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Hyperfunctions, formal groups and generalized Lipschitz summation formulas

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ABSTRACT

A construction relating the theory of hyperfunctions with the theory of formal groups and generalizations of the classical Lipschitz summation formula is proposed. It involves new polylogarithmic rational functions constructed via the Fourier expansion of certain sequences of Bernoulli-type polynomials, related to the Lazard formal group. Related families of one-dimensional hyperfunctions are also constructed.

1. Introduction

The purpose of this paper is to establish a connection between the theory of hyperfunctions on one side and some topics of mathematical physics and analytic number theory on the other side. Precisely, we will provide a natural setting which allows to construct new classes of hyperfunctions by means of a generalization of the Lipschitz summation formula to negative powers. In this construction, the notion of formal group plays a special role.

In complex analysis, hyperfunctions have been introduced as generalizations of functions in order to describe a jump from one holomorphic function to another, in the presence of a boundary. Hyperfunctions are relevant in several contexts of functional analysis, in particular, in connection with the existence of weak solutions of the Cauchy problem for Laplacian operators [1,2], in Fourier analysis, in microlocal analysis [3], and in mathematical physics in relation with small divisor problems [4] and with the theory of $(2+1)$-dimensional integrable systems [5]. Hyperfunctional cohomology groups have also been related to automorphic forms and period functions [6].

We recall that the classical Lipschitz summation formula gives the Fourier series expansion of the periodic analytic function obtained by summation over integer translates of the power z^{-k}, where $z \in \mathbb{C}^+$ (the complex upper half plane) and $k > 1$ is a positive integer. The case $k = 1$ has been studied in [7].

Each of the generalized formulas we propose will provide a hyperfunctional equation involving a specific two-variable polylogarithm series.

The relevance of formal groups first of all relies on their close connection with the theory of Lie groups. Indeed, it is well known that, over a field of characteristic zero, there exists an equivalence of categories, given by

$$\text{Lie Algebras} \longleftrightarrow \text{Formal Groups}.$$

Also, if A is the valuation ring of a complete ultrametric field F, a commutative diagram of functors holds [8]:

$$\begin{array}{ccc} \text{Analytic Groups}/F & \longleftrightarrow & \text{Formal Groups}/F \\ & \nwarrow\ \nearrow & \\ & \text{Formal Groups}/A & \end{array}$$

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More specifically, any n-dimensional formal group law defines an n-dimensional Lie algebra over the same valuation ring A by means of the identification

$$[x, y] = F_2(X, Y) - F_2(Y, X), \tag{1}$$

where $F_2(X, Y)$ denotes the quadratic part of the formal group law $F(X, Y)$.

However, in a field of characteristic $p \neq 0$, the equivalence (1) no longer holds, and formal groups can be thought of as intermediate objects between Lie groups and Lie algebras.

A first result of this paper is the connection we propose between the theory of formal groups and that of hyperfunctions, that relies on the following. We define suitable sequences of Appell polynomials of Bernoulli type: they share several arithmetic properties with the Bernoulli polynomials, including certain famous congruences (see the Appendix). They can be viewed as polynomial realizations of the universal Bernoulli polynomials [9], related with the Lazard universal formal group. Indeed, each polynomial sequence corresponds to a choice of the coefficients appearing in the definition of the universal formal group. Their periodic versions provide primitives of the periodic delta function. To each sequence we will associate a natural extension of the notion of polylogarithm of order n defined in the unit disk. We call this new function an *extended delta rational function* (see Definition 11). It is a two-variable Dirichlet series, extending to the whole Riemann sphere as a meromorphic function. The case of the classical Bernoulli polynomials corresponds to the standard polylogarithms, and it is treated thoroughly. Then we will show that the new Appell polynomial sequences we construct, $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$ (depending on certain parity properties) and the extended delta rational functions introduced in this work satisfy interesting hyperfunctional equations, as we will prove in Lemmas 6 and 12.

The main result of the paper is of a number theoretical nature. The hyperfunctional equations so obtained allow to construct the generalized Lipschitz summation formulas, as we prove in Theorem 9, and in a more general setting in formula (52). For instance, in the case of polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ our summation formula reads

$$\sum_{k\in\mathbb{Z}} \varphi_{\overline{P}_n}(\tau + k) = 2i\,(2\pi i)^{-n} \begin{cases} \Delta_{-n}(q) & \text{if } |q| < 1, \text{ i.e. } \Im\tau > 0 \\ (-1)^{n-1}\,\Delta_{-n}\left(q^{-1}\right) & \text{if } |q| > 1, \text{ i.e. } \Im\tau < 0, \end{cases} \tag{2}$$

where $\overline{P}_n$ are the hyperfunctions associated to the polynomial P_n and $\varphi_{\overline{P}_n}$ is the function in $O^1\left(\overline{\mathbb{C}} \setminus [0, 1]\right)$ representing $\overline{P}_n$.

