Linearization of generalized interval exchange maps

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joint work with Pierre Moussa (CEA Saclay)
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The slides of this talk will be available on my webpage
http://homepage.sns.it/marmi/

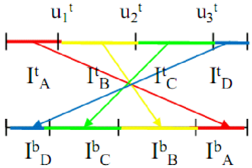
Plan of the talk

- Interval exchange maps
- Simple deformations and the linearization theorem
- Combinatorial data, Keane property and Rauzy diagrams
- Rauzy-Veech algorithm
- Irrational i.e.m. and Poincaré's theorem

Interval exchange maps

I bounded open interval, $I = \bigsqcup_{\mathcal{A}} I_\alpha^t = \bigsqcup_{\mathcal{A}} I_\alpha^b$ partitions (mod 0) in d subintervals, indexed by the alphabet $\mathcal{A}$.

A *standard (resp. generalized, resp. generalized C^r) interval exchange map* (i.e.m) with these data is a 1-to-1 map (mod 0) T on I sending each I_α^t onto I_α^b through a translation (resp. orientation-preserving homeomorphism, resp. a piecewise C^r diffeo $\overline{I_\alpha^t} \to \overline{I_\alpha^b}$ for each $\alpha \in \mathcal{A}$). Parameter space for standard normalized i.e.m is $(d-1)$-dimensional. If $d = 2$ then a standard i.e.m. is a rotation a generalized i.e.m. is an orientation-preserving circle homeomorphism.



Combinatorial data and genus of an i.e.m.

The order in which the I^t_α, I^b_α appear is encoded by a pair $\pi = (\pi_t, \pi_b)$ of bijections from $\mathcal{A}$ to $\{1, \cdots, d\}$, the *combinatorial data* of the i.e.m T. t=TOP, b=BOTTOM

Irreducibility: for any $1 \leqslant k < d$

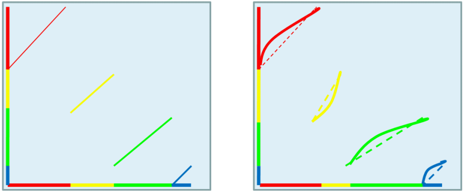
$$\pi_t^{-1}(1, \cdots, k) \neq \pi_b^{-1}(1, \cdots, k).$$

$$\Omega_{\alpha\beta} = \begin{cases} +1 & \text{if } \pi_t(\beta) > \pi_t(\alpha)\ ,\ \pi_b(\beta) < \pi_b(\alpha), \\ -1 & \text{if } \pi_t(\beta) < \pi_t(\alpha)\ ,\ \pi_b(\beta) > \pi_b(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

$\operatorname{rank} \Omega = 2g$. g is the *genus of the map*

Standard vs. generalized interval exchange maps

All translation surfaces obtained by suspension *à la Veech* from standard i.e.m. with a given Rauzy diagram have the same genus g, and the same number s of marked points and $d = 2g + s - 1$.



A standard i.e.m. with $g = 2, s = 1, d = 4$; an affine (broken line) and a generalized i.e.m. with the same combinatorial data.

Question: What part of the theory of circle diffeomorphisms generalize to i.e.m.s?

Overview of results: topological (semi)-conjugacy

- Use the Rauzy–Veech "continued fraction" algorithm to define an analogue of the rotation number. To each standard i.e.m. with *no connection*, the RV algorithm associates an ∞–complete path in a Rauzy diagram that can be viewed as a "rotation number".
- A generalized i.e.m. T is *irrational* if its associated path is ∞–complete; then T is semi–conjugated to any standard i.e.m. with the same rotation number (Poincaré's theorem). MY TALK WILL STOP HERE! MUCH MORE (and better) IN YOCCOZ'S TALK...
- Regarding Denjoy's theorem, results on the existence of wandering intervals for affine g.i.e.m. (Camelier-Gutierrez, Bressaud, Hubert and Maas, MMY) go in the negative direction and suggest that topological conjugacy to a standard i.e.m. has positive codimension in genus $g \geq 2$.

Overview of results: small divisors

- A first step in the direction of extending small divisor results beyond the torus case was achieved by Forni's theorem on the existence and regularity of solutions cohomological equation associated to linear flows on surfaces of higher genus.
- In an earlier paper, we (MMY) considered the cohomological equation $\psi \circ T_0 - \psi = \varphi$ for a standard i.e.m. T_0. We found explicitly in terms of the Rauzy-Veech algorithm a full measure class of standard i.e.m. (which we called Roth type i.e.m.) for which the cohomological equation has bounded solution provided that the datum φ belongs to a finite codimension subspace of the space of functions having on each continuity interval a continuous derivative with bounded variation.
- The cohomological equation is the linearization of the conjugacy equation $T \circ h = h \circ T_0$ for a generalized i.e.m. T close to the standard i.e.m. T_0.

Simple deformations and a linearization theorem

We say that a generalized i.e.m. T is a *simple deformation of class* C^r of a standard i.e.m. T_0 if

- T and T_0 have the same discontinuities;
- T and T_0 coincide in the neighborhood of the endpoints of I and of each discontinuity;
- T is a C^r diffeomorphism on each continuity interval onto its image.

