There is only one KAM curve

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based on joint works with David Sauzin and Carlo Carminati
The slides of this talk will be available on my webpage
http://homepage.sns.it/marmi/

Plan of the talk

- The standard family
- (Perhaps) the oldest open problem in KAM theory: a little bit of history
- A uniqueness result for KAM curves
- A natural boundary
- Open Problems

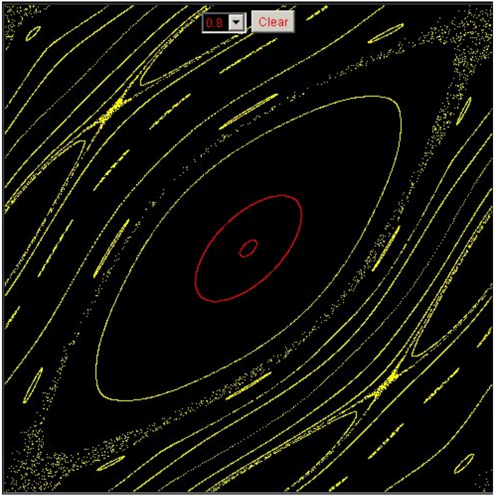
The standard family

Let f be a 1-periodic real analytic function with zero mean value. The standard family is discrete dynamical system defined by

$$T_\varepsilon\colon (x,y) \mapsto (x_1,y_1), \qquad \begin{cases} x_1 = x + y + \varepsilon f(x) \\ y_1 = y + \varepsilon f(x) \end{cases} \tag{1}$$

in the phase space $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and ε is a real parameter (when $f(x) = \cos(2\pi x)$, T_ε is the so-called standard map). For ε close to 0, this is an exact symplectic map that we can view as a perturbation of the integrable twist map $(x,y) \mapsto (x+y,y)$.

The standard map



picture generated from
http://www.dynamical-systems.org/twist/Applet.html

Invariant rotational curves for the standard family

For $\omega \in \mathbb{R} - \mathbb{Q}$, we call *invariant graph of frequency* ω *for* T_ε the graph $G = \{(x, \varphi(x))\} \subset \mathbb{T} \times \mathbb{R}$ of a continuous map $\varphi \colon \mathbb{T} \to \mathbb{R}$ such that T_ε leaves G invariant and the restriction of T_ε to G is conjugate to the translation $x \mapsto x + \omega$ by a homeomorphism of $\mathbb{R}$ of the form $\mathrm{id} + u$, where u is a 1-periodic function.

There is a natural way of viewing an invariant graph of frequency ω as a parametrized curve $G = \gamma(\mathbb{T})$: one looks for a continuous function $u \colon \mathbb{T} \to \mathbb{R}$ such that the curve γ is parametrized as

$$\gamma(\theta) = \big(\theta + u(\theta), \omega + v(\theta)\big)$$

where $v(\theta) = u(\theta) - u(\theta - \omega)$ and u is a solution of

$$u(\theta + \omega) - 2u(\theta) + u(\theta - \omega) = \varepsilon f\big(\theta + u(\theta)\big)$$

KAM theory for the standard family

- If it exists, the invariant graph of irrational frequency ω is unique thanks to the positive twist map condition verified by T_ε. The corresponding parametrization γ is unique up to a shift in the variable θ.
- Normalize u so that it has zero mean value: *finding u is equivalent to finding an invariant graph of frequency ω.*
- Moser theorem for twist maps guarantees the existence of an invariant graph of frequency ω for every Diophantine ω provided $|\varepsilon|$ is small enough. The corresponding curve γ_ω is analytic in the angle θ and depends analytically on ε.
- We want to investigate the regularity of the map $\omega \mapsto \gamma_\omega$, which for the moment is defined on the set of real Diophantine numbers. More specifically, we are interested in the quasianalytic properties of this map; this will lead us to extend it to certain complex values of the frequency ω.

(Perhaps) The oldest problem in KAM theory

- In 1954, in his Amsterdam ICM conference, the same where he stated his theorem, Kolmogorov asked whether the regularity of the solutions of small divisor problems with respect to the frequency could be investigated using appropriate analytical tools, suggesting a connection with the theory of "monogenic functions" in the sense of Émile Borel
- Borel's uniform monogenic functions (1917) are a generalization of analytic functions to closed sets (including Swiss cheeses with empty interior). They are built using Cauchy's integral theory rather than series expansions (which typically will diverge) and they may or may not be quasianalytic spaces.