The future research plans include an extension of the proposed construction to the multidimensional case (see also [10] for a different generalization), as well as possible applications of the proposed Lipschitz formulas to the study of Eisenstein series and periods of modular forms.

The paper is organized as follows. In Section 2, the theory of Hyperfunctions is recalled briefly. In Section 3, the notion of delta rational function is studied and a generalization of the Lipschitz summation formula is proved. In Section 4, the theory of formal groups is discussed. In Section 5 it is proposed a generalization of the previous approach, related to the Lazard group. The extended delta rational functions and Lipschitz-type summation formulas are introduced. In the Appendix, interesting congruences related to our construction are reviewed.

2. Hyperfunctions: some preliminaries

In this section, we will provide a brief and self-consistent introduction to the theory of hyperfunctions of a single variable, following closely [1], Chapter IX of [2], [11] and [4]. For a more extensive treatment and further details, the reader is invited to consult these books, as well as [12] for interesting applications.

Let us denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}$, and denote by $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper and lower half planes of $\mathbb{C}$.

Definition 1. The space of hyperfunctions $\mathcal{B}$ on the real line $\mathbb{R}$ is

$$\mathcal{B}(\mathbb{R}) := H^1_{\mathbb{R}}(\mathbb{C}, \mathcal{O}), \tag{3}$$

i.e. the first sheaf cohomology group on $\mathbb{R}$.

Now, since $\mathbb{C}^+ \cup \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}$, we have the following decomposition:

$$H^1_{\mathbb{R}}(\mathbb{C}, \mathcal{O}) = \left[H^0\left(\mathbb{C}^+, \mathcal{O}\right) \oplus H^0\left(\mathbb{C}^-, \mathcal{O}\right)\right] / H^0(\mathbb{C}, \mathcal{O}). \tag{4}$$

In other words, since $H^0(\mathcal{O})$ represents nothing but the global sections of the sheaf (i.e. holomorphic functions), a hyperfunction can be thought of as a pair of holomorphic functions on the upper and lower half planes respectively, modulo an entire function. This geometric definition immediately generalizes to the case of hyperfunctions in several variables. If $\Omega \subset \mathbb{R}$ is an open set, and U an arbitrary complex neighborhood of Ω, then clearly

$$\mathcal{B}(\Omega) = H^1_{\Omega}(U, \mathcal{O}). \tag{5}$$

Still denoting by $\mathcal{O}(U)$ the space of holomorphic functions in U, another equivalent definition is

$$\mathcal{B}(\Omega) := \varinjlim U \supset \Omega\, \mathcal{O}(U \setminus \Omega)\ /\mathcal{O}(U) \tag{6}$$

where the inductive limit with respect to the family of complex neighborhoods $U \supset \Omega$ is considered. A hyperfunction $f(x)$ is therefore an equivalence class $[F(z)]$, whose representative is $F(z) \in \mathcal{O}(U \setminus \Omega)$. The representative $F(z)$ is said to be a defining function of $f(x)$. Since

$$\Omega \subset U, \qquad U \setminus \Omega = U_+ \cup U_-, \qquad U_\pm = U \cap \{\Im z \lessgtr 0\},$$

often the following boundary-value representation is used:

$$f(x) = F_+(x + i0) - F_-(x - i0),$$

with $F_\pm(z) = F(z)|_{U_\pm}$.

Alternatively, one can construct a theory of hyperfunctions based on analytic functionals. Let $K \subset \mathbb{C}$ be a non empty compact set, and denote by A the space of entire analytic functions in $\mathbb{C}$.

Definition 2. The space $A'(K)$ of the analytic functionals carried by K is the space of linear forms u acting on A such that for every neighborhood V of K there is a constant $C_V > 0$ such that

$$|u(\varphi)| \leq C_V \sup_V |\varphi|, \quad \forall \varphi \in A. \tag{7}$$

Observe that $A'(K)$ is a Fréchet space, since a seminorm is associated to each neighborhood V of K. One can define

$$\mathcal{B}(\Omega) := A'\left(\overline{\Omega}\right)/A'(\partial\Omega). \tag{8}$$

It is interesting to notice that the space of hyperfunctions $\psi \in B(\Omega)$ with compact support $K \subset \Omega$ can be identified with analytic functionals in $A'(\mathbb{R})$ with support K. Indeed, an analytic functional u on $\cup_{i=1}^r K_i$ can always be decomposed into a sum $u = u_1 + \cdots + u_r$, with each of the functionals $u_j \in A'\left(K_j\right)$. Consequently, since supp $\psi \subset K \cup \partial\Omega$, the contribution of ψ on $\partial\Omega$ can be factored out and ψ is identified with a uniquely defined functional with support in Ω.

