Roth type interval exchange maps

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Roth type irrational numbers

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $(a_n)_{n\in\mathbb{N}}$ be the sequence of the partial quotients of its continued fraction expansion and let $(q_n)_{n\in\mathbb{N}}$ be the sequence of the denominators of the convergents

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

Roth type irrationals have several equivalent characterizations:

- *rate of approximation by rational numbers*: for all $\varepsilon > 0$ there exists a positive constant C_ε such that $|q\alpha - p| \geq C_\varepsilon q^{-(1+\varepsilon)}$ for all p/q;
- *growth rate of denominators*: $q_{n+1} = O\left(q_n^{1+\varepsilon}\right)$ for all $\varepsilon > 0$;
- *growth rate of partial quotients*: $a_{n+1} = O\left(q_n^{\varepsilon}\right)$ for all $\varepsilon > 0$.

Cohomological equation for circle rotations and Roth type

Let $r \geq 0$, $r = [r] + \{r\}$ and let $C^r(\mathbb{T})$ be the subspace of $C(\mathbb{T})$ consisting of functions which are $[r]$ times differentiable and whose $[r]$-th derivative is Hölder continuous of exponent $\{r\}$.

Usual norm $\|\phi\|_{C^r} := \|\phi\|_{C^{[r]}} + \sup_{x \neq y} \frac{|D^{[r]}\phi(x) - D^{[r]}\phi(y)|}{|x-y|^{\{r\}}}$.

$C_0^r(\mathbb{T}) := \{f \in C^r(\mathbb{T}) \ : \ \int_{\mathbb{T}} f(x)dx = 0\}$.

Theorem *Let $r > 1$. α is of Roth type $\Leftrightarrow$ for all $\phi \in C_0^r(\mathbb{T})$ the cohomological equation*

$$\psi(x+\alpha) - \psi(x) = \phi(x)$$

has a unique solution ψ which belongs to $C_0^{r-1-\tau}(\mathbb{T})$ for all $\tau > 0$, $\tau \leq r-1$.

Herman's problem (proceedings ICM 1978)

- Coboundaries of $\mathcal{C}^\infty$ functions on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ under a diophantine translation have codimension $= 1$ in $\mathcal{C}^\infty(\mathbb{T}^d)$
- M compact manifold, f a $\mathcal{C}^\infty$ diffeomorphism of M
- Let Σ be a finite codimension linear subspace of $\mathcal{C}^\infty(M)$
- *suppose* that for all $\Phi \in \Sigma$ the cohomological equation

$$\Psi \circ f - \Psi = \Phi$$

  has a solution $\Psi \in \mathcal{C}^\infty(M)$
- *is it true that M is smoothly diffeomorphic to $\mathbb{T}^d$ and that f is smoothly conjugate to a diophantine translation of $\mathbb{T}^d$?*

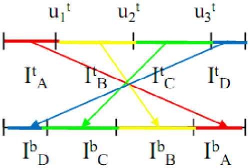
Interval exchange maps

Let

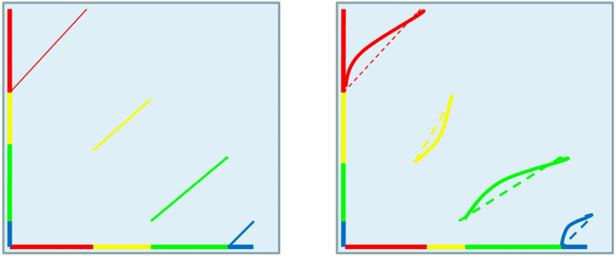
$$I = \bigsqcup_{\mathcal{A}} I^t_\alpha = \bigsqcup_{\mathcal{A}} I^b_\alpha$$

be two partitions (mod 0) of a bounded open interval I in the same number d of subintervals, indexed by the same alphabet $\mathcal{A}$.

A *(standard) interval exchange map* (i.e.m) with these data is a 1-to-1 map (mod 0) T on I sending each I^t_α onto I^b_α through a translation.



Standard vs. generalized interval exchange maps



A standard i.e.m. with $d = 4$; an affine (broken line) and a generalized i.e.m. with the same combinatorial data.

