Some Open Problems Related To Small Divisors

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0. Introduction

What follows is a redaction of a (memorable) three–hours–long open problem session which took place during the workshop in Cetraro. Each of the five lecturers (H. Eliasson, M. Herman, S. Kuksin, J.N. Mather and J.–C. Yoccoz) spent about half an hour briefly introducing some open problems. This redaction grew from the notes that the first author took during the session : whereas we mention who suggested each of the problems of the list we give below (with the exception of the authors of this text) we are the only responsible for any mistake the reader may find in their description or formulation. Moreover the list of references is very far from being complete and has not been updated. We tried to make this text self–contained, but see [KH] for terminology and further information and [Yo2] for a short survey of classical results concerning small divisor problems.

1. One–Dimensional Small Divisor Problems (On Holomorphic Germs and Circle Diffeomorphisms)

1.1 Linearization of the quadratic polynomial. Size of Siegel disks.

Let us consider the linearization problem for the quadratic polynomial $P_\lambda(z) = \lambda(z - z^2)$ ([Yo3], Chapter II) where $z \in \mathbb{C}$, $\lambda = e^{2\pi i\alpha}$ and $\alpha \in \mathbb{C}/\mathbb{Z}$. We say that P_λ is *linearizable* if there exists a holomorphic map tangent to the identity $h_\lambda(z) = z + \mathcal{O}(z^2)$ such that $h_\lambda(\lambda z) = P_\lambda(h_\lambda(z))$. Then h_λ is unique and we will denote r_λ its radius of convergence (when $|\lambda| = 1$ this measures the "size" of the Siegel disk of P_λ).

The second author proved the following results :

(1) there exists a bounded holomorphic function $U : \mathbb{D} \to \mathbb{C}$ such that for all $\lambda \in \mathbb{D}$, $|U(\lambda)|$ is equal to r_λ ;

(2) for all $\lambda_0 \in \mathbb{S}^1$, $|U(\lambda)|$ has a non–tangential limit in λ_0, which is still equal to r_{λ_0} ;

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(3) if $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, P_λ is linearizable if and only if α is a Brjuno number : if $(p_n/q_n)_{n\geq 0}$ denotes the sequence of the convergents of the continued fraction expansion of α then being a *Brjuno number* means that $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty$;
(4) There exists a universal constant $C_1 > 0$ and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all Brjuno numbers α one has

$$(1-\varepsilon)B(\alpha) - C_\varepsilon \leq -\log |U(e^{2\pi i\alpha})| \leq B(\alpha) + C_1 .$$

Problem 1.1.1 *Does the function (defined on the set of Brjuno numbers)* $\alpha \mapsto B(\alpha) + \log |U(e^{2\pi i\alpha})|$ *belong to* $L^\infty(\mathbb{R}/\mathbb{Z})$ *?*

There is a good numerical evidence [Mar] in support to a positive answer to the following much stronger property :

Problem 1.1.2 *Does the function* $\alpha \mapsto B(\alpha) + \log |U(e^{2\pi i\alpha})|$ *extend to a* $1/2$*–Hölder continuous function as* α *varies in* $\mathbb{R}$ *?*

These two problems are discussed to some extent in [MMY1], [MMY2]. For some related analytical and numerical results concerning some area–preserving maps, including the standard family, we refer to [Mar], [BPV], [MS], [Da], [BG], [CL].

1.2 Herman rings. Differentiable conjugacy of diffeomorphisms of the circle

The second author [Yo5] proved that the Brjuno condition is also necessary and sufficient in the local conjugacy problem for analytic diffeomorphisms of the circle. In this case the simplest non–trivial model is provided by the Blaschke products $Q_{a,\alpha}(z) = \rho_\alpha z^2 \frac{z+a}{1+az}$. Here $a \in (3, +\infty)$ and $\rho_\alpha \in \mathbb{S}^1$ is chosen in such a way that the rotation number of the restriction of $Q_{a,\alpha}$ to $\mathbb{S}^1$ is exactly α. Under these assumptions $Q_{a,\alpha}$ induces an orientation–preserving analytic diffeomorphism of $\mathbb{S}^1$. Note that when $a \to +\infty$ then $Q_{a,\alpha}(z) \to e^{2\pi i\alpha}z$.
When α is a Brjuno number if a is large enough then $Q_{a,\alpha}$ is analytically conjugated to the rotation $R_\alpha(z) = e^{2\pi i\alpha}z$ in a neighborhood of $\mathbb{S}^1$. If α satisfies the more restrictive artihmetical condition $\mathcal{H}$ (we refer to Yoccoz's lectures in this volume for its definition) then $Q_{a,\alpha}$ is conjugated to the rotation for all $a > 3$. This leads to the following

Problem 1.2.1*Let* α *be a Brjuno number not satisfying condition* $\mathcal{H}$ *: does there exist an* $a > 3$ *such that* $Q_{a,\alpha}$ *is not analytically conjugated to the rotation* R_α *?*

Concerning this problem Herman showed that there exists at least a Brjuno number α not satisfying $\mathcal{H}$ such that the answer to the previous question is positive. In general one expects to exist a maximal interval $(a_0, +\infty)$, $a_0 > 3$, such that $Q_{a,\alpha}$ is analytically conjugated to a rotation for all $a \in (a_0, +\infty)$ whereas $Q_{a_0,\alpha}$ is not analytically conjugated.