Therefore, we can also think of $A'(K)$ as the space of hyperfunctions with support in K. The link between the two approaches to the theory of hyperfunctions is now provided by the following lemma. Let $\mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$ denotes the space of holomorphic functions on $\overline{\mathbb{C}} \setminus K$ and vanishing at infinity.

Lemma 3. *The spaces $A'(K)$ and $\mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$ are canonically isomorphic. To each $u \in A'(K)$ there corresponds a function $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$ given by*

$$\varphi(z) = u(c_z), \quad \forall z \in \mathbb{C} \setminus K,$$

where $c_z(x) = \frac{1}{\pi}\frac{1}{x-z}$. Conversely, to each $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus K)$ there corresponds the hyperfunction

$$u(\psi) = \frac{i}{2\pi}\int_\gamma \varphi(z)\psi(z)dz, \quad \forall \psi \in A, \tag{9}$$

where γ is any piecewise $\mathcal{C}^1$ path winding around K in the positive direction.

For future purposes, we also briefly describe *periodic hyperfunctions*. Let $\mathbb{T}^1 = \mathbb{R} \setminus \mathbb{Z} \subset \mathbb{C} \setminus \mathbb{Z}$. A hyperfunction on $\mathbb{T}^1$ is a linear functional Ψ on the space $\mathcal{O}\left(\mathbb{T}^1\right)$ of functions analytic in a complex neighborhood of $\mathbb{T}^1$ such that for all neighborhood V of $\mathbb{T}^1$ there exists a constant $C_V > 0$ such that

$$|\Psi(\Phi)| \leq C_V \sup_V |\Phi|, \quad \forall \Phi \in \mathcal{O}(\mathcal{V}). \tag{10}$$

We will denote by $A'\left(\mathbb{T}^1\right)$ the Fréchet space of hyperfunctions with support in $\mathbb{T}^1$. Let $\mathcal{O}_\Sigma$ denote the complex vector space of holomorphic functions $\Phi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, 1-periodic, bounded at $\pm i\infty$ and such that $\Phi(\pm i\infty) := \lim_{\Im z \to \pm\infty} \Phi(z)$ exist and satisfy $\Phi(+i\infty) = -\Phi(-i\infty)$.

Lemma 4. *The spaces $A'\left(\mathbb{T}^1\right)$ and $\mathcal{O}_\Sigma$ are canonically isomorphic: to each $\Psi \in A'\left(\mathbb{T}^1\right)$ there corresponds a function $\Phi \in \mathcal{O}_\Sigma$ given by*

$$\Phi(z) = \Psi(C_z), \quad \forall z \in \mathbb{C} \setminus K,$$

where $C_z(x) = \cot\pi(x - z)$. Conversely, to each $\Phi \in \mathcal{O}_\Sigma$ there corresponds the hyperfunction

$$\Psi(\Xi) = \frac{i}{2}\int_\Gamma \Phi(z)\Xi(z)dz, \quad \forall \Xi \in A'\left(\mathbb{T}^1\right), \tag{11}$$

where Γ is any piecewise $\mathcal{C}^1$ path winding around a closed interval $I \subset \mathbb{R}$ of length 1 in the positive direction.

Given a compactly supported hyperfunction, making infinitely many copies of it translating its support leads to a periodic hyperfunction. This point of view is systematically exploited in [4], from which the following commutative diagram is taken:

$$\begin{array}{lll} A'([0,1]) & \longrightarrow & O^1\left(\overline{\mathbb{C}}\setminus[0,1]\right) \\ \Sigma_{\mathbb{Z}}\downarrow & & \downarrow\Sigma_{\mathbb{Z}} \\ A'\left(\mathbb{T}^1\right) & \longrightarrow & \mathcal{O}_{\Sigma}. \end{array} \tag{12}$$

The horizontal lines are the above mentioned isomorphisms, and $\sum_{\mathbb{Z}}$ denotes the summation over integer translates.

3. Polylogarithms and summation formulas

3.1. Analytic properties

In this section we discuss some number theoretical properties of classical polylogarithms [13,14] of order n, that we also call *delta rational functions*. We will show that they are related with Bernoulli polynomials via the Fourier expansion formula (15); they also enter a generalized Lipschitz summation formula providing new classes of hyperfunctions.

Extended delta rational functions will be introduced in Section 5.2.