For simple deformations our main result can be summarized as follows:

Theorem. *For almost all standard i.e.m. T_0 and for any integer $r \geq 2$, amongst the C^{r+3} simple deformations of T_0, those which are C^r-conjugate to T_0 by a diffeomorphism C^r close to the identity form a C^1 submanifold of codimension $d^* = (g-1)(2r+1)+s$.*

An open problem

Prove the theorem for $r = 1$: for a.a. s.i.e.m. T_0, C^4 simple deformations which are C^1-conjugate to T_0 by a C^1-close-to-Id diffeo form a C^1 submanifold of codim$= d^* = 3g - 3 + s = (d-1) + (g-1)$.

$(d-1)$ parameters fix the rotation number (see later)

Some remarks

- The standard i.e.m. T_0 considered in the theorem are of *restricted Roth type* (i.e. Roth type + nonvanishing KZ-cocycle Lyapunov exponents). They still form a full measure set by a theorem of Forni.
- To extend this result to non-simple C^r deformations T of a standard i.e.m. T_0, there are gluing problems of the derivatives of T at the discontinuities. *Indeed there is a conjugacy invariant which is an obstruction to linearization*.
- An earlier result is presented in an unpublished manuscript of De La Llave and Gutierrez. They consider standard i.e.m. with periodic paths for the Rauzy-Veech algorithm (for $d = 2$, this corresponds to rotations by a quadratic irrational). They prove that, amongst piecewise analytic generalized i.e.m. , the bi-Lipschitz conjugacy class of such a standard i.e.m. contains a submanifold of finite codimension.

Singularities and connections

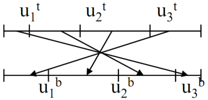
- The *singularities* of T are the $d-1$ points $u_1^t < \cdots < u_{d-1}^t$ separating the I_α^t.
- The *singularities* of T^{-1} are the $d-1$ points $u_1^b < \cdots < u_{d-1}^b$ separating the I_α^b.
- A *connection* is a relation $T^m(u_i^b) = u_j^t$ with $1 \leqslant i, j < d$ and $m \geqslant 0$.

Theorem (Keane) If the length data are rationally independent, a standard i.e.m. T has no connections.
If a standard i.e.m. T has no connections, T is minimal.

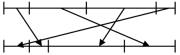
Definition A (generalized) i.e.m. T has the *Keane property* if it has no connections.

The elementary step of the Rauzy–Veech algorithm

Let T be a g.i.e.m with no connection. Then $u^t_{d-1} \neq u^b_{d-1}$. Set $\widehat{u}_d := \max(u^t_{d-1}, u^b_{d-1})$, $\widehat{I} := (u_0, \widehat{u}_d)$, and denote by $\widehat{T}$ the first return map of T in $\widehat{I}$. The return time $= 1$ or 2.



T

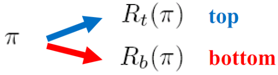


$\widehat{T}$

$\widehat{T}$ is a g.i.e.m on $\widehat{I}$ with combinatorial data $\widehat{\pi}$ labeled by the same alphabet $\mathcal{A}$. Moreover $\widehat{T}$ has no connection $\Longrightarrow$ one can iterate the algorithm! We say that $\widehat{T}$ is deduced from T by an elementary step of the Rauzy–Veech algorithm. We say that the step is of top (resp. bottom) *type* if $u^t_{d-1} < u^b_{d-1}$ (resp. $u^t_{d-1} > u^b_{d-1}$). One then writes $\widehat{\pi} = R_t(\pi)$ (resp. $\widehat{\pi} = R_b(\pi)$).

Rauzy diagrams

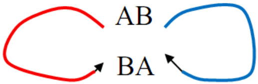
Rauzy class: nonempty set of irreducible combinatorial data invariant under R_t, R_b and minimal w.r.t. this property. *Rauzy diagram*: graph whose vertices are elements of a Rauzy class and whose arrows connect a vertex π to its images $R_t(\pi)$ and $R_b(\pi)$. Each vertex is therefore the origin and the endpoint of two arrows.



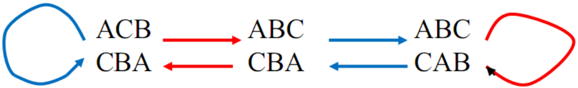
$\pi = (\pi_t, \pi_b)$; denote by α_t, α_b the letters such that $\pi_t(\alpha_t) = \pi_b(\alpha_b) = d$.

- The *winner* of the arrow of top (resp. bottom) type from π is α_t (resp. α_b); the *loser* is α_b (resp. α_t).
- The winner also gives a name to the arrow
- A path in the diagram is a word in the alphabet
- A path γ in a Rauzy diagram is *complete* if each letter in $\mathcal{A}$ appears in the corresponding word
- γ is k–*complete* if it is the concatenation of k complete paths.
- An infinite path is ∞–*complete* if it is the concatenation of infinitely many complete paths.

