The Nori correspondence

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Grothendieck defined the *algebraic fundamental group* $\pi_1^{\mathrm{alg}}(X, x_0)$ of a connected scheme X, relative to a geometric point $x_0\colon \operatorname{Spec}\Omega \to X$; it is a profinite group. If G is a finite group, the continuous homomorphisms $\pi_1^{\mathrm{alg}}(X, x_0) \to G$ correspond to (not necessarily connected) Galois covers $Y \to X$ with group G, with a fixed geometric point $y_0\colon \operatorname{Spec}\Omega \to Y$ over x_0.

An important generalization of Galois covers is given by *finite torsors*, i.e., G-torsors for a finite group scheme G over a base field k. It is a natural question whether there is a Galois theory for finite torsors. This is given by Nori's theory.

We will fix a base field k, over which all schemes and morphisms will be defined. In contrast with Grothendieck's Galois theory, Nori's theory is relative to k.

The category of finite groups embeds into the category of finite group schemes over k by sending each finite group G to the constant group scheme $\coprod_{g \in G} \operatorname{Spec} k$. The category of affine group schemes is closed under projective limits; a *profinite* group scheme is an affine group scheme that is, in some way, a projective limit of finite group schemes.

The embedding of the category of finite groups into that of finite group schemes extends to an embedding of the category of profinite groups into that of profinite group schemes. If k is algebraically closed of characteristic 0 these are equivalences.

If k has characteristic 0, but is not algebraically closed, (pro-)finite group schemes are twisted forms of (pro-)finite groups. If $\operatorname{char} k > 0$, there are non-smooth finite group schemes over k.

Let X be a geometrically reduced connected scheme over a field k, and let $x_0 \in X(k)$ be a rational point. Nori defined the *fundamental group scheme* $\pi_1^{\mathrm{N}}(X, x_0)$; it is a profinite group scheme with the property that, given a finite group scheme G on k, the homomorphisms $\pi_1^{\mathrm{N}}(X, x_0) \to G$ correspond to G-torsors $Y \to X$ with a fixed rational point $y_0 \in Y(k)$ lying over x_0.

On an algebraically closed field of characteristic 0 this coincides with Grothendieck's fundamental group. More generally, if k has characteristic 0 the group $\pi_1^{\mathrm{N}}(X, x_0)$ is a twisted form of the fundamental group of $X_{\overline{k}}$, obtained by the obvious action of the Galois group of $\overline{k}/k$. In positive characteristic this is completely false, because of the existence of non-smooth finite group schemes.

The main point of the theory is the tannakian interpretation of $\pi_1^{\mathrm{N}}(X, x_0)$.

Let G be an affine group scheme, and let $\mathscr{C} = \operatorname{Rep} G$ be the category of representations of G. It is a *neutral tannakian category*, that is:

(1) It is an abelian k-linear category with finite-dimensional Hom's.

(2) It has a *symmetric monoidal structure* $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$, given by tensor product, which is associative and symmetric, and has an identity $\mathbf{1}$ (the trivial representation of G on k).

(3) Every representation V has a dual $V^\vee$, with functorial isomorphisms $\operatorname{Hom}(V \otimes X, Y) \simeq \operatorname{Hom}(X, V^\vee \otimes Y)$.

(4) $\operatorname{Hom}(\mathbf{1}, \mathbf{1}) = k$.

(5) We have a fixed *fiber functor* $\Phi \colon \mathscr{C} \to \operatorname{Vect}_k$, which is k-linear, exact, and preserves the tensor product.

Conversely, given a neutral tannakian category $\mathscr{C}$ with fiber functor $\Phi \colon \mathscr{C} \to \mathrm{Vect}_k$, one can define an affine group scheme as the group scheme of automorphisms of Φ.

Theorem (Grothendieck, Saavedra Rivano, Deligne). These constructions give an equivalence between the category of affine group schemes on k and the category of neutral tannakian categories.

When X is complete, the category of representations of $\pi_1^{\mathrm{N}}(X, x_0)$ has an interesting tannakian interpretation.

Let $\pi_1^{\mathrm{N}}(X, x_0) \to \mathrm{GL}(V)$ be a representation; this has a factorization $\pi_1^{\mathrm{N}}(X, x_0) \to G \to \mathrm{GL}(V)$, where G is a finite quotient of $\pi_1^{\mathrm{N}}(X, x_0)$. The quotient $\pi_1^{\mathrm{N}}(X, x_0) \to G$ corresponds to a G-torsor $Y \to X$; we associate with these data a vector bundle $(Y \times V)/G \to Y/G = X$.