Definition 5. A delta rational function is a polylogarithm of integer order n, i.e. is the series expressed for any $n \in \mathbb{Z}$ and $q \in \mathbb{D}$ by

$$\delta_n(q) = \sum_{k=1}^{\infty} k^n q^k. \tag{13}$$

Observe that, if $n \geq 0$, δ_n extends to the whole Riemann sphere as a rational function of degree $n+1$ with just a simple pole of order $n+1$ at $q=1$. We have that $\delta_{-1}(q) = -\log(1-q)$. If $n \leq -1$ then δ_n extends to the whole $\mathbb{C}\setminus[1,+\infty)$ and as a multivalued function to the whole $\mathbb{C}\setminus\{0,1,\infty\}$. Indeed, since

$$q\partial_q\delta_n(q) = \delta_{n+1}(q),$$

one can define the analytic continuation of δ_n. For instance, the continuation of δ_{-2} is obtained by means of the integral formula

$$\delta_{-2}(q) = -\int_0^q \frac{\log(1-t)}{t}dt = \int_0^q\left(\int_0^t \frac{d\zeta}{1-\zeta}\right)\frac{dt}{t}.$$

Note that $[1,+\infty)$ is a branch cut. For all $n \in \mathbb{Z}$ and $q \in \mathbb{D}$, one has

$$\delta_n\left(q^k\right) = k^{-1-n}\sum_{\Lambda^k=1}\delta_n(\Lambda q), \tag{14}$$

where Λ denotes a k-th root of unity. The equality extends to the closed disk if $n \leq -2$. One can directly prove the following result.

Lemma 6. *The fundamental inversion equation*

$$\delta_n(q) + (-1)^n\delta_n\left(q^{-1}\right) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\dfrac{(2\pi i)^{-n}}{(-n)!}B_{-n}\left(\dfrac{\log q}{2\pi i}\right) & \text{if } n \leq 0, \end{cases} \tag{15}$$

holds for all $q \neq 1$ if $n \geq 0$, and q in $\mathbb{C}\setminus[0,+\infty]$ if $n \leq -1$. Here B_k is the k-th Bernoulli polynomial.

3.2. Periodic hyperfunctions

The inversion relation (15) has a beautiful interpretation in terms of hyperfunctions, as we will show below.

Definition 7. The periodic Bernoulli functions and distributions are expressed by

$$\widetilde{B}_n(x) = \begin{cases} -\dfrac{B_n(x-[x])}{n!} & \text{if } n \geq 1 \\ -1+\delta_{\mathbb{T}}(x) & \text{if } n = 0 \\ \delta_{\mathbb{T}}^{(-n)}(x) & \text{if } n < 0 \end{cases}, \quad x \in \mathbb{R} \tag{16}$$

where $[x]$ is the integer part of x, $\delta_{\mathbb{T}}$ is the periodic delta distribution and $\delta_{\mathbb{T}}^{(k)}$ its derivative of order $k \geq 0$.

Explicitly,

$$\delta_{\mathbb{T}}(x) = \sum_{k=-\infty}^{+\infty} e^{2\pi ikx}. \tag{17}$$

From the well-known property of the periodic Bernoulli functions

$$\frac{d^n}{dx^n}\left(B_n\left(x-[x]\right)\right) = n!\left(1-\delta_{\mathbb{T}}\right), \tag{18}$$

we see that we can consider the Bernoulli functions as primitives of the periodic delta function. It is immediate to check that the following hyperfunctional equation holds.

Proposition 8. *Let $q^{\pm} = e^{2\pi i(x\pm i0)}$, $x \in \mathbb{R}$. For all $n \in \mathbb{Z}$ we have*

$$\delta_n\left(q^+\right) + (-1)^n \delta_n\left(\left(q^-\right)^{-1}\right) = (2\pi i)^{-n} \widetilde{B}_{-n}(x). \tag{19}$$

Note that on both sides one can apply derivatives $q\partial_q$ and $(2\pi i)^{-1}\partial_x$.

3.3. Generalized Lipschitz summation formula

As is well known, the classical Lipschitz formula is a consequence of the Poisson summation formula, relating sums over integers of pairs of Fourier transforms. It states that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1}e^{2\pi irz}, \quad z \in \mathbb{H},\ k \in \mathbb{Z}_{\geq 2} \tag{20}$$

(for a proof of (20), see, for instance, Zagier, Chapter 4 in [15]).

In this section, we will give a hyperfunctional generalization of (20). We make use of the fact that periodic hyperfunctions with compact support are the result of a summation over integer translates of hyperfunctions with support on [0, 1], according to the commutative diagram (12). By applying the results described in the previous section, we get a functional version of the Lipschitz formula.