Rauzy diagrams for 2 and 3 intervals

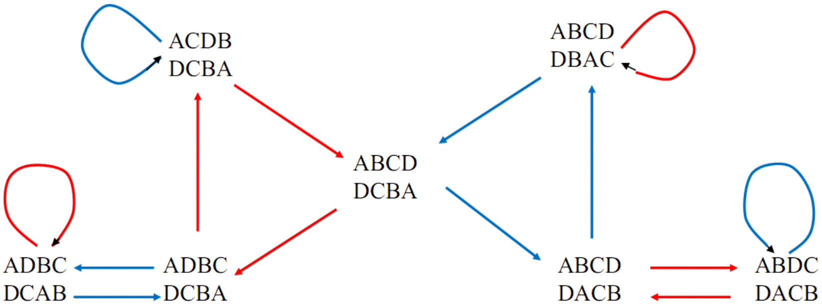


g=1, s=1,d=2



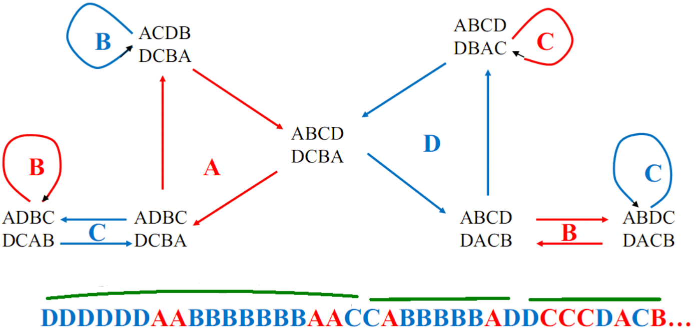
g=1,s=2,d=3

Rauzy diagrams for 4 intervals and genus 2



g=2,s=1,d=4

Rauzy diagram for d=4 and complete paths in the Rauzy-Veech algorithm



The Rauzy-Veech algorithm

Let $T = T^{(0)}$ be an i.e.m. with no connection, $\pi^{(0)}$ its combinatorial data and $\mathcal{D}$ the Rauzy diagram on $\mathcal{A}$ having $\pi^{(0)}$ as a vertex.
The i.e.m. $T^{(1)}$, with combinatorial data $\pi^{(1)}$, is deduced from $T^{(0)}$ by the elementary step of the Rauzy–Veech algorithm has also no connection.
Iterating the procedure one gets an infinite sequence $T^{(n)}$ of i.e.m. with combinatorial data $\pi^{(n)}$, acting on a decreasing sequence $I^{(n)}$ of intervals and a sequence $\gamma(n, n+1)$ of arrows in $\mathcal{D}$ from $\pi^{(n)}$ to $\pi^{(n+1)}$ associated to the successive steps of the algorithm.
If $m < n$, we also write $\gamma(m, n)$ for the path from $\pi^{(m)}$ to $\pi^{(n)}$ made of the concatenation of the $\gamma(l, l+1)$, $m \leqslant l < n$.
We write $\gamma(T)$ for the infinite path starting from $\pi^{(0)}$ formed by the $\gamma(n, n+1)$, $n \geqslant 0$.

Irrational generalized i.e.m.

If T is standard then $\gamma(T)$ is ∞-complete and, conversely, an ∞-complete path is equal to $\gamma(T)$ for some standard i.e.m. T with no connection. However for a generalized i.e.m. T the path $\gamma(T)$ is not always ∞-complete.

A g.i.e.m. T is *irrational* if it has no connection and $\gamma(T)$ is ∞-complete. We then call $\gamma(T)$ the *rotation number* of T.

When $d = 2$ (circle) the Rauzy diagram has one vertex and two arrows. If the rotation number

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

the associated ∞-complete path takes a_1 times the first arrow, then a_2 times the second arrow, a_3 times the first arrow, ...

Poincaré's theorem

A standard i.e.m. is irrational iff it has no connection and two s.i.e.m. with no connection are topologically conjugated iff they have the same rotation number.

More generally, if T is an irrational generalized i.e.m. with the same rotation number of a standard i.e.m T_0, then there is, as in the circle case, a semiconjugacy from T to T_0, i.e. a continuous nondecreasing surjective map h from the interval I of T onto the interval I_0 of T_0 such that $T_0 \circ h = h \circ T$ (Poincaré's theorem).

Another open problem

Describe the set of generalized C^r interval exchange maps which are semi-conjugate to a given standard i.e.m. T_0 (with no connections).

In the circle case, for a diophantine rotation number, one has a C^∞ submanifold of codimension 1. In the Liouville case one has still a topological manifold of codimension 1 which is transverse to all 1-parameter strictly increasing families. One can therefore dare to ask:

1. *Is the above set a topological submanifold of codimension $d-1$?*
2. *if the answer is positive, does there exist a (smooth) field of "transversal" subspaces of dimension $d-1$?*

The questions make sense for any T_0, but the answer could depend on the diophantine properties of T_0.