Problem 1.2.2 *How smooth is the conjugacy for a_0 ? How does this smoothness depend on α ?*

This leads naturally to the study of conjugacy classes of orientation–preserving diffeomorphisms of the circle with finitely many continuous derivatives. Here the relevant arithmetical conditions on the rotation number are of Diophantine type. Let $\tau \geq 0$; we denote $\mathrm{DC}(\tau)$ the set of irrational numbers α whose denominators q_n of the continued fraction expansion satisfy $q_{n+1} = \mathcal{O}(q_n^{1+\tau})$ for all $n \geq 0$.
Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; $r, s \in \{0, +\infty, \omega\} \cup [1, +\infty)$. Let $\mathrm{Diff}^r_+(\mathbb{T})$ be the group of $\mathcal{C}^r$ diffeomorphisms[1] of $\mathbb{T}$ which are orientation–preserving. We denote $D^r(\mathbb{T})$ the group of $\mathcal{C}^r$ diffeomorphisms of the real line such that $\tilde{f} - \mathrm{id}$ is $\mathbb{Z}$–periodic. We consider the linearization problem $f \circ h = h \circ R_\alpha$ where R_α denotes the rotation of α on $\mathbb{T}$, α is the rotation number of $f \pmod 1$ and $h \in \mathrm{Diff}^s_+(\mathbb{T})$, with $r \geq s$. One must distinguish the *local* conjugacy problem from the *global* one : thus we define

$$\begin{aligned}\mathcal{C}_{r,s} = \{&\alpha \in \mathbb{R} \setminus \mathbb{Q}\ ,\ \text{every}\, f \in \mathrm{Diff}^r_+(\mathbb{T})\,\text{with rotation number}\,\alpha \bmod 1\\ &\text{is conjugated to}\, R_\alpha \text{with a conjugacy}\, h \in \mathrm{Diff}^s_+(\mathbb{T})\}\\ \mathcal{C}^{loc}_{r,s} = \{&\alpha \in \mathbb{R} \setminus \mathbb{Q}\ ,\ \text{every}\, f \in \mathrm{Diff}^r_+(\mathbb{T})\,\text{with rotation number}\,\alpha \bmod 1\\ &\mathcal{C}^r\text{–close to}\, R_\alpha \,\text{is conjugated to}\, R_\alpha \,\text{with a conjugacy}\, h \in \mathrm{Diff}^s_+(\mathbb{T})\}\end{aligned}$$

Note that in the definition of $\mathcal{C}^{loc}_{r,s}$ the neighborhood in the $\mathcal{C}^r$ topology which measures the distance of f from R_α depends on α.
Let $s < r - 1 < \infty$. The following inclusions are known after [He2, KO1, KO2, KS, Yo1, Yo4], etc.

$$\mathrm{DC}(r - s - 1 - \varepsilon) \subset \mathcal{C}_{r,s} \subset \mathcal{C}^{loc}_{r,s} \subset \mathrm{DC}(r - s - 1 + \varepsilon)\ \ \text{for all}\,\varepsilon > 0$$

The third inclusion is due to Herman [He2]. One also knows from [SK] that :
- if $1 < s < 2 < s + 1 < r < 3$ then $\mathrm{DC}(r - s - 1) \subset \mathcal{C}_{r,s}$;
- $\mathrm{DC}(0) \subset \mathcal{C}_{r,r-1}$ provided that $r > 2$, $r \notin \mathbb{N}$.

Problem 1.2.3 *Determine $\mathcal{C}_{r,s}$ and $\mathcal{C}^{loc}_{r,s}$. Are they different ?*

1.3 Gevrey classes

In the case of the conjugacy of germs of formal diffeomorphisms of $(\mathbb{C}, 0)$ one can consider a problem similar to the local one above requiring the formal germs to belong to some ultradifferentiable class, for example Gevrey classes.
Consider two subalgebras $A_1 \subset A_2$ of $z\mathbb{C}[[z]]$ closed with respect to the composition of formal series. In addition to the usual cases $z\mathbb{C}[[z]]$ (formal germs) and $z\mathbb{C}\{z\}$ (analytic germs) one can for example consider Gevrey–s classes G_s, $s > 0$ (i.e.

[1] If $r = 0$ it is the group of homeomorphisms of $\mathbb{T}$; if $r \geq 1$, $r \in \mathbb{R} \setminus \mathbb{N}$, it is the group of $\mathcal{C}^{[r]}$ diffeomorphisms whose $[r]$–th derivative satisfies a Hölder condition of exponent $r - [r]$; if $r = \omega$ it is the group of $\mathbb{R}$–analytic diffeomorphisms.

series $F(z)=\sum_{n\geq 0} f_n z^n$ such that there exist $c_1, c_2 > 0$ such that $|f_n| \leq c_1 c_2^n (n!)^s$ for all $n \geq 0$). Let $F \in A_1$ being such that $F'(0) = \lambda \in \mathbb{C}^*$. We say that F is linearizable in A_2 if there exists $H \in A_2$ tangent to the identity and such that $F \circ H = H \circ R_\lambda$ where $R_\lambda(z) = \lambda z$. Let $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
One knows that if α is a Brjuno number then for all $s > 0$ all germs $F \in G_s$ have a linearization $H \in G_s$ (see [CM]). Let $r > s > 0$ and denote

$$\mathcal{G}_{r,s} = \{\alpha \in \mathbb{R}\setminus\mathbb{Q}\ ,\ \text{every}\, F \in G_s \,\text{is conjugated to}\, R_\alpha \text{with a conjugacy} H \in G_r\}$$

One knows that a condition weaker than Brjuno is sufficient [CM].

Problem 1.3.1 *Determine* $\mathcal{G}_{r,s}$.