Theorem 9. *For all $n \in \mathbb{Z}$ we have*

$$\sum_{k\in\mathbb{Z}} \varphi_{\overline{B}_n}(\tau+k) = 2i\,(2\pi i)^{-n}\begin{cases}\delta_{-n}(q) & \text{if } |q|<1, \text{ i.e. } \Im\tau>0\\ (-1)^{n-1}\delta_{-n}\left(q^{-1}\right) & \text{if } |q|>1, \text{ i.e. } \Im\tau<0,\end{cases} \tag{21}$$

whereas the usual Lipschitz formula corresponds to $n \leq -1$. Here $\overline{B}_n$ is the restriction to [0, 1] of $\widetilde{B_n}$ and $\varphi_{\overline{B}_n}$ is the function in $O^1\left(\overline{\mathbb{C}} \setminus [0,1]\right)$ which represents the hyperfunction $\overline{B}_n$

$$\varphi_{\overline{B}_n}(\tau) = \left\langle \overline{B}_n, \frac{1}{\pi}\frac{1}{x-\tau}\right\rangle_{[0,1]}. \tag{22}$$

Proof. It is enough to remark that an inverse of the operator $\sum_{\mathbb{Z}}$ on the hyperfunctions is just the restriction of a periodic hyperfunction to the interval [0, 1]. Thus the r.h.s. of (19) reads

$$\sum_{k\in\mathbb{Z}} (2\pi i)^{-n}\,\overline{B}_{-n}(x+k) \tag{23}$$

and from (15) and (19) one obtains the desired formula by considering the two associated holomorphic functions. □

One of the main results we obtain is the explicit expression for formula (22), which reads

$$\varphi_{\overline{B}_n}(\tau) = \begin{cases}\left\langle \delta^{(-n)}(x), \frac{1}{\pi}\frac{1}{x-\tau}\right\rangle_{[0,1]} = (-1)^{-n+1}\frac{(-n)!}{\pi\tau^{-n+1}} & \text{if } n<0\\ \left\langle -1+\delta(x), \frac{1}{\pi}\frac{1}{x-\tau}\right\rangle_{[0,1]} = -\frac{1}{\pi\tau}\left[1+\tau\log\left(1-1/\tau\right)\right] & \text{if } n=0\\ \left\langle -\frac{1}{n!}B_n(x), \frac{1}{\pi}\frac{1}{x-\tau}\right\rangle_{[0,1]} = -\frac{1}{\pi n!}\left[B_n(\tau)\log\left(1-1/\tau\right)+R_n(\tau)\right] & \text{if } n>0,\end{cases} \tag{24}$$

where R_n is the polynomial of degree n such that $B_n(\tau)\log(1-1/\tau)+R_n(\tau) \in O^1\left(\overline{\mathbb{C}}\setminus[0,1]\right)$. Here are the first six polynomials

$$R_1(\tau)=1,\qquad R_2(\tau)=\tau-\frac{1}{2},\qquad R_3(\tau)=\tau^2-\tau+\frac{1}{12},$$
$$R_4(\tau)=\tau^3-\frac{3}{2}\tau^2+\frac{1}{3}\tau+\frac{1}{12},\qquad R_5(\tau)=\tau^4-2\tau^3+\frac{3}{4}\tau^2+\frac{1}{4}\tau-\frac{13}{360},$$
$$R_6(\tau)=\tau^5-\frac{5}{2}\tau^4+\frac{4}{3}\tau^3+\frac{1}{2}\tau^2-\frac{13}{60}\tau-\frac{7}{120}. \tag{25}$$

Remark. One could object that we should have used the simplest Appell sequence $\{x^n\}_{n\in\mathbb{N}}$ and their associated hyperfunctions on $[0, 1]$ to write the generalized Lipschitz summation formula. But this would trivially give zero, since

$$x^n=\frac{1}{n+1}\left[B_{n+1}(x+1)-B_{n+1}(x)\right] \tag{26}$$

and the function $B_{n+1}(x+1)-B_{n+1}(x)$ clearly belongs to the kernel of $\sum_{\mathbb{Z}}$. Instead, other less trivial Appell sequences can be used to provide useful generalizations of the construction we proposed, as will be illustrated in the subsequent sections.

4. The connection with the Lazard formal group

We describe here briefly an interesting connection between formal groups, hyperfunctions, and the so-called universal Bernoulli polynomials.

Given a commutative ring with identity R, we will denote by $R[\![x_1, x_2, \ldots]\!]$ the ring of formal power series in $x_1, x_2, \ldots$ with coefficients in R. Following [16,17], we recall that a commutative one-dimensional formal group law over R is a two-variable formal power series $\Phi(x, y) \in R[\![x, y]\!]$ such that

(1) $\Phi(x,0)=\Phi(0,x)=x$

and

(2) $\Phi(\Phi(x,y),z)=\Phi(x,\Phi(y,z))$.

When $\Phi(x,y) = \Phi(y,x)$, the formal group is said to be commutative. The existence of an inverse formal series $\varphi(x) \in R[\![x]\!]$ such that $\Phi(x,\varphi(x))=0$ follows from the previous definition.