Affine i.e.m.'s will be the object of Yoccoz's seminar on thursday

Combinatorial data

The order in which the I^t_α, I^b_α appear is encoded by a pair $\pi = (\pi_t, \pi_b)$ of bijections from $\mathcal{A}$ to $\{1, \cdots, d\}$, the *combinatorial data* of the i.e.m T.

t=TOP, b=BOTTOM

We always assume that $\pi = (\pi_t, \pi_b)$ is *irreducible*:
for any $1 \leq k < d$

$$\pi_t^{-1}(1, \cdots, k) \neq \pi_b^{-1}(1, \cdots, k).$$

Otherwise, the recurrent part of the dynamics occurs in two disjoint intervals.

The case $d = 2$ corresponds to rotations

Lengths and translation vector

- Besides combinatorial data, one only needs to know the *length data* $\lambda_\alpha = |I^t_\alpha| = |I^b_\alpha|$.
- The *translations vector* $\delta = (\delta_\alpha)_{\alpha\in\mathcal{A}}$ is related to λ by

$$\delta = \Omega\lambda$$

- The antisymmetric matrix Ω is given by

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t(\beta) > \pi_t(\alpha)\ ,\ \pi_b(\beta) < \pi_b(\alpha), \\ -1 & \text{if } \pi_t(\beta) < \pi_t(\alpha)\ ,\ \pi_b(\beta) > \pi_b(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

- $\operatorname{rank}\Omega = 2g$, g is the genus of the map. This is also the genus of all translation surfaces obtained from T by Veech's zippered rectangles construction

Singularities and connections

- The *singularities* of T are the $d-1$ points $u_1^t < \cdots < u_{d-1}^t$ separating the I_α^t.
- The *singularities* of T^{-1} are the $d-1$ points $u_1^b < \cdots < u_{d-1}^b$ separating the I_α^b.
- A *connection* is a relation $T^m(u_i^b) = u_j^t$ with $1 \leq i, j < d$ and $m \geq 0$.

Theorem (Keane) If the length data are rationally independent, T has no connections.
If T has no connections, T is minimal.

Definition An i.e.m. T has the *Keane property* if it has no connections.

Roth type interval exchange maps and the cohomological euqation

We will denote $\mathrm{BV}^1_*(\sqcup I_\alpha))$ the space of functions φ such that

- are absolutely continuous on each I_α
- whose first derivative is of bounded variation on each I_α
- the integral of $D\varphi$ on the disjoint union $\sqcup I_\alpha$ vanishes

Theorem A. *Let T be an interval exchange map with the Keane property and of Roth type. Let $\Phi \in BV^1_*(\sqcup I_\alpha)$. There exists a function χ constant on each interval I_α and a bounded function Ψ such that*

$$\Psi - \Psi \circ T = \Phi - \chi \,.$$

Theorem B. *Roth type interval exchange maps form a full measure set in the space of all interval exchange maps.*

The theorem of Forni on solutions of cohomological equations for linear flows on translation surfaces

- (M, ω) translation surface, X vertical vector field
- A Poincaré section of (X, M, ω) is an i.e.m., the suspension of an i.e.m. gives a triple (X, M, ω)
- The cohomological equation in this case is simply $X\Psi = \Phi$
- One can rotate X of an angle θ: the vector field becomes $X(\theta)$ and the equation is now $X(\theta)\Psi = \Phi$
- Forni (Ann. Math. 1995) shows that for a.e. direction θ one can solve the cohomological equation provided Φ belongs to the kernel of a certain number of invariant distributions.
- The loss of differentiability here is at least 3 derivatives
- Methods are completely different (Forni develops new harmonic analysis tools, uses an analogue of Fatou lemma, in the spirit of Yoccoz's "blind" proof of the Siegel linearization theorem for the quadratic polynomial)

Our proof: Birkhoff sums and cohomological equations

$$S_T(n+1)\phi := \phi + \phi \circ T + \ldots + \phi \circ T^n$$

If ϕ is the coboundary of ψ then the sum is bounded since

$$S_T(n+1)\phi = \psi - \psi \circ T^n$$

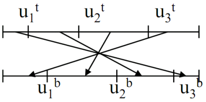
The theorem of Gottschalk-Hedlund gives a converse: if the Birkhoff sums of a continuous observable ϕ are bounded under a minimal homeomorphism T of a compact space then ϕ is the coboundary of a continuous function ψ

What is a Roth type i.e.m.?