Of course one can ask a similar question in the circle case, distinguishing the local from the global case.

2. Finite–Dimensional Small Divisor Problems

2.1 Linearization of germs of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$

Let $n \geq 2$ and let $f \in (\mathbb{C}[[z_1, \dots z_n]])^n$ be a germ of formal diffeomorphism of $(\mathbb{C}^n, 0)$, $z = (z_1, \dots, z_n)$, $f(z) = Az + O(z^2)$ with $A \in \mathrm{GL}(n, \mathbb{C})$. Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of A, $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, $\lambda^k = \lambda_1^{k_1} \dots \lambda_n^{k_n}$ and $|k| = \sum_{j=1}^n |k_j|$. Assume that the eigenvalues are all distinct. If

$$\lambda^k - \lambda_j \neq 0 \ \text{ for all } j = 1, \dots, n \ \text{ and } k \in \mathbb{N}^n\ ,\ |k| \geq 2 \qquad (NR)$$

then f is formally linearizable, i.e. there exists a unique germ h of formal diffeomorphism of $(\mathbb{C}^n, 0)$, tangent to the identity, such that $h^{-1} \circ f \circ h = A$.
Let f be $\mathbb{C}$–analytic and assume that A satisfies (NR). For $m \in \mathbb{N}$, $m \geq 2$ let

$$\Omega(m) = \inf_{2\leq |k| \leq m\ 1 \leq j \leq n} |\lambda^k - \lambda_j|\ .$$

Then Brjuno [Br] proved that if A is diagonalizable, satisfies (NR) and the condition

$$\sum_{k=0}^{\infty} 2^{-k} \log(\Omega(2^{k+1}))^{-1} < +\infty \qquad (B)$$

then f is analytically linearizable, i.e. the formal germ h defines a germ of $\mathbb{C}$–analytic diffeomorphism of $(\mathbb{C}^n, 0)$. The proof uses the classical majorant series method used by Siegel [S, St] to prove that h is $\mathbb{C}$–analytic under the stronger assumption that $\lambda_1, \dots, \lambda_n$ satisfy a diophantine condition.

Problem 2.1.1 (M. Herman) *What is the optimal arithmetical condition on the eigenvalues of the linear part which assures that f is analytically linearizable ? Can one obtain it by direct majorant series method ?*

It seems unlikely that condition (B) is optimal.
Concerning the problem of linearization of germs of holomorphic diffeomorphisms near a fixed point [He4] contains many other questions, most of which are still open.

2.2 Elliptic fixed points and KAM theory

If one replaces the assumption of being conformal with the assumption of preserving the standard symplectic structure of $\mathbb{R}^{2n}$ one can consider the following problem.
Let f be a real analytic symplectic diffeomorphism of $\mathbb{R}^{2n}$ which leaves the origin fixed $f(0) = 0$. Let $z \in \mathbb{R}^{2n}$ and assume that $f(z) = Az + O(z^2)$, where $A \in \mathrm{Sp}\,(2n, \mathbb{R})$ is conjugated in $\mathrm{Sp}\,(2n, \mathbb{R})$ to $r_{\alpha_1} \times \ldots \times r_{\alpha_n}$, where $r_{\alpha_i} = \begin{pmatrix} \cos 2\pi\alpha_i & \sin 2\pi\alpha_i \\ -\sin 2\pi\alpha_i & \cos 2\pi\alpha_i \end{pmatrix}$ and the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ satisfies a diophantine condition. Herman [He6] stated the following conjecture.

Problem 2.2.1 (M. Herman) Show that there exists $\varepsilon_0 > 0$ such that in the ball $\|z\| \le \varepsilon_0$ there is a set of positive Lebesgue measure of invariant Lagrangian tori.

Note that here the assumption of f being real analytic is essential since in the $\mathcal{C}^\infty$ case the conjecture is true for $n = 1$, open if $n = 2$ but one knows the existence of counterexamples if $n \ge 3$. Using Birkhoff normal form one can prove that the conjecture is true in many special cases (some twist condition, see [BHS]).

2.3 $\mathbb{Z}^k$–actions

Let $k \ge 1$ and $F_1, \ldots, F_k$ be commuting diffeomorphisms in $D^r(\mathbb{T})$. Let $\alpha_1, \ldots, \alpha_k$ be the rotation numbers of $F_1, \ldots, F_k$. Then the diffeomorphisms $f_1, \ldots, f_k$ of the circle induced by $F_1, \ldots, F_k$ generate a $\mathbb{Z}^k$–action on $\mathbb{T}$. If α_1 is irrational and $F_1 = R_{\alpha_1}$, then we must have $F_i = R_{\alpha_i}$ for $1 \le i \le k$, because the centralizer of R_{α_1} in $D^0(\mathbb{T})$ is the group of translations. Therefore, if α_1 is irrational and F_1 is conjugated to R_{α_1} by a diffeomorphism $h \in D^s(\mathbb{T})$, the full $\mathbb{Z}^k$–action is linearized by h.
J. Moser [M2] has shown that, if for some γ, τ and all $q > 0$

$$\max_{1 \le i \le k} \|q\alpha_i\| \ge \gamma q^{-\tau}$$

then there exists a neighborhood V of the identity in $D^\infty(\mathbb{T})$ such that if $F_i \in D^\infty(\mathbb{T})$, $\rho(F_i) = \alpha_i$ and $F_i \circ R_{-\alpha_i} \in V$ then the action is linearizable in $D^\infty(\mathbb{T})$. The simultaneous approximation condition above is probably optimal in the $\mathcal{C}^\infty$ category.
More generally, one can define, for $1 \le s \le r \le \infty$, or $r = s = \omega$

$$C^k_{r,s} = \{(\alpha_1, \ldots, \alpha_k)\,,\ \text{any } k\text{–uple}\,(F_1, \ldots, F_k) \text{ of commuting diffeomorphisms in } (D^r(\mathbb{T}))^k \text{with rotation numbers } (\alpha_1, \ldots, \alpha_k) \text{ is linearizable in } D^s(\mathbb{T})\}\ ,$$

and similarly $C^{k,loc}_{r,s}$ if one assumes furthermore that $F_i \circ R_{-\alpha_i}$, $1 \le i \le k$, are $\mathcal{C}^r$–close to the identity.