Let us consider the polynomial ring $\mathbb{Q}[c_1, c_2, \ldots]$ and the formal group logarithm

$$F(s)=s+c_1\frac{s^2}{2}+c_2\frac{s^3}{3}+\cdots \tag{27}$$

Let $G(t)$ be the associated inverse series, i.e. the formal group exponential

$$G(t)=t-c_1\frac{t^2}{2}+\left(3c_1^2-2c_2\right)\frac{t^3}{6}+\cdots \tag{28}$$

so that $F(G(t))=t$.

The formal group law related to these series is provided by

$$\Phi(s_1,s_2)=G(F(s_1)+F(s_2))$$

and it represents the so called *Lazard Universal Formal Group* [16]. It is defined over the Lazard ring L, i.e. the subring of $\mathbb{Q}[c_1, c_2, \ldots]$ generated by the coefficients of the power series $G(F(s_1)+F(s_2))$ (following [17]).

In [9], the *universal Bernoulli polynomials* $B_n^G(x, c_1, \ldots, c_k, \ldots) \equiv B_k^G(x)$ *related to the formal group exponential* G have been introduced. They are defined by

$$\frac{t}{G(t)}e^{xt}=\sum_{n\geq 0}B_n^G(x)\frac{t^k}{n!},\quad x\in\mathbb{R}. \tag{29}$$

The corresponding numbers by construction coincide with the universal Bernoulli numbers discovered in [18], and generated by

$$\frac{t}{G(t)}=\sum_{n\geq 0}\widehat{B}_n\frac{t^k}{n!},\quad x\in\mathbb{R}. \tag{30}$$

In algebraic topology, the numbers c_i are identified with the cobordism classes of $\mathcal{C}P^N$. Consider a 2-periodic generalized cohomology theory E, with a complex orientation in degree zero. In this case, given a stably almost complex manifold M, whose fundamental class $[M] \in E^0$, the formal logarithm defines a canonical formal group law over E^0.

In particular, when $a = 1$, $c_i = (-1)^i$, then $F(s) = \log(1+s)$, $G(t) = e^t - 1$, and the universal Bernoulli polynomials and numbers reduce to the standard ones. Many other examples of Bernoulli-type polynomials considered in the literature are obtaining by specializing the rational coefficients c_i. The reasons to consider such a generalization of Bernoulli polynomials are manifold. First, to any choice of the coefficients $c_1, c_2, \ldots$ there corresponds a sequence of polynomials of Appell type sharing with the standard Bernoulli polynomials many algebraic and combinatorial properties [19]. In particular, the associated numbers satisfy universal Clausen–von Staudt and Kummer congruences, as shown in the Appendix. Also, in the same way as the Riemann zeta function is associated with the Bernoulli polynomials, it is possible to relate the polynomials (29) with a large class of absolutely convergent L-series. Their values at negative integers correspond to the generalized Bernoulli numbers, and they possess as well several interesting number-theoretical properties [9,20]. Observe that the polynomials (39) belong to the class (29), i.e. the coefficient c_i are all rational, if the functions $g(t)$ verify the further condition $g^{(n)}(0) \in \mathbb{Q}$ for any n. This is the case, for instance, for the proposed examples of the polynomials (40) and (42). As an immediate consequence, the numbers generated by

$$\frac{t}{(e^t - 1)\, g(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \tag{31}$$

with the prescribed conditions on $g(t)$ do satisfy by construction the Clausen–von Staudt and Kummer congruences and many others. These considerations enable us to associate one-variable formal groups with suitable classes of hyperfunctions, as a consequence of the previous construction.

5. The general case: sequences of Appell polynomials, generalized polylogarithms and hyperfunctions

5.1. Appell sequences

By analogy with the theory developed in the previous section, we study a more general class of polynomials of Bernoulli type, with the aim of constructing associated families of hyperfunctions. The main requirement is that they belong to the class of Appell polynomials, i.e. polynomials satisfying the property

$$\frac{d}{dx} A_n(x) = n A_{n-1}(x), \tag{32}$$

with the condition

$$A_0(x) = \text{const}. \tag{33}$$

Lemma 10. *The Fourier series expansion of the polynomials (32), for $0 < x < 1$ and $n \geq 1$ has the form*

$$A_n(x) = \sum_{k=-\infty}^{\infty} \gamma_k(n) e^{2\pi i k x}, \tag{34}$$

with

$$\gamma_k(n) = \int_0^1 A_n(t) e^{-2\pi i k t}\, dt. \tag{35}$$

We get

$$A_n(x) = -n! \sum_{k=1}^{\infty} \left[\sum_{j=1}^{n} \frac{1}{j!} \frac{\varphi_j}{(2\pi i k)^{n+1-j}} e^{2\pi i k x} + \sum_{j=1}^{n} (-1)^{n+1-j} \frac{1}{j!} \frac{\varphi_j}{(2\pi i k)^{n+1-j}} e^{-2\pi i k x} \right], \tag{36}$$

with $0 < x < 1$, and $\varphi_j = A_j(1) - A_j(0)$, $j = 1, \ldots, n$. The standard Bernoulli polynomials correspond to the case $\varphi_1 = 1, \varphi_j = 0,\ j = 2, 3, \ldots$.