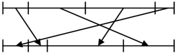
- The first condition is a *growth rate condition* for the matrices appearing in an accelerated version of the *Zorich continued fraction algorithm*. This condition is the precise analogue of the third of the equivalent arithmetical characterizations of Roth type irrational numbers given above.
- The second condition is a *spectral condition* which guarantees unique ergodicity of Roth type i.e.m.'s. This condition does not follow from condition (a) but is automatically satisfied if the i.e.m. is of *constant type* (i.e. the matrices considered in (a) have bounded norm).
- The third and last condition is a *coherence condition*.

The elementary step of the Rauzy-Veech algorithm

Let T be an i.e.m on an interval $I = (0, \ell)$ with the Keane property ($\Rightarrow u^t_{d-1} \neq u^b_{d-1}$). Let $\hat{\ell} := \max(u^t_{d-1}, u^b_{d-1})$, $\widehat{I} = (0, \hat{\ell})$.



T



$\widehat{T}$

The first return map $\widehat{T}$ of T on $\widehat{I}$ is an i.e.m whose combinatorial data $\hat{\pi} = (\hat{\pi}_t, \hat{\pi}_b)$ are naturally written with the same alphabet $\mathcal{A}$ than for T: one has $\hat{\pi}_t = \pi_t$ if $u^t_{d-1} < u^b_{d-1}$, $\hat{\pi}_b = \pi_b$ if $u^t_{d-1} > u^b_{d-1}$. Write $\hat{\pi} = R_t(\pi)$ in the first case, $\hat{\pi} = R_b(\pi)$ in the second case.

The relation between old and new length data is

$$\lambda = V\hat{\lambda}\,, \quad \text{where } V = \mathbf{I} + E_{\alpha_\varepsilon \alpha_{1-\varepsilon}} \in \mathrm{SL}\left(\mathbb{Z}^{\mathcal{A}}\right)$$

Rauzy classes and Rauzy diagrams

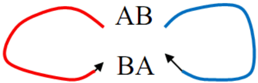
The i.e.m $\widehat{T}$ also has the Keane property; thus it is possible to iterate indefinitely the elementary step $T \to \widehat{T}$.

A *Rauzy class* is a set of combinatorial data (over the same alphabet $\mathcal{A}$) which is invariant under R_t and R_b, and minimal with respect to this property. All maps with combinatorial data in the same Rauzy class have the same genus.

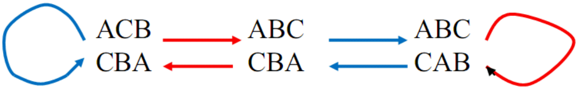
The associated *Rauzy diagram* is the directed graph whose vertices are the elements of the Rauzy class, and which has two arrows starting at each vertex π, joining this vertex to $R_t(\pi)$ and $R_b(\pi)$.

Iterating the Rauzy-Veech algorithm for an i.e.m T with combinatorial data π and no connections produces an infinite path γ_T starting at π in the Rauzy diagram having π as a vertex.

Rauzy diagrams for 2 and 3 intervals

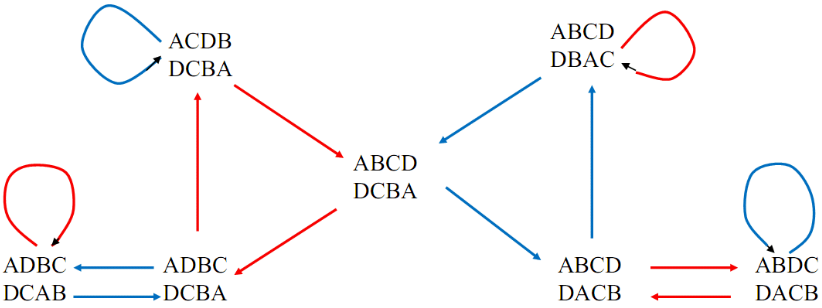


g=1, s=1,d=2



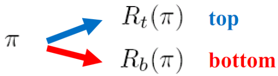
g=1,s=2,d=3

Rauzy diagrams for 4 intervals and genus 2



g=2,s=1,d=4

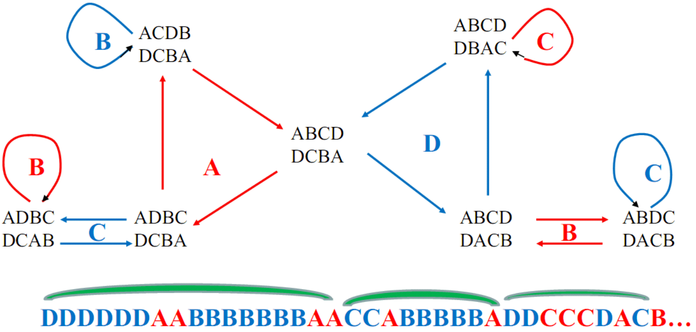
Paths in Rauzy diagrams and acceleration of the algorithm



The arrow from π to $R_t(\pi)$ (resp. $R_b(\pi)$) is said to be of top type (resp. bottom type).