Komlogorov 1954 ICM talk

In order to clear up this matter more comprehensively, we shall investigate again the equations of motion over the two-dimensional torus, by introducing into them the parameter θ, variable in some kind of limit $[\theta_1 ; \theta_2]$:

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, x_2, \theta).$$

We shall assume that the functions $F_\alpha(x_1, x_2, \theta)$ are analytic. It is obvious that the ratio of mean frequencies $\gamma(\theta)$ will analytically depend on θ. If $\gamma(\theta)$ is not constant, the set R of those θ for which the system can be analytically transformed into a form

$$\frac{d\xi_\alpha}{dt} = \lambda_\alpha,$$

will take up almost the entire segment $[\theta_1, \theta_2]$. Eigen functions

$$\varphi_{mn} = e^{i(m\lambda_1 + n\lambda_2)}$$

Komlogorov 1954 ICM talk

upon being returned to the initial coordinates x_1, x_2 will serve for $\theta \in R$ as analytical functions from x_1 and x_2. However, generally speaking, even on R they will be in this set with respect to θ totally disconnected, and also this discontinuity cannot be destroyed by the rejection from R of the set of measure zero. These circumstances are more significantly essential than this in that $\varphi_{mn}(x_1, x_2, \theta)$ can be determined also in some points of the residual set $[\theta_1, \theta_2]-R$ of measure zero at the expense of the assumption of their non-analyticity and discontinuity with respect to x_1 and x_2.

It is possible that the relation of $\varphi_{mn}(x_1, x_2, \theta)$ the parameter θ on R is attributed to the class of functions of the type pf monogenic Bord functions (24) and, in spite of totally disconnected character, it admits the investigation with proper analytical means.

A timeline

- Arnold (1961) complexifies the rotation number and investigates the monogenic regularity of the conjugacy of circle maps to an irrational rotation, but he cannot prove it because of a technical difficulty (at each step of the KAM iteration the domain shrinks by a finite amount)
- Herman (1985) proves the local conjugacy theorem for circle maps using the Schwarzian derivative trick and Schauder's fixed point theorem. He complexifies the rotation number ω and proves the $\mathcal{C}^1$-holomorphic regularity (monogenic regularity) of the conjugacy w.r.t. to ω.
- The results of Pöschel on Whitney regularity are also related to the problem. Whitney smoothness implies the existence of a smooth extension, which is highly non-unique.

Arnold 1961 circle diffeomorphisms article

Newton's method was applied for a similar purpose by A. N. Kolmogorov [6]. Theorem 2 of the present paper is in a way a discrete analogue of his theorem on the preservation of conditionally periodic motions under small changes of the Hamilton's function. In distinction to [6] we have no analytic integral invariants at our disposal, but rather we seek them. Moreover, we prove (in Theorem 2) the analyticity of the dependence on a small parameter ϵ, from which there follows the convergence of all the series in powers of ϵ that are usual in the theory of perturbations.

A direct proof of the convergence of these series has not been achieved, and A. N. Kolmogorov has even conjectured* (before studying the paper [7] of K. L. Siegel) that they might diverge.

Another conjecture of Kolmogorov, stated by him in the report [8], turned out to be true: questions in which small denominators play a role are connected with the monogenic functions of Borel [9]. For our case this is established in §§7,8 and used in §11.

Arnold 1961: complex frequencies

when $z \longrightarrow A(z, \epsilon, \Delta(\epsilon))$. Theorem 2 is proved.

§7. On monogenic functions

7.1. **The concept of monogeneity.** In the investigation of the dependence of the solutions of equation (1) of §2 on the parameter μ we encounter functions analytic in the upper and in the lower half-plane, and everywhere discontinuous on the real axis. All the functions, Δ_n, g_n, ϕ_n, F_n, Φ_n constructed in §6, considered as functions of μ, have these properties (see §8). These functions belong to the type called by Borel [9] monogenic.

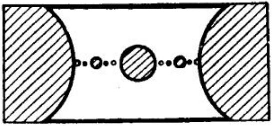


Figure 2

The monogenic functions of Borel are defined on the set $E = \bigcup_{k=1}^{\infty} E_k$, where $E_k \subseteq E_{k+1}$ are perfect compact subsets of the complex plane. In our case E_k is the set M_K^R of points μ of the rectangle $|\mathrm{Im}\ \mu| \leq R$, $0 \leq \mathrm{Re}\ \mu \leq 1$ of the

Arnold 1961: monogenic regularity of the solution of the linearized equation

omitted.