This yields a functor from $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0) \to \operatorname{Vect} X$ to the category of vector bundles on X. Its essential image consists of vector bundles with a reduction of structure group to a finite group scheme. If X is smooth and $k = \mathbb{C}$, then E is in the image if and only if it admits a flat holomorphic connection with finite monodromy. These bundles have an alternate characterization as *essentially finite* bundles.

Let $\mathrm{SS}_0 X \subseteq \operatorname{Vect} X$ be the category of those vector bundles that are semistable of degree 0 when restricted to the normalization of an arbitrary irreducible curve in X. The category $\mathrm{SS}_0 X$ is abelian, and contains the image of $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0)$.

If $f \in \mathbb{N}[x]$ is a polynomial and E is a vector bundle on X, we can define $f(E)$, interpreting the sum as a direct sum and the powers as tensor powers. A vector bundle E is *finite* if there exist f and g in $\mathbb{N}[x]$ with $f \neq g$ and $f(E) \simeq g(E)$.

In the category Vect X the Krull–Schmidt theorem holds, that is, the decomposition of a bundle as a direct sum of indecomposable bundles is unique up to isomorphisms. A bundle E is finite if and only if the set of isomorphism classes of the indecomposable components of all the powers $E^{\otimes n}$ is finite.

Hence:

(1) $E \oplus F$ is finite if and only if E and F are both finite.

(2) If E and F are finite, then $E \otimes F$ is finite.

(3) A line bundle is finite if and only if it is torsion.

So finite bundles on $\mathbb{P}^n$ are trivial: this follows from the structure theorem for vector bundles on $\mathbb{P}^1$, and the fact that a bundle on $\mathbb{P}^n$ that is trivial on each line is in fact trivial.

In characteristic 0, every bundle in the image of $\operatorname{Rep} \pi_1^{\mathrm{N}}(X, x_0)$ is finite.

The category $\mathrm{Fin}\, X$ of finite vector bundles in contained in $\mathrm{SS}_0\, X$.

A bundle is *essentially finite* if it is a subquotient in $\mathrm{SS}_0\, X$ of a finite bundle. The category $\mathrm{EFin}\, X \subseteq \mathrm{SS}_0\, X$ of essentially finite bundles is abelian.

Theorem (Madhav Nori). The functor $\mathrm{Rep}\, \pi_1^{\mathrm{N}}(X, x_0) \to \mathrm{Vect}\, X$ induces an equivalence of $\mathrm{Rep}\, \pi_1^{\mathrm{N}}(X, x_0)$ with $\mathrm{EFin}\, X$.

Since in characteristic 0 the essential image of $\mathrm{Rep}\, \pi_1^{\mathrm{N}}(X, x_0)$, which is $\mathrm{EFin}\, X$, is contained in $\mathrm{Fin}\, X$, we deduce that $\mathrm{EFin}\, X = \mathrm{Fin}\, X$.

Since every finite bundle on $\mathbb{P}^n$ is trivial, the same is true for essentially finite bundles. Hence $\pi_1^{\mathrm{N}}(\mathbb{P}^n, x_0) = \{1\}$.

Borne and I extend the theory, removing the dependence on a base point, and giving a simpler and more direct approach to the proof of the correspondence, which does not use semistable bundles.

One substitutes the fundamental group with a *gerbe*.

Recall Grothendieck's functorial point of view: a scheme X over k is identified with the functor $\mathsf{h}_X \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ it represents, via Yoneda's lemma. We need to extend the formalism to (pseudo)-functors $(\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Groupoid})$ to the category of groupoids (categories in which all arrows are isomorphisms).

A key example: if $G \to \operatorname{Spec} k$ is an algebraic group, we have the "classifying stack" $\mathscr{B}_k G \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Groupoid})$, sending each k-scheme S into the category $\mathscr{B}_k G(S)$ of G-torsors over S, which is a groupoid.

A version of Yoneda's lemma says that if $\Gamma \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Groupoid})$ is a pseudo-functor and X is a k-scheme, natural transformations ("morphisms") $X = \mathsf{h}_X \to \Gamma$ form a category equivalent to $\Gamma(X)$. Thus, for example, morphisms $X \to \mathscr{B}_k G$ correspond to G-torsors over X.

We are interested in *affine gerbes* over k. These are pseudo-functors $\Gamma\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Groupoid})$ such that:

(1) They are stacks in the fpqc topology.

(2) There exists some field extension k'/k such that $\Gamma(k') \neq \emptyset$.