- Let $\pi = (\pi_t, \pi_b)$ be a vertex; denote by α_t, α_b the letters such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$. The *winner* of the arrow of top (resp. bottom) type from π is α_t (resp. α_b); the *loser* is α_b (resp. α_t).
- The winner also gives a name to the arrow
- A path in the diagram is a word in the alphabet
- Keane property $\Longrightarrow$ every name is taken infinitely many times by the sequence of arrows
- Acceleration of the Rauzy-Veech algorithm: we group together consecutive arrows corresponding to words which are maximal with respect to this property:
  all letters of the alphabet but one appear in the word

Rauzy diagram for d=4 and acceleration of the Rauzy-Veech algorithm



Acceleration of the Rauzy-Veech algorithm II

- When $d = 2$ (and only in this case) this is the same as saying that you group together arrows with the same name
- When $d = 2$ Rauzy-Veech algorithm leads to the Farey map, the acceleration leads to the Gauss map
- For all d, the Rauzy-Veech map preserves an absolutely continuous ergodic invariant measure with infinite mass
- For all d, the acceleration we constructed preserves an absolutely continuous invariant *probability* measure

Acceleration leads to matrices with positive entries!

Let $l \geq k > 0$, denote by $T^{(k)}$ the i.e.m. obtained by iterating k times the accelerated algorithm starting from $T^{(0)} = T$.

The length data of $T^{(k)}$ and $T^{(k+1)}$ are related by

$$\lambda^{(k)} = Z(k+1)\lambda^{(k+1)} .$$

Let $Q(k,l) = Z(k+1)\cdots Z(l)$, so that $\lambda^{(k)} = Q(k,l)\lambda^{(l)}$.
$T^{(l)}$ is the first return map of $T^{(k)}$ on a subinterval $I^{(l)} \subset I^{(k)}$
$Q_{\alpha\beta}(k,l)$ is the time spent by $I^{t(l)}_{\beta}$ in $I^{t(k)}_{\alpha}$ until it returns in $I^{(l)}$.
Lemma. *If $l \geq k + 2d - 3$ (if $d = 2$ assume $l \geq k + 2$) then $Q_{\alpha\beta}(k,l) > 0$ for all $\alpha, \beta \in \mathcal{A}$.*

Roth type interval exchange maps: condition (a)

- (a) *Growth rate of partial quotients* $Z(k+1)$: for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $k \geq 0$ we have

$$\|Z(k+1)\| \leq C_\varepsilon \|Q(k)\|^\varepsilon .$$

  Here $Q(k) = Q(0,k)$.

- When $d = 2$, this amounts exactly to the classical Roth type, which is equivalent to have, for all $\varepsilon > 0$

$$a_{k+1} = O\left(q_k^\varepsilon\right) .$$

Condition (a) and size of intervals under renormalization

We can reformulate (a) in terms of the lengths $\lambda_\alpha^{(k)}$. It is convenient here to take as norm of a matrix the sum of all coefficients (in absolute value; the matrices that we consider here have nonnegative entries).

Proposition *We have always, for $k \geq 0$*

$$\mathrm{Max}_{\alpha\in\mathcal{A}}\lambda_\alpha^{(k)} \geq \lambda^* \|Q(k)\|^{-1} \geq \mathrm{Min}_{\alpha\in\mathcal{A}}\lambda_\alpha^{(k)} .$$

Condition (a) is equivalent to the following converse estimate: for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\mathrm{Max}_{\alpha\in\mathcal{A}}\lambda_\alpha^{(k)} \leq C_\varepsilon \mathrm{Min}_{\alpha\in\mathcal{A}}\lambda_\alpha^{(k)} \|Q(k)\|^{\varepsilon} .$$

Further consequences of condition (a)

