mathrm{alg}}(X_{\overline{k}},\xi)$. Every rational point $x_0 \in X(k)$ yields a section

$$\mathrm{Gal}(\overline{k}/k) \longrightarrow \pi_1^{\mathrm{alg}}(X,\xi)\,,$$

well defined up to conjugacy.

In characteristic 0 we show that we have an equivalence between sections $\mathrm{Gal}(\overline{k}/k) \to \pi_1^{\mathrm{alg}}(X,\xi)$ and $\Pi_{X/k}(k)$.

Conjecture (Grothendieck). If X is a smooth geometrically connected projective curve of genus at least 2 over a field k that is a finitely generated extension of $\mathbb{Q}$, then this function from $X(k)$ to conjugacy classes of sections $\mathrm{Gal}(\overline{k}/k) \to \pi_1^{\mathrm{alg}}(X, \xi)$ is bijective.

Injectivity is known.

Thus, the conjecture can be restated as follows.

Conjecture. Let X be a smooth geometrically connected projective curve of genus at least 2 over a field k that is a finitely generated extension of $\mathbb{Q}$. Then the morphism $X \to \Pi_{X/k}$ induces a bijection between $X(k)$ and isomorphism classes of objects in $\Pi_{X/k}(k)$.

Theorem (essentially due to Tamagawa). Suppose that k is a finitely generated extension of $\mathbb{Q}$. If for every smooth geometrically connected projective curve X of genus at least 2 with $X(k) = \emptyset$ we have $\Pi_{X/k}(k) = \emptyset$, then the Section Conjecture holds.

To prove that $\Pi_{X/k}(k) = \emptyset$ it is enough to produce a finite gerbe Γ with $\Gamma(k) = \emptyset$, and a morphism $X \to \Gamma$.

Let Q be a nontrivial Brauer–Severi variety of degree n over k. The Picard group $\operatorname{Pic} Q$ is generated by a line bundle $\mathscr{O}_Q(r)$ with $r > 1$, that becomes isomorphic to $\mathscr{O}_{\mathbb{P}^{n-1}}(r)$ on $Q_{\overline{k}} \simeq \mathbb{P}^{n-1}_{\overline{k}}$. The integer r is the *exponent* of Q.

Theorem (Borne–V.). Let $f\colon X \to Q$ be a morphism. Assume that there exists a prime p dividing r and an invertible sheaf Λ on X, such that $\Lambda^{\otimes p} \simeq f^*\mathscr{O}_P(r)$. Then $\Pi_{X/k}(k) = \emptyset$.

Sketch of proof. Let $Q^\vee$ be the dual Brauer–Severi variety, that is, the Hilbert scheme of hyperplanes in P. Then $Q^\vee$ has also exponent r. Let $\Gamma \to (\mathrm{Alg}/k) \to (\mathrm{Groupoid})$ be the affine gerbe, whose sections over an affine k-scheme S consist of invertible sheaves E on $S \times P^\vee$, with an isomorphism $E^{\otimes p} \simeq \mathrm{pr}_2^* \mathscr{O}_{P^\vee}(r)$. Using Λ, one produces a morphism $X \to \Gamma$. Clearly $\Gamma(k) = \emptyset$

One easily produces examples of smooth curves X with the property above, as ramified covers of curves in Q.