(3) Any two objects are fpqc-locally isomorphic, that is, given two objects ξ and η in $\Gamma(S)$, where S is an affine k-scheme, there exists a faithfully flat morphism $f\colon T \to S$ with T affine, such that $f^*\xi \simeq f^*\eta$.

(4) If k'/k is a field extension and ξ is in $\Gamma(k')$, the functor $\underline{\mathrm{Aut}}_{k'}\,\xi\colon (\mathrm{Sch}/k')^{\mathrm{op}} \to (\mathrm{Grp})$ sending each k-scheme $f\colon S \to \operatorname{Spec} k'$ into the automorphism group of $f^*\xi$ is represented by an affine group scheme.

If G is an affine group scheme over k, then $\mathscr{B}_k G$ is an affine gerbe; the trivial torsor $G \to \operatorname{Spec} k$ gives a distinguished element of $\mathscr{B}_k G(k)$, or, more suggestively, a section $\operatorname{Spec} k \to \mathscr{B}_k G$ of the structure morphism $\mathscr{B}_k G \to \operatorname{Spec} k$.

Conversely, let Γ be an affine gerbe, and $\xi \in \Gamma(k)$, or $\xi \colon \operatorname{Spec} k \to \Gamma$. We obtain an affine group scheme $G \overset{\text{def}}{=} \underline{\mathrm{Aut}}_k\, \xi$; descent theory gives an isomorphism $\mathscr{B}_k G \simeq \Gamma$.

So, $\{$affine group schemes$\} = \{$affine gerbes with sections$\}$.

There are gerbes with $\Gamma(k) = \emptyset$.

Also, different sections $\operatorname{Spec} k \to \Gamma$ can give rise to non-isomorphic groups; equivalently, there may be non-isomorphic affine group scheme G and H with $\mathscr{B}_k G \simeq \mathscr{B}_k H$.

One can define vector bundles on gerbes; roughly, a vector bundle V on a gerbe Γ consists of a vector bundle $V_{S,\xi}$ for any pair (S, ξ), where S is a k-scheme S and $\xi \in \Gamma(S)$, plus isomorphism $V_{(T,f^{\xi})} \simeq f^* V_{(S,\xi)}$ for any morphisms $f\colon T \to S$ of k-schemes, satisfying a number of compatibility conditions. These form an abelian category $\operatorname{Vect}\Gamma$.

If G is an affine group scheme over k, we have a natural functor $\operatorname{Rep} G \to \operatorname{Vect}\mathscr{B}_k G$; if V is a representation of G and $P \to S$ is a G-torsor, we have that $(V \times P)/G \to S$ is a vector bundle over S. This functor is an equivalence; so, $\operatorname{Rep} G$ only depends on $\mathscr{B}_k G$. If Γ is a gerbe, we set $\operatorname{Rep}\Gamma \stackrel{\text{def}}{=} \operatorname{Vect}\Gamma$.

If G is an affine group scheme, the fiber functor $\operatorname{Rep}\mathscr{B}_k G = \operatorname{Rep} G \to \operatorname{Vect}_k$ is the pullback along the section $\operatorname{Spec} k \to \mathscr{B}_k G$ given by the trivial torsor $G \to \operatorname{Spec} k$.

Here is an interesting example. Assume that char $k \neq 2$, fix a positive integer n and let $\mathscr{Q}_n \colon (\mathrm{Sch}/k) \to (\mathrm{Groupoid})$ be the pseudo-functor such that $\mathscr{Q}_n(S)$ is the groupoid of vector bundles $E \to S$ of rank n with a non-degenerate quadratic form; the arrows are given by isometries. A section $\operatorname{Spec} k \to \mathscr{Q}_n$ corresponds to a pair (V, q), where V is an n-dimensional vector space and q is a non-degenerate quadratic form on V. The group $\underline{\mathrm{Aut}}_k(V, q)$ is the orthogonal group $\mathrm{O}(V, q)$.

Thus, if (V, q) and (V', q') are non-degenerate quadratic forms, $\mathscr{B}_k\mathrm{O}(V, q) \simeq \mathscr{B}_k\mathrm{O}(V', q')$, and $\operatorname{Rep}\mathrm{O}(V, q) \simeq \operatorname{Rep}\mathrm{O}(V', q')$.

If Γ is an affine gerbe, the category $\operatorname{Rep}\Gamma$ is a *non-neutral tannakian category*, that is:

(1) It is an abelian k-linear category with finite-dimensional Hom's.

(2) It has a *symmetric monoidal structure* $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$, given by tensor product, which is associative and symmetric, and has an identity $\mathbf{1}$ (the trivial representation of G on k).