Proof. It is a direct consequence of the conditions (32)–(35) and of the formula of integration by parts. □

If $\varphi_j = 0$ for j even or j odd, then formula (36) can be written in a more compact form. The corresponding polynomial sequences will be denoted by $\{p_n(x)\}_{n\in\mathbb{N}}$ and $\{q_n(x)\}_{n\in\mathbb{N}}$, respectively.

(a) If $\varphi_j = 0$ for j even, we define

$$p_n(x) =: -n! \sum_{k=1}^{\infty} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n} \frac{1}{j!} \frac{\varphi_j}{(2\pi i k)^{n+1-j}} \left[e^{2\pi i k x} + (-1)^n e^{-2\pi i k x} \right], \quad 0 < x < 1. \tag{37}$$

(b) If $\varphi_j = 0$ for j odd, we introduce

$$q_n(x) =: -n! \sum_{k=1}^{\infty} \sum_{\substack{j=1 \\ j \text{ even}}}^{n} \frac{1}{j!} \frac{\varphi_j}{(2\pi ik)^{n+1-j}} \left[e^{2\pi ikx} + (-1)^{n+1} e^{-2\pi ikx}\right], \quad 0 < x < 1. \tag{38}$$

Remark. The previous conditions on φ_j are not particularly restrictive. Indeed, one can easily construct infinitely many polynomial sequences possessing the prescribed parity properties. A possible generating function for the polynomials defined in the case (a), for instance, has the form

$$\frac{te^{xt}}{(e^t - 1)\, g(t)} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \tag{39}$$

where $g(t)$ is any real analytic function in a suitable domain, satisfying the parity condition $g(t) = g(-t)$.

Examples.

(a) Take

$$\frac{t^2 e^{xt}}{(e^t - 1)\sin t} = \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!}. \tag{40}$$

We easily deduce

$$a_0(x) = 1, \qquad a_1(x) = x - \frac{1}{2}, \qquad a_2(x) = x^2 - x + \frac{1}{2},$$
$$a_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{2},$$
$$a_4(x) = x^4 - 2x^3 + 3x^2 - 2x + \frac{23}{30}, \ldots. \tag{41}$$

(b) Consider the generating function

$$\frac{(1 - \cos t + t)e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}. \tag{42}$$

We immediately get

$$b_0(x) = 1, \qquad b_1(x) = x, \qquad b_2(x) = x^2 - \frac{1}{3},$$
$$b_3(x) = x^3 - x,$$
$$b_4(x) = x^4 - 2x^2 + \frac{7}{15}. \tag{43}$$

5.2. Extended delta rational functions

Let $n \in \mathbb{Z}$, $q \in \mathbb{D}$. We introduce here a straightforward generalization of the notion of polylogarithm of order n adapted to the chosen Appell polynomials: the extended delta rational function $\Delta_n(q)$.

Definition 11. The extended delta rational function, for any $n \in \mathbb{Z}$ and $q \in \mathbb{D}$, is defined by

$$\Delta_n(q) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} k^n q^k & n > 0, \\ \displaystyle\sum_{k=1}^{\infty} a_k(-n) q^k & n \leq 0, \end{cases} \tag{44}$$

where

$$a_k(n) = \sum_{\substack{j \\ j \text{ even or odd}}}^{n} \frac{1}{j!} \frac{\varphi_j}{k^{n+1-j}}. \tag{45}$$

In Eq. (45), the summation should be understood either over the even values of j or over the odd ones, depending on the choice of the polynomials (37) or (38), respectively. The above definition is motivated by the following result, which provides an extension of the construction proposed in Section 3. Our aim is to obtain the hyperfunctional equations associated to the proposed Appell polynomials. As a consequence of the Fourier expansion (36) and of relations (37)–(38) we get the relation between extended delta rational functions and Appell sequences.

Lemma 12. *The following inversion equations, generalizing relation (15), hold:*

$$\Delta_n(q) + (-1)^n \Delta_n\left(q^{-1}\right) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\dfrac{(2\pi i)^{-n}}{(-n)!} P_{-n}\left(\dfrac{\log q}{2\pi i}\right) & \text{if } n \leq 0, \end{cases} \tag{46}$$

and

$$\Delta_n(q) + (-1)^{n+1} \Delta_n\left(q^{-1}\right) = \begin{cases} 0 & \text{if } n \geq 1 \\ -\dfrac{(2\pi i)^{-n}}{(-n)!} Q_{-n}\left(\dfrac{\log q}{2\pi i}\right) & \text{if } n \leq 0, \end{cases} \tag{47}$$

where P_n and Q_n are respectively the Appell polynomials (37) and (38). The inversion relations hold for all $q \neq 1$ if $n \geq 0$, whereas for $n \leq -1 q$ can be taken in $\mathbb{C} \setminus [0, +\infty]$.