Let $r > 0$, let $\tau_r(x)$ be the return time to the r-neighbourhood of x

$$\tau_r(x) = \min\{j \geq 1\,,\, |T^j(x) - x| < r\}$$

One can prove (D.H. Kim, S.M., 2008) that if T is an i.e.m. which verifies condition (a) then for almost every x one has

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1$$

For a circle irrational rotation, $T(x) = x + \alpha$, one knows that

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \frac{1}{\eta}\,, \quad \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1$$

where $\eta = \sup\{\beta > 0\,,\, \liminf_{j\to\infty} j^\beta\{j\alpha\} = 0\}$ is the *type* of the rotation number α. Of course α is of Roth type if and only if $\eta = 1$

Roth type interval exchange maps: condition (b)

- For each $k \geq 0$, let $\Gamma^{(k)}$ be the space of functions on $\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{t(k)}$ which are constant on each $I_\alpha^{t(k)}$. For $0 \leq k \leq l$, let $S(k,l)$ be the linear map from $\Gamma^{(k)}$ to $\Gamma^{(l)}$ whose matrix in the canonical basis is ${}^tQ(k,l)$. This can be interpreted as a special Birkhoff sum.
- For $\varphi = (\varphi_\alpha)_{\alpha \in \mathcal{A}} \in \Gamma^{(k)}$, define

$$I_k(\varphi) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \varphi_\alpha \ ;$$

  we have then

$$I_l(S(k,l)\varphi) = I_k(\varphi) \ .$$

- Denote by $\Gamma_*^{(k)}$ the kernel of the linear form I_k
- (b) *Spectral gap*: There exists $\theta > 0$, $C > 0$ such that, for all $k \geq 0$, we have

$$\| S(k) \mid_{\Gamma_*^{(0)}} \| \leq C \| S(k) \|^{1-\theta} = C \| Q(k) \|^{1-\theta} \ .$$

Spectral gap, unique ergodicity and constant type i.e.m.'s

- $k \geq 2d - 3$ ($k \geq 2$ if $d = 2$) $\implies Q_{\alpha\beta}(k) > 0$ for all α, β
- One expects the positive cone to be more expanded by $Q(k)$ than the other directions (Perron–Frobenius)
- This is not automatic! Minimal non uniquely ergodic i.e.m. 's exist (a minimal i.e.m. is uniquely ergodic $\Leftrightarrow$ the positive cone converges to a ray under $Q(k)$ as $k \to \infty$).
- condition (b) ensures that this weird behaviour does not occur. It is indeed possible to construct families of topologically conjugate non uniquely ergodic i.e.m.'s with $d = 4, g = 2$ and verifying our first condition (but not the second)
- For *bounded type i.e.m.'s*, i.e. such that (ã) *the sequence* $Z(k)$ *is bounded* then condition (b) follows. Indeed, each $Q(k, k + 2d - 3)$ ($Q(k, k + 2)$ when $d = 2$) will contract by a definite factor < 1 the Hilbert metric of the projective positive cone.

(c): Coherence condition

- Special Birkhoff sums $S(k,l) : \Gamma^{(k)} \to \Gamma^{(l)}$ acting on piecewise constant functions.
- $\Gamma_s^{(k)}$ be the *stable subspace*: $\|S(k,l)v\| \leq C\|S(k,l)\|^{-\sigma}\|v\|$ for all $l \geq k$
- $\Gamma_s^{(k)}$ always contains the translation vector $(\delta_\alpha^{(k)})_{\alpha\in\mathcal{A}}$.
- The operator $S(k,l)$ maps $\Gamma_s^{(k)}$ onto $\Gamma_s^{(l)}$.
- quotient operator $S_\flat(k,l) : \Gamma^{(k)}/\Gamma_s^{(k)} \to \Gamma^{(l)}/\Gamma_s^{(l)}$.
- As we have quotiented out the stable directions, it is not unreasonable to expect that the norm of the inverse of $S_\flat(k,l)$ is not too large:
- (c) *Coherence:* *for any* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that, for all* $l \geq k$, *we have*

$$\|[S_\flat(k,l)]^{-1}\| \leq C_\varepsilon \|Q(l)\|^\varepsilon \, , \|S(k,l)\,|_{\Gamma_s^{(k)}}\| \leq C_\varepsilon \|Q(l)\|^\varepsilon \, .$$