(3) Every representation V has a dual $V^\vee$, with functorial isomorphisms $\operatorname{Hom}(V \otimes X, Y) \simeq \operatorname{Hom}(X, V^\vee \otimes Y)$.

(4) $\operatorname{Hom}(\mathbf{1}, \mathbf{1}) = k$.

(5) There exists a fiber functor $\mathscr{C} \to \mathrm{Vect}_{k'}$ for some field extension k' of k.

Theorem (Grothendieck, Saavedra Rivano, Deligne). The 2-category of affine gerbes over k is equivalent to the 2-category of tannakian categories.

An affine gerbe is (pro-)finite if for some (hence for all) ξ in $\Gamma(k')$, where k' is an extension of k, the group scheme $\underline{\mathrm{Aut}}_{k'}\,\xi$ is (pro-)finite. A profinite gerbe is a projective limit of finite gerbes.

Theorem (Borne, —). Suppose that X is a geometrically connected and geometrically reduced scheme (or a stack, or a more general object) over k. Then X has a *fundamental gerbe*, that is, a profinite gerbe $\Pi_{X/k}$ with a morphism $X \to \Pi_{X/k}$ such that any morphism $X \to \Gamma$, where Γ is a profinite gerbe, factors uniquely through $\Pi_{X/k}$.

If $x_0 \in X(k)$, its image in $\Pi_{X/k}$ gives an isomorphism $\Pi_{X/k} \simeq \mathscr{B}_k\pi_1^{\mathrm{N}}(X, x_0)$.

If $\operatorname{char} k = 0$, then $\Pi_{X/k}$ is the gerbe associated with Deligne's relative fundamental groupoid.

There are two approaches to the proof.

Bertrand Töen: It is obvious.

Niels, Angelo: Do some work.

As in Nori's case, the really interesting part is the tannakian interpretation of $\Pi_{X/k}$.

Let X be a scheme (or a more general object) over k that is geometrically connected and geometrically reduced. We say that X is *pseudo-proper* if it is quasi-compact and for every locally free sheaf E on X we have $\dim_k \mathrm{H}^0(X, E) < \infty$. This condition ensures that the Krull–Schmidt holds in $\mathrm{Vect}\, X$.

We say that a vector bundle on X is *essentially finite* if is the kernel of a homomorphism of finite bundles. When X is proper, this coincides with Nori's definition. Here I am cheating, because this is proved using Nori's theorem.

Theorem. Let X be pseudo-proper. The pullback $\mathrm{Rep}\, \Pi_{X/k} \to \mathrm{Vect}\, X$ induces an equivalence with the category $\mathrm{EFin}\, X$ of essentially finite bundles on X.

Recall *Grothendieck's section conjecture*.

Let X be a proper variety, geometrically connected and geometrically reduced on k, with a geometric point $\xi\colon \operatorname{Spec} \overline{k} \to X$. The natural morphism

$$\pi_1^{\mathrm{alg}}(X, \xi) \longrightarrow \pi_1^{\mathrm{alg}}(\operatorname{Spec} k, \operatorname{Spec} \overline{k}) = \mathrm{Gal}(\overline{k}/k)$$

is surjective, with kernel $\pi_1^{\mathrm{alg}}(X_{\overline{k}}, \xi)$. Every rational point $x_0 \in X(k)$ yields a section

$$\mathrm{Gal}(\overline{k}/k) \longrightarrow \pi_1^{\mathrm{alg}}(X, \xi)\,,$$

well defined up to conjugacy.

Conjecture (Grothendieck). If X is a smooth geometrically connected projective curve of genus at least 2 over a field k which is a finitely generated extension of $\mathbb{Q}$, then this function from $X(k)$ to conjugacy classes of sections $\mathrm{Gal}(\overline{k}/k) \to \pi_1^{\mathrm{alg}}(X, \xi)$ is bijective.

Injectivity is known.

This is false in positive characteristic, for example, when k is a finite field: in this case $\mathrm{Gal}(\overline{k}/k)$ is free, so a section always exists, while $X(k)$ could be empty.

In characteristic 0 we have an equivalence between sections $\mathrm{Gal}(\overline{k}/k) \to \pi_1^{\mathrm{alg}}(X, \xi)$ and $\Pi_{X/k}(k)$.

Conjecture (Borne, —). Let X be a smooth geometrically connected projective curve of genus at least 2 over a finitely generated field k. Then the morphism $X \to \Pi_{X/k}$ induces a bijection between $X(k)$ and isomorphism classes of objects in $\Pi_{X/k}(k)$.

We can prove injectivity.

In characteristic 0 this is equivalent to the section conjecture.