By analogy with formulas (16), the associated periodic functions and distributions are defined as

$$\widetilde{P}_n(x) = \begin{cases} -\dfrac{P_n(x - [x])}{n!} & \text{if } n \geq 1 \\ -1 + \delta_{\mathbb{T}}(x) & \text{if } n = 0 \\ \delta_{\mathbb{T}}^{(-n)}(x) & \text{if } n < 0 \end{cases} \tag{48}$$

with an analogous definition for the case of polynomials $Q_n(x)$. For all $n \in \mathbb{Z}$ we have the hyperfunctional equations

$$\Delta_n\left(q^+\right) + (-1)^n \Delta_n\left(\left(q^-\right)^{-1}\right) = (2\pi i)^{-n} \widetilde{P}_{-n}(x), \tag{49}$$

and

$$\Delta_n\left(q^+\right) + (-1)^{n+1} \Delta_n\left(\left(q^-\right)^{-1}\right) = (2\pi i)^{-n} \widetilde{Q}_{-n}(x). \tag{50}$$

The proof of these relations is again a direct consequence of the previous definitions. Also, by denoting with $\overline{P}_n$ the restriction to $[0, 1]$ of $\widetilde{P}_n$ and with $\varphi_{\overline{P}_n}$ the function in $O^1\left(\overline{\mathbb{C}} \setminus [0, 1]\right)$ which represents the hyperfunction $\overline{P}_n$, namely

$$\varphi_{\overline{P}_n}(\tau) = \left\langle \overline{P}_n, \frac{1}{\pi} \frac{1}{x - \tau} \right\rangle_{[0,1]}, \tag{51}$$

we find, for any choice of the sequence $\{P_n(x)\}_{n \in \mathbb{N}}$ and $\{Q_n(x)\}_{n \in \mathbb{N}}$ our main result, i.e. the *generalized Lipschitz formula*

$$\sum_{k \in \mathbb{Z}} \varphi_{\overline{P}_n}(\tau + k) = 2i\,(2\pi i)^{-n} \begin{cases} \Delta_{-n}(q) & \text{if } |q| < 1, \text{ i.e. } \Im\tau > 0 \\ (-1)^{n-1} \Delta_{-n}\left(q^{-1}\right) & \text{if } |q| > 1, \text{ i.e. } \Im\tau < 0, \end{cases} \tag{52}$$

and the corresponding formula for $\overline{Q}_n$ (again the usual Lipschitz formula corresponds to $n \leq -1$). The explicit computation of $\varphi_{\overline{P}_n}(\tau)$ and $\varphi_{\overline{Q}_n}(\tau)$ is completely analogous to the proposed construction for the Bernoulli polynomials and we will not repeat it here.

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Appendix. Congruences related with the Lazard group

The Clausen–von Staudt congruence [21], one of the most beautiful of mathematics, states that

$$B_n + \sum_{p|n} \frac{1}{p} \in \mathbb{Z}, \tag{53}$$

where B_n denotes the n-th Bernoulli number. This proves the strict link between Bernoulli numbers and prime numbers. Many generalizations of this result have been obtained in the literature in the last decades. In an attempt to clarify the deep connection between these congruences and algebraic topology, in [18] Clarke proposed the notion of universal Bernoulli numbers $\widehat{B_n}$, defined as (30), and proved the remarkable universal von Staudt congruence.

If n is even, we have

$$\widehat{B_n} \equiv - \sum_{\substack{p-1|n \\ p \text{ prime}}} \frac{c_{p-1}^{n/(p-1)}}{p} \mod \mathbb{Z}[c_1, c_2, \ldots]. \tag{54}$$

If n is odd and greater than 1, we have

$$\widehat{B_n} \equiv \frac{c_1^n + c_1^{n-3} c_3}{2} \mod \mathbb{Z}[c_1, c_2, \ldots]. \tag{55}$$

Like the classical ones, the universal Bernoulli numbers as well play an important role in several branches of mathematics, in particular in complex cobordism theory (see e.g. [22,23]), where the coefficients c_n are identified with the cobordism classes of $\mathbb{C}P^n$.

Kummer congruences are also relevant in algebraic topology, and in defining p-adic extensions of zeta functions. We also have a universal Kummer congruence [24]. Suppose that $n \neq 0, 1 \pmod{p-1}$. Then

$$\frac{\widehat{B}_{n+p-1}}{n+p-1} \equiv \frac{\widehat{B}_n}{n} c_{p-1} \mod p\mathbb{Z}_p[c_1, c_2, \ldots]. \tag{56}$$

Other related congruences can be found in [24,25].

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