Problem 2.3.1 *Determine $C^k_{r,s}$, $C^{k,loc}_{r,s}$.*

Progress in this direction is contained in the papers [DL], [Kra], [PM], etc.
The previous problem generalizes to the study of $\mathbb{Z}^k$ actions on $\mathbb{R}^n$. Let $k > n \ge 1$ and consider an action $\mathbb{Z}^k \hookrightarrow \mathrm{Diff}^\omega_+(\mathbb{R}^n)$. Among them the actions by translations $R_{\alpha_i} : x \mapsto x + \alpha_i$, $1 \le i \le k$, where $\alpha_i \in \mathbb{R}^n$, play a distinguished role. Assume that $\alpha_1, \dots, \alpha_k$ generate $\mathbb{R}^n$ and consider those actions whose generators $f_1, \dots, f_k \in \mathrm{Diff}^\omega_+(\mathbb{R}^n)$ are $\mathcal{C}^\omega$–close to $R_{\alpha_1}, \dots, R_{\alpha_k}$ and which are $\mathcal{C}^0$–conjugated to it (thus one can call $\alpha_1, \dots, \alpha_k$ the rotation numbers of the action.

Problem 2.3.2 *For which rotation numbers is this $\mathcal{C}^0$–conjugacy indeed analytic ?*

2.4 Diffeomorphisms of compact manifolds

Let M be a $\mathcal{C}^\infty$ compact connected manifold. We denote $\mathrm{Diff}^\infty(M)$ the group of $\mathcal{C}^\infty$ diffeomorphisms of M with the $\mathcal{C}^\infty$ topology and $\mathrm{Diff}^\infty_+(M)$ the group of $\mathcal{C}^\infty$ diffeomorphisms $\mathcal{C}^\infty$–isotopic to the identity (i.e. the connected component of $\mathrm{Diff}^\infty(M)$ which contains the identity).
The general problem that one may address is to study the structure (conjugacy classes, centralizers, etc.) of $\mathrm{Diff}^\infty(M)$. In the special case of the n–dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ one has a KAM theorem which describes the local structure of $\mathrm{Diff}^\infty_+(\mathbb{T}^n)$ near diophantine translations $R_\alpha : x \mapsto x + \alpha$, $\alpha \in \mathbb{T}^n$: there exists a neighborhood U_α of R_α in $\mathrm{Diff}^\infty_+(\mathbb{T}^n)$ such that if $f \in U_\alpha$ there exists $\lambda \in \mathbb{T}^n$ and $g \in \mathrm{Diff}^\infty_+(\mathbb{T}^n)$ such that $g(0) = 0$ and $f = R_\lambda \circ g^{-1} \circ R_\alpha \circ g$. Moreover this decomposition is locally unique.
In [He1] a "converse" of this theorem is asked :

Problem 2.4.1 (M. Herman) *Let V be a compact $\mathcal{C}^\infty$ manifold, $f \in \mathrm{Diff}^\infty(V)$, U a $\mathcal{C}^\infty$ neighborhood of the identity, $\mathcal{O}_{f,U} = \{g \circ f \circ g^{-1}\ ,\ g \in U\}$. If $\mathcal{O}_{f,U}$ is a finite codimension manifold is it true that $V = \mathbb{T}^n$ and f is $\mathcal{C}^\infty$–conjugate to a Diophantine translation ?*

In the torus case one proves KAM theorem by means of an implicit function theorem in Fréchet spaces (see [Ha, Bo]). The main point is that a translation R_α being diophantine is equivalent to ask that for all $\varphi \in \mathcal{C}^\infty(\mathbb{T}^n)$ there exist $\psi \in \mathcal{C}^\infty(\mathbb{T}^n)$ and $\lambda \in \mathbb{R}$ such that the linearized conjugacy equation

$$\psi \circ R_\alpha - \psi = \varphi + \lambda$$

holds. Then one can ask the analogue of Problem 2.4.1 for the linearized conjugacy equation

Problem 2.4.2 (M. Herman) *If for all $\varphi \in \mathcal{C}^\infty(V)$ there exists $\psi \in \mathcal{C}^\infty(V)$ and $\lambda \in \mathbb{R}$ such that $\psi \circ f - \psi = \varphi + \lambda$ is it true that $V = \mathbb{T}^n$ and f is $\mathcal{C}^\infty$–conjugate to a Diophantine translation ?*