Lemma 10. *Suppose that the function* $f(z, \epsilon, \Delta, \mu) = \tilde{f}$ *is analytic with respect to* z *in the region* $|\mathrm{Im}\, z| \le R$; ϵ, $|\epsilon| \le \epsilon_0$; $|\Delta| \le \Delta_0$ *and is monogenic with respect to* $\mu \in N_K^R$, *and suppose that in the indicated region*

$$|f| \leqslant C, \quad \left|\frac{\partial f}{\partial \mu}\right| \leqslant L.$$

Then the solution of the equation

$$g(z + 2\pi\mu, \varepsilon, \Delta, \mu) - g(z, \varepsilon, \Delta, \mu) = f(z, \varepsilon, \Delta, \mu)$$

is monogenic with respect to $\mu \in N_K^R$ *and analytic with respect to* z *in the*

Arnold 1961: the region of monotonicity contracts at each iteration

Arnold cannot conclude because at each step of the KAM proof his control of complex frequencies deteriorates by a *finite* amount.

§8. On the dependence of the constructions of Theorem 2 on μ

8.1. We have seen, in subsection 7.4, that the solution of the linear equation (1) of §2 depends on μ monogenically. In the present section we shall prove the monogenicity with respect to μ of the functions Δ_n, F_n, Φ_n, g_n, $\Delta^{(n)}$ constructed in §6.

It turns out that the region of monotonicity contracts as n increases (by $|\text{Im } 2\pi\mu|$ at each step) and the author has not been able to establish whether the solution of equation (1) of §4 depends monogenically on μ.

Herman's Schwarzian derivative trick I

The Schwarzian derivative of a C^3 orientation preserving diffeomorphism f is

$$Sf := D^2 \mathrm{Log} Df - \frac{1}{2}(D \mathrm{Log} Df)^2 \ .$$

The Schwarzian derivative vanishes on linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$, $ad - bc = \pm 1$.
The composition rule for Schwarzian derivatives is

$$S(f \circ g) = Sf \circ g \ (Dg)^2 + Sg \ .$$

Let f be a circle diffeomorphisms of rotation number ω. Let R_ω be the corresponding rotation of the circle. Assume that ω is diophantine.

Herman's Schwarzian derivative trick II

Taking Schwarzian derivatives, the conjugacy equation

$$f \circ h = h \circ R_\omega$$

becomes

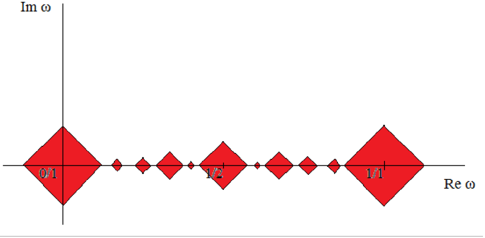
$$(Sh) \circ R_\omega - Sh = ((Sf) \circ h)(Dh)^2$$

This is a linear difference equation in the Schwarzian derivative Sh of the conjugacy (but the r.h.s. depends also on h). Given a diffeomorphism h, one computes the r.h.s. $((Sf) \circ h)(Dh)^2$, solves the equation $\psi \circ R_\omega - \psi = ((Sf) \circ h)(Dh)^2$ and then finds a diffeomorphism $\tilde{h} = \Phi(h)$ as smooth as h with $S\tilde{h} = \psi$. One can even use the contraction principle to conclude (at the cost of one more derivative for f).

Complex frequencies (Herman style)

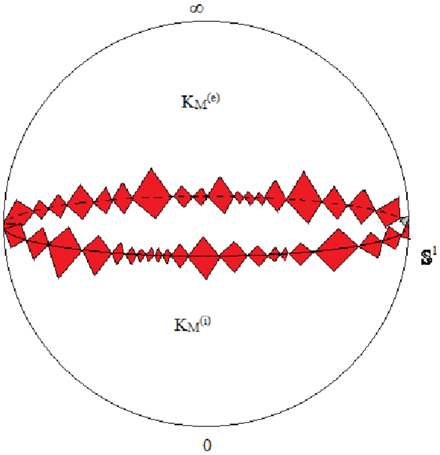
$$A_M^{\mathbb{R}} = \left\{ \omega \in \mathbb{R} \mid \forall (n,m) \in \mathbb{Z} \times \mathbb{N}^*, \ |\omega - \frac{n}{m}| \geq \frac{1}{Mm^{2+\tau}} \right\}, \tau > 0 \text{ fixed}$$

$$A_M^{\mathbb{C}} = \{ \omega \in \mathbb{C} \mid \ \exists \omega_* \in A_M^{\mathbb{R}} \ \text{ such that } \ |\Im m\, \omega| \geq |\omega_* - \Re e\, \omega| \}$$



A butchered complex sphere

$$q = \mathrm{e}^{2\pi \mathrm{i}\omega} \in K_M = \mathrm{e}^{2\pi \mathrm{i} A_M^{\mathbb{C}}} \cup \{0, \infty\} \subset \hat{\mathbb{C}},$$