3. KAM Theory and Hamiltonian Systems

3.1 Twist maps

We let $\mathbb{T}$ denote the circle $\mathbb{R}/\mathbb{Z}$ and $\theta(\text{mod}\,1)$ denote the standard parameter of $\mathbb{T}$ and x the corresponding parameter of its universal cover $\mathbb{R}$. We will let $y \in \mathbb{R}$ denote the standard parameter of the second factor of $\mathbb{T} \times \mathbb{R}$. Let U be an open subset of $\mathbb{T} \times \mathbb{R}$ which intersects each vertical line $\{\theta\} \times \mathbb{R}$ in an open non empty interval. We consider maps f which are diffeomorphisms from U onto an open subset $f(U) \subset \mathbb{T} \times \mathbb{R}$ which also intersects each vertical line in an open non empty interval. We assume that f is *orientation preserving* and *area preserving*. Since we are in two dimensions the area preserving condition is the same as requiring that f be *symplectic*. Let $\tilde{f}$ denote the lift of f to the universal cover so that $\tilde{f}(x+1, y) = \tilde{f}(x, y) + (1, 0)$. We also set $(x', y') = \tilde{f}(x, y)$.
An orientation preserving symplectic $\mathcal{C}^1$ diffeomorphism f satisfies a *positive* (resp. *negative*) *monotone twist* condition if $\frac{\partial x'}{\partial y} > \varepsilon$ (resp. $< -\varepsilon$) for some fixed $\varepsilon > 0$ and for all (x, y). Geometrically this condition states that the image of a segment $x =$constant under $\tilde{f}$ forms a graph over the x axis. An *integrable* twist map has the form $\tilde{f}(x, y) = (x + r(y), y)$.
From the area preserving property of f it follows that $y'dx' - ydx$ is a closed 1–form and therefore there exists a *generating function* (or *variational principle*) $h = h(x, x')$ such that

$$y = -\partial_1 h(x, x')\ , \quad y' = \partial_2 h(x, x')\ .$$

The generating function is unique up to the addition of a constant and its invariance under translations $(x, x') \mapsto (x+1, x'+1)$ is equivalent to the condition that $y'dx' - ydx$ is exact on $\mathbb{T} \times \mathbb{R}$. Moreover, from the positive twist condition one has $\partial_{12} h(x, x') < 0$.
A *rotational* invariant curve is a homotopically non–trivial f–invariant curve. By Birkhoff's theory (see [He3], Chapitre I), such a curve is the graph of a Lipschitz function. For near–to–integrable twist maps KAM theory provides the existence of many rotational invariant curves. Herman [He3, He5] proved that rotational invariant curves persist in twist diffeomorphisms which are $\mathcal{C}^3$–close to an integrable map.

Problem 3.1.1 (J. Mather) *Does there exist an example of a $\mathcal{C}^r$ twist area–preserving map with a rotational invariant curve which is not $\mathcal{C}^1$ (separate question fo each $r \in [1, \infty] \cup \{\omega\}$).*

Problem 3.1.2 (J. Mather) *Given a $\mathcal{C}^\infty$ twist diffeomorphism and a rotational invariant curve which is not $\mathcal{C}^\infty$ is it possible to destroy it by an arbitrarily small $\mathcal{C}^\infty$ perturbation ?*

One knows, following [Ma3] that if the rotation number is not diophantine this is indeed possible even in the case the circle is $\mathcal{C}^\infty$. [Ma3] contains also destruction results for $\mathcal{C}^r$ twist maps (see p. 212) and [Fo] for analytic maps but in both cases there is a gap between the destruction results and the persistence results given by KAM theory.

It is a classical counterexample of Arnold that there exist analytic diffeomorphisms of the circle, with irrational rotation number, whose conjugacy to a rigid rotation is not absolutely continuous. Since every diffeomorphism of the circle can be embedded as rotational invariant curve of an area–preserving monotone twist map of the annulus with the same degree of smoothness, this example, and those constructed by Denjoy [De1, De2] and Herman [He2] in the differentiable case, give examples of "regular" twist maps f having "regular" rotational invariant curves γ such that $f|_\gamma$ is topologically conjugate to a rigid rotation but the conjugacy is not absolutely continuous. In these examples the irrational rotation number has extremely good approximations by rational numbers.

The classical results of Denjoy on diffeomorphisms of the circle show that given an invariant curve γ, if the rotation number α of $f|_\gamma$ is irrational then one has the following :

- if $f|_\gamma \in \mathcal{C}^2$ then $f|_\gamma$ is topologically conjugate to R_α and every orbit is dense in γ ;
- there exist examples of $f|_\gamma \in \mathcal{C}^{2-\varepsilon}$, $\varepsilon > 0$, such that no orbit is dense in γ and the limit set of the orbit of every point of γ is the same Cantor set (*Denjoy minimal set*).

Thus even if f is smooth, limit sets different from γ may appear provided that the invariant curve γ loses smoothness.

Problem 3.1.3 (J. Mather) *Does there exist a $\mathcal{C}^3$ area–preserving twist map of the annulus with a rotational, not topologically transitive, invariant curve of irrational rotation number ? Same problem with $r \geq 3$ or even analytic.*

M. Herman [He3] gave an example of class $\mathcal{C}^{3-\varepsilon}$. Hall and Trupin [HT] gave a $\mathcal{C}^\infty$ example without the area–preserving condition.

The most important progress towards the understanding of these problems has come through the introduction of *Aubry–Mather sets* [AL, Ma1, Ma2, Ma4]. These are closed invariant sets given by a parametric representation

$$x = u(\theta) \ , \quad y = -\partial_1 h(u(\theta), u(\theta + \alpha)) \ ,$$

where u is monotone (but not necessarily continuous) and $u - \theta$ is $\mathbb{Z}$–periodic. They do exist for all rotation numbers α and they are subsets of a closed Lipschitz graph. For rational α one obtains periodic orbits, whereas for irrational numbers one has rotational invariant curves if u is continuous (in fact Lipschitz by Birkhoff's theorem) or an invariant Cantor set if u has countably many discontinuities. In this case the Aubry–Mather set can be viewed as a Cantor set drawn on a Lipschitz graph. Another important property of Aubry–Mather sets is that the "ordering" of an orbit is the same as for the rotation by α of a circle.

Mather based his proof on the variational problem

$$\int_0^1 h(u(\theta), u(\theta+\alpha))d\theta$$

minimizing this functional in the class of weakly monotone functions, thus they are also called *action minimising sets.*

Problem 3.1.4 (M. Herman) *For a $\mathcal{C}^r$ twist area–preserving diffeomorphism does the union of the action–minimising sets which are not closed curves and do not contain periodic orbits have Hausdorff dimension* 0 *?*

Here $r \geq 3$; otherwise Herman himself has a counterexample.

3.2 Euler–Lagrange flows

It is proved in [M1] that any monotone twist map can be obtained as the time–1 map of the Hamiltonian flow associated to a time–dependent, $\mathbb{Z}$–periodic in time Hamiltonian $H : T^*(\mathbb{T}\times\mathbb{R})\times\mathbb{R} \to \mathbb{R}$ satisfying Legendre condition $H_{yy}(\theta, y, t) > 0$. This assures that f can also be interpolated by the time–1 map of the Euler–Lagrange flow associated to the Lagrangian function $L : T(\mathbb{T}\times\mathbb{R})\times\mathbb{R} \to \mathbb{R}$ obtained from H by Legendre transform.
More generally, let M be a a closed Riemannian manifold M and consider Lagrangians of the form kinetic energy + time periodic potential $V \in \mathcal{C}^r(M\times\mathbb{T})$ (see [Ma5] for a more general setting). Assume that M has dimension at least 2.

Problem 3.2.1 (J. Mather) *Is there a residual set (in the sense of Baire category) in $\mathcal{C}^r(M\times\mathbb{T})$ such that there exists a corresponding trajectory of the Euler–Lagrange flow with kinetic energy growing to ∞ as $t\to\infty$?*

Of course these systems are very far from integrable ones. De La Llave has some results concerning this problem.

3.3 n–body problem

Let $n \geq 2$. Consider $n+1$ point masses $m_0, \dots, m_n$ moving in an inertial reference frame $\mathbb{R}^3$ with the only forces acting on them being their mutual gravitational attraction. If the i–th particle has position vector q_i then the Newtonian equations of motion are

$$m_i\ddot{q}_i = -\frac{\partial V}{\partial q_i}\ ,\ \ i = 1, \dots, n$$

where $V(q_0, \dots, q_n) = -\sum_{0\leq i<j\leq n} \frac{Gm_im_j}{|q_i-q_j|}$ and G is the universal gravitational constant. The size of the system is measured by the moment of inertia $I = \frac{1}{2}\sum_{i=0}^n m_i|q_i|^2$. The total energy of the system $E = T + V = \sum_{i=0}^n m_i\frac{|\dot{q}_i|^2}{2} + V(q_0, \dots q_n)$, the total angular momentum $A = \sum_{i=0}^n m_iq_i \wedge \dot{q}_i$ and the total momentum $P = \sum_{i=0}^n m_i\dot{q}_i$ are integrals of the motion. We will assume that the center of mass $C = \sum_{i=0}^n m_iq_i$ is *fixed* at the origin, thus $P = 0$.

J. Mather recalled a problem proposed by G.D. Birkhoff [Bi] :

Problem 3.3.1 (G.D. Birkhoff) *Let $n = 2$ (three body problem) and assume that the total energy is negative. One can even assume that the three particles move on a fixed plane. Is it true that the moment of inertia of the system can grow to ∞ as $t \to \infty$ for a dense open set of initial conditions ? One can ask the same question for $n > 2$.*

The restriction to negative energy is necessary since from the identity of Jacobi–Lagrange $\ddot{I} = E + T$ follows that if the energy $E \geq 0$ all orbits which are defined for all times are wandering. The only thing which is known is that even for negative energy wandering sets do exist (as Birkhoff and Chazy showed long ago, see [Al]). The fact that the bounded orbits form a positive Lebesgue–measure set for the planar three–body problem is a consequence of KAM theory (see [Ar]). According to M. Herman [He6] "what seems not an unreasonable question to ask (and possibly prove in a finite time with a lot of technical details) is"

Problem 3.3.2 (M. Herman) *If one of the masses $m_0 = 1$ and all the other masses $m_j << 1$ are sufficiently small, are there wandering domains in any neighborhood of fixed distinct circular orbits around the mass m_0 and moving in the same direction in a plane ?*

4. Linear Quasiperiodic Skew–Products, Spectral Theory and Hamiltonian Partial Differential Equations

4.1 Reducibility of skew–products

Let $\omega \in \mathbb{R}^d$, $\theta \in \mathbb{T}^d$ (here $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), $V : \mathbb{T}^d \to \mathbb{R}$ be a continuous function. The *time–continuous one–dimensional quasiperiodic Schrödinger equation*

$$-\ddot{y}(t) + \varepsilon V(\theta + \omega t) y(t) - E y(t) = 0 \ , \qquad (CQS)$$

as well as the *time–discrete quasiperiodic Schrödinger equation*

$$-(u_{n+1} + u_{n-1}) + \varepsilon V(\theta + n\omega) u_n - E u_n = 0 \qquad (DQS)$$

are both examples of *linear quasiperiodic* equations.
In the *periodic case* $\omega = 2\pi \left(\frac{p_1}{q}, \dots \frac{p_d}{q}\right)$, $\frac{p_i}{q} \in \mathbb{Q}$ for all $i = 1, \dots, l$, according to Floquet theory, the time evolutions of the solutions of (DQS) is determined by the eigenvalues of the matrix

$$A_q(\theta) = \prod_{j=0}^{q-1} \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon V(\theta + j\omega) - E \end{pmatrix} \in \mathrm{SL}\,(2, \mathbb{R}) \ ,$$