$\mathcal{C}^1$-holomorphic invariant curves

- Our main result is that the parametrization of KAM curves $\omega \mapsto u(\omega; \theta, \varepsilon)$ extends to a $\mathcal{C}^1$-holomorphic function from K_M to $\mathbb{B}_{R,\rho}$, where $\mathbb{B}_{R,\rho} = H^\infty(S_R \times \mathbb{D}_\rho)$ is the complex Banach space of all bounded holomorphic functions of $S_R \times \mathbb{D}_\rho$, $S_R = \{x \in \mathbb{C}/\mathbb{Z} \mid |\Im m\, x| < R\}$ and $\mathbb{D}_\rho = \{\varepsilon \in \mathbb{C} \mid |\varepsilon| < \rho\}$.
- $\mathcal{C}^1$-holomorphy essentially means *complex differentiability in the sense of Whitney*, *i.e.* real Whitney differentiability on a closed subset with partial derivatives which satisfy the Cauchy-Riemann equations. They were introduced into the subject by Michel Herman and are a generalization of Borel's theory of uniform monogenic functions.

Statement of the main theorem

Theorem

Assume that f is real analytic and holomorphic in a neighbourhood of $\overline{S}_{R_0}$, with zero mean value. Then there exist $c > 0$ and, for each $M > 2\zeta(1+\tau)$, a function

$$\tilde{u}_M \in \mathcal{C}^1_{hol}(K_M, \mathbb{B}_{R,\rho}), \quad \textit{with } \rho = cM^{-8}$$

such that, for each $\omega \in A_M^{\mathbb{R}}$ and $\varepsilon \in (-\rho, \rho)$, the function $\theta \in \mathbb{T} \mapsto \tilde{u}_M(\mathrm{e}^{2\pi \mathrm{i}\omega})(\theta, \varepsilon)$ has zero mean value and parametrizes an invariant graph of frequency ω for T_ε.

There is only one KAM curve

- The previous Theorem provides a function $\tilde{u}_M$ on K_M which is an extension to complex frequencies of the parametrization of KAM curves existing for real diophantine frequencies belonging to the Cantor set $A_M^{\mathbb{R}}$.
- *This extension is unique* thanks to a quasianalyticity property of the space of $\mathcal{C}^1$-holomorphic functions on K_M (S.M. and D. Sauzin, Bull Bras. Math. Soc 2012).
- Indeed the space of $\mathcal{C}^1$-holomorphic functions on K_M is $\mathcal{H}^1$*-quasianalytic: any subset of* K_M *of positive* $\mathcal{H}^1$*-measure is a uniqueness set* (*i.e.* the only function of the space vanishing identically on it $\equiv 0$). In other words, a function of $\mathcal{L}$ is determined by its restriction to any subset of C of positive $\mathcal{H}^1$-measure.

A natural boundary

Indeed, we have also proved that from the point of view of classical analytic continuation, the real axis in frequency space appears as a natural boundary, because of the density of the resonances, but our quasianalyticity result is sufficient to prove that some sort of "generalized analytic continuation" through it is indeed possible: the knowledge of the parametrizations on a set of positive linear measure of rotation numbers (real or complex) is sufficient to determine all the parametrized KAM curves: in this sense there is only one KAM curve, parametrized by one monogenic function of the rotation number.

Open problems: I

Herman's question: Hadamard's quasianaliticity at diophantine points.

sequence $(C_n)_n$ in an appropriate way depending on the function considered (in the example if the radii of the disks D_n^p and the sequence $|A_p|$ decrease fast enough), wanted his monogenic functions to have quasi-analytic properties (i.e. monogenic continuation) (cf. [D, p. 139-146] and [C, ch. IX]). We believe that this last point is one of the main reasons of E. Borel's work on monogenic functions (which is anterior to the work of Denjoy-Carleman on quasi-analytic functions [D], [C]).

In this respect we can ask the following question: Let $t \in I_{\delta'} \to G(t)$ be the function defined in 8.

Question. *Is the function $G_1(t)$ always "determined" by its Taylor series at $t = 0$?*

(I think that the answer is negative, for the linearized equation does not seem to belong to any quasi-analytic class.)

Open problems II

- *More on Herman's question* Concerning the linearized equation: in (S.M. and D. Sauzin Memoirs of the AMS 2000) we prove that the solution of the linearized equation cannot belong to any quasianalytic Carleman class but nevertheless there is Hadamard's quasianaliticity at diophantine points and indeed there is even more: resurgence at resonances.
- *Question: Can one replacing positive Hausdorff measure with Hausdorff dimension equal to one in the quasianaliticity result we prove?* If this were true, then KAM curves would be uniquely determined by their knowledge on constant type numbers.
- *Question (even more ambitious!): Can one prove that algebraic (or, better, quadratic irrationals) are a uniqueness set for the space of $\mathcal{C}^1$-holomorphic function from K_M to $\mathbb{B}_{R,\rho}$?*