(for the (CQS), if $\Phi_t(\theta, E)$ denotes the monodromy matrix, the evolution is determined by $\Phi_q(\theta, E)$). Moreover one can make a change of variables which transforms the system to a constant coefficient system (this is the so–called *reducible* case, which was first considered by Lyapunov [NS]).
Not all quasiperiodic systems are reducible : indeed one has to overcome a small divisors problem in order to show that a quasiperiodic Schrödinger equation is reducible [DS], [MP]. For un up–to–date review we refer to [El2].
The problem of reducibility can be formulated more generally for any linear quasiperiodic *skew–product* system. Let G denote some matrix subgroup of $\mathrm{GL}\,(D, \mathbb{R})$, $\mathfrak{g}$ its Lie algebra. Let $A : \mathbb{T}^d \to G$ and $B : \mathbb{T}^d \to \mathfrak{g}$ be continuous functions. Then

$$(\theta, X) \mapsto (\theta + \omega, A(\theta)X) \ ,$$
$$\dot{X}(t) = B(\theta + t\omega)X \ ,$$

are (respectively) discrete and continuous time skew–products. Of course two examples are given by (DQS) (with $G = \mathrm{SL}\,(2, \mathbb{R})$) and (CQS) (in this case $B \in \mathfrak{sl}\,(2, \mathbb{R})$). Again, a skew–product system will be reducible if there exists a change of variable which transforms it to a constant coefficient skew–product, i.e. A (resp. B in the continuous case) is constant.

Problem 4.1 (L.H. Eliasson) [El1, El2] *Show that any generic analytic one–parameter family of skew–systems sufficiently close to constant coefficients is reducible for almost every parameter value.*

A result very close to the previous problem has been obtained in [Kr] for general compact matrix groups.

4.2 Spectral theory and integrated density of states

An important notion in the study of quasiperiodic Schrödinger equations is the *rotation number* (or *integrated density of states*, see [JM]) : for all E and for all initial data the following limit exists

$$\alpha(E) := \lim_{t \to \infty} \frac{1}{t} \arg(y(t) + i\dot{y}(t)) \ , \quad \text{for the (CQS)} \ .$$

If the system is reducible then the imaginary parts of the eigenvalues of the constant coefficient matrix $B \in \mathfrak{sl}\,(2, \mathbb{R})$ to which it is transformed are $\pm\alpha(E)$.
When ε is small the properties of the rotation number as E varies are known : it is an absolutely continuous function. Generically it is constant outside a Cantor set. When ε^{-1} is small "localized" solutions can exist.

Problem 4.2.1 (H. Eliasson) *What are the properties of the map $E \mapsto \alpha(E)$ when ε^{-1} is small ? Is it singular continuous ? Absolutely continuous ?*

Problem 4.2.2 (H. Eliasson) *Consider the (DQS) and suppose that there exist a measurable function $E_\infty(\alpha)$ and for a.e. α a function $\mathcal{U}_\alpha : \mathbb{T}^d \to \mathbb{R}$ ($\mathcal{U}_\alpha$ belongs* *to* $L^2(\mathbb{T}^d)$ *and is measurable in* α*) such that* $u_n = e^{in\alpha}\mathcal{U}_\alpha(\theta + n\omega)$, $n \in \mathbb{Z}$, $E = E_\infty(\alpha)$ *is a solution of (DQS). Assume that if for some subset* Y *of* $\mathbb{T}$ *one has* $E_\infty(Y) = 0$ *then the Haar measure of* Y *is also null. Is the spectrum purely absolutely continuous ?*

4.3 Nonlinear Hamiltonian PDEs

Here we consider nonlinear Hamiltonian Partial Differential Equations (PDEs) in the finite–volume case. This means that we are concerned with equations for (vector) functions $u = u(x,t)$ where the space variable x belongs to some bounded domain (with Dirichlet or periodic boundary conditions). The idea is to treat Hamiltonian PDEs as ordinary differential equations in some infinite–dimensional function spaces formed by functions of the space variable x and to assume that they can be written in the form

$$\dot{u}(t) = J\nabla H(u(t)) \;, \qquad (H)$$

where J is an anti self–adjoint operator in L^2 and ∇ denotes the L^2–gradient. Equation (H) is Hamiltonian w.r.t. the symplectic structure

$$\omega(v_1(x), v_2(x)) = \langle (-J)^{-1} v_1(x), v_2(x)\rangle_{L^2} \;.$$

A typical example is provided by the nonlinear Schrödinger equation

$$\dot{u} + i\Delta u - iu|u|^{2p} = 0 \qquad (NLS)$$

with $x \in \mathbb{T}^n$, $p \in \mathbb{N}$. (NLS) can be rewritten in Hamiltonian form taking $J = i$ and $H(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2p+2}\int |u|^{2p+2}$. Both $H(u)$ and the L^2–norm $\int |u|^2$ are preserved under the flow. The case $p = 1$ (cubic NLS) in one spatial dimension is integrable.

Another classical example of integrable Hamiltonian PDE is the Kortweg–de Vries (KdV) equation under zero–meanvalue periodic boundary conditions [GKM, No, La, DMN, MT]

$$\dot{u} = \frac{\partial}{\partial x}(-u_{xx} + 3u^2) \;, \quad x \in \mathbb{T}^1 \;, \quad \int_0^{2\pi} u(x,t)dx = 0 \;. \qquad (KdV)$$

In this case integrability means the following : for all $n \in \mathbb{N}$ the space $\mathcal{H} = \cap_{s>0} H_0^s(\mathbb{T}^1)$ (here $H_0^s(\mathbb{T}^1)$ is the Sobolev space formed by zero–meanvalue functions on $\mathbb{T}^1$) contains a smooth invariant $2n$–dimensional manifold $\mathcal{T}^{2n}$ such that

a) the restriction of (KdV) to $\mathcal{T}^{2n}$ defines a Liouville–Arnold integrable Hamiltonian system ;
b) $\mathcal{T}^{2n} \subset \mathcal{T}^{2m}$ if $m > n$;
c) $\cup \mathcal{T}^{2n}$ is dense in each $H_0^s(\mathbb{T}^1)$.

Moreover the invariant manifolds $\mathcal{T}^{2n}$ are filled with time–quasiperiodic solutions (the so–called *n–gap solutions*, see the lectures of S. Kuksin in this volume).

Also the Sine–Gordon (SG) equation under Dirichlet boundary conditions

$$u_{tt} = u_{xx} - \sin u \ , \quad u(t,0) = u(t,\pi) = 0 \ , \qquad (SG)$$

is integrable (i.e. it has n–gap solutions) but the manifolds $\mathcal{T}^{2n}$ have algebraic singularities and their union is proved to be dense only near the origin.
We refer to [B] and [Ku] for an introduction to the recent progress on the theory of nonlinear Hamiltonian PDEs.

Problem 4.3.1 (S. Kuksin) *Find a Lyapounov Theorem (not of KAM type) for Hamiltonian PDEs. I.e. prove that (under reasonable assumptions) most of time–periodic solutions from any non–degenerate one–parameter family persist under small Hamiltonian perturbations of the equation.*

For further information and some progress in this direction see [Ba2].

Problem 4.3.2 (S. Kuksin) *Find a non–perturbative way to obtain time–periodic solutions of an Hamiltonian PDE (the solutions should not be travelling waves).*

The main object of the Lectures of Kuksin in this volume is the proof of a KAM–type theorem wich assures the persistence under small Hamiltonian perturbations of most finite–gap solutions of Lax–integrable Hamiltonian PDEs like (KdV) or (SG). Nothing is known on the infinite–gap solutions (almost periodic solutions, see [AP] for an introduction). Following [MT] (see also [BKM]), one can ask :

Problem 4.3.3 (S. Kuksin) *Any Sobolev space $H_0^m(\mathbb{T}^1)$, $m \geq 1$, is filled by almost periodic solutions of (KdV). Do most of them persist under small Hamiltonian perturbations of the equation ?*

Presumably they do not.
From the KAM theorem described in Kuksin's lectures it also follows [BK] that most small amplitude finite–gap solutions of the φ^4 equation

$$u_{tt} = u_{xx} - mu + \gamma u^3 \ , \quad m, \gamma > 0 \ , \qquad (\varphi^4)$$

persist under small Hamiltonian perturbations. The (φ^4) equation can be indeed obtained from the (SG) equation developing $\sin u$ into Taylor series at the third order and truncating. If one sets $m = 0$ into (φ^4) the existence of small–amplitude time–quasiperiodic solutions is not known.

Problem 4.3.4 (S. Kuksin) *Construct small amplitude time quasiperiodic solutions for the massless φ^4 equation $u_{tt} = u_{xx} - u^3$, $x \in \mathbb{T}^1$.*

Let us consider again the (NS) equation. As we have seen the time–flow preserves the Hamiltonian and the L^2–norm of the solutions. One can study the following properties of the long–time behaviour of the solutions :

a) $\lim_{t\to\infty} \|u(t)\|_m = \infty$ (here m is sufficiently large) ;
a') $\limsup_{t\to\infty} \|u(t)\|_m = \infty$;

b) $u(t) \to 0$ weakly in H^s as $t \to \infty$;
b') $u(t_n) \to 0$ weakly in H^s for a suitable sequence $t_n \to \infty$.

Problem 4.3.5 (S. Kuksin) *Which properties among a)–b') hold for the solutions of (NS) with a typical initial condition u_0, $u(0,x) = u_0(x) \in H^m$?*

Note the following facts :

- a)–b') are *all* wrong for *any* u_0 if $p = 1$, $n = 1$;
- a)–b') are wrong for *some* u_0 (for any p, n) since (NS) has time–periodic solutions ;
- b) $\Rightarrow$ a) and b') $\Rightarrow$ a') due to the conservation of the L^2–norm.

Problem 4.3.6 (S. Kuksin) *Does (NS) have an invariant measure in H^s such that its support (in H^s) is a set with non–empty interior (here s is sufficiently large) ?*

Not that a positive answer to Problem 4.9 implies that a) is wrong almost everywhere with respect to the measure.

Problem 4.3.7 (S. Kuksin) *In the spirit of Nekhoroshev's theorem one can ask the following : does an arbitrary solution of a ε–perturbed KdV equation remain ε^a–close to some finite gap torus during a time interval $0 \le t \le T_\varepsilon$ with $T_\varepsilon \ge C_M \varepsilon^{-M}$ for all $M > 0$?*

Probably a crucial step is to prove this statement for $T = \varepsilon^{-2}$. A first result in this direction has been obtained in [Ba1].

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