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Scaling near resonances and complex rotation numbers for the standard map

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Abstract. We consider the radius of convergence $\rho(\omega)$ of the Lindstedt series for the standard map and study its scaling behaviour as the rotation number tends to a rational value p/q both through real, diophantine numbers and through *complex* values. We compute numerically $\rho(\omega)$ by means of Padé approximants, and therefore are able to plunge deeply into the asymptotic regime by computing $\rho(\omega)$ very close to resonances. The scaling law $\rho(\omega) \sim |\omega - p/q|^{\beta/q}$, with $\beta = 2$, is observed; this is consistent with the conjecture that $\rho(\omega) \sim e^{-2B(\omega)}$, where $B(\omega)$ is a purely arithmetical function called *Brjuno's function*. In the case of the first two resonances ($p/q = 0/1$ and $p/q = 1/2$) we prove that the conjugating function to rotations (Lindstedt series) $u(\theta)$ tends to a limit $u^{(p/q)}(\theta)$ as ω tends to the resonance and ε is scaled in such a way to keep the radius of convergence fixed; this limit is analytically computed and its singularities in the complex θ and ε planes are found to agree with the results obtained by Padé approximants. The relevance of these results for a perturbative approach to renormalization theory is discussed.

AMS classification scheme numbers: 54Cxx

1. Introduction

In this paper we continue the study of the standard map at complex rotation numbers that we began in [1]; complex rotation numbers were considered in the very first works on the topic (see e.g. [2]) and appear naturally when discussing small denominators problems; see also [3, 4] for the study of Hamiltonian maps in the complex plane.

To fix the notation, we recall that the standard map is an area-preserving twist diffeomorphism of the cylinder $\mathbb{T} \times \mathbb{R}$ into itself given by:

$$x' = x + y + \varepsilon \sin x$$

$$y' = y + \varepsilon \sin x$$

with $x \in \mathbb{T}$ and $y \in \mathbb{R}$. We regard the nonlinear term $\varepsilon \sin x$ as a 'perturbation' of the linear map $(x, y) \mapsto (x' = x + y, y' = y)$, for which the dynamics is given by rotation on the

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cylinder: $x_n = 2\pi n\omega + x_0$, $y_n = 2\pi\omega$. We note that the variable y may be eliminated to give *one second order* recurrence:

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon \sin x_n \tag{1}$$

which may be seen as the *Lagrangian form* of the standard map.

It is well known (see e.g. [5], more references can be found in [6]) that if ω is irrational and satisfies a diophantine condition then the dynamics of the perturbed map can be analytically conjugated to the dynamics of the unperturbed, linear map provided that the perturbation is small enough, i.e. there exists a function $u_\omega(\theta, \varepsilon)$, jointly analytic in (θ, ε) for $|\operatorname{Im}\theta| < \bar{\theta}$, $|\varepsilon| < \bar{\varepsilon}$ such that if:

$$x = \theta + u_\omega(\theta, \varepsilon)$$

$$y = 2\pi\omega + u_\omega(\theta, \varepsilon) - u_\omega(\theta - 2\pi\omega, \varepsilon)$$

then in the new variable θ the dynamics is the one of the unperturbed, linear map $\theta_n = 2\pi n\omega + \theta_0$. We note also that the existence of a smooth conjugation to a rotation of rotation number ω is the same thing as the existence of a homotopically non-trivial invariant curve of the same rotation number and of the same smoothness class, as a simple application of Birkhoff's theorem and of the implicit function theorem shows.

The diophantine condition which, following [5], guarantees the existence of an analytic conjugation $u_\omega(\theta, \varepsilon)$ may be stated in terms of the continued fraction expansion of ω. In fact, let:

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_1, a_2, \ldots]$$

and let p_k/q_k be the sequence of rational approximants to ω obtained by truncating its continued fraction expansion; then the diophantine condition mentioned above takes the form:

$$q_{k+1} = O(q_k^\gamma)$$

with $\gamma \geqslant 2$.

It is immediate to verify that the function $u_\omega(\theta, \varepsilon)$ satisfies the equation:

$$(D_\omega^2 u_\omega)(\theta, \varepsilon) \equiv u_\omega(\theta + 2\pi\omega, \varepsilon) - 2u_\omega(\theta, \varepsilon) + u_\omega(\theta - 2\pi\omega, \varepsilon) = \varepsilon \sin(\theta + u_\omega(\theta, \varepsilon)). \tag{2}$$

Imposing the condition that the mean of u_ω over θ must be 0 the solutions are formally unique. Note that $D_\omega = D_{\omega+1}$, so $u_{\omega+1} = u_\omega$ for all ω.

We can then study the analyticity properties of the function u_ω expanding it in a Taylor series in ε and in a Fourier series in θ (the so called *Lindstedt series* [7]):

$$u_\omega(\theta, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n u_n(\theta) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{k \in \mathbb{Z}} \hat{u}_{n,k} e^{ik\theta}. \tag{3}$$

The coefficients $\hat{u}_{n,k}$ may be easily computed recursively by inserting this expansion in equation (2); we obtain:

$$\begin{aligned} D_\omega^2 u_1(\theta) &= \sin\theta \\ D_\omega^2 u_{n+1}(\theta) &= \sum_{h=1}^{n} \left(\frac{1}{h!} \frac{d^h \sin\theta}{d\theta^h}\right) \sum_{n_1 + \ldots + n_h = n} u_{n_1}(\theta) \cdots u_{n_h}(\theta) \end{aligned} \tag{4}$$

(in particular, it is easy to see that for the standard map the sum over k in (3) is actually restricted to $|k| \leqslant n$). It is therefore clear how, due to the unboundedness of the operator D_ω^{-2} which appears in (4), *small divisors* enter into the Lindstedt series and make it difficult to study the analyticity properties of the conjugating function u. To control these small divisors, the machinery of superconvergent methods leading to KAM theory [8] has been devised (but see [9, 10] for an alternative, direct approach).

In [12] the analytic properties of the conjugating function to the golden mean rotation were studied by means of Padé approximants; evidence for a natural boundary on a circle of radius equal to the breakdown threshold for the golden mean invariant curve (as computed, e.g., by Greene's residues method) was found.

The origin of this natural boundary was clarified in [1] by studying the conjugating function at *complex* rotation numbers: in fact, if ω has a non-zero imaginary part, the small divisors disappear and the convergence of the series (3) can be easily proved directly. In this way, the series (3) is regarded as an expansion of an analytic function $u(\theta, \varepsilon, \omega)$ of *three* variables; the existence of an analytic conjugation to a rotation of given rotation number ω_0 at a given ε_0 is therefore related to the existence and regularity of the limit of $u(\theta, \varepsilon_0, \omega)$ as ω tends to ω_0. In other words, there is only one 'complex' KAM torus, which analytically continues to real, homotopically non-trivial invariant curves as ω tends to sufficiently irrational, real ω, provided ε is small enough. Using Padé approximants, we show that the natural boundaries appearing at real, diophantine ω are due to the accumulations of lines of complex singularities related to resonances. In particular, if we take the real part of ω to be diophantine and we let the imaginary part tend to 0, the radius of convergence of the Lindstedt series tends to the critical breakdown threshold and the number of lines of singularities increases, creating the natural boundary.

The situation is made clearer if we look at the behaviour of the radius of convergence as ω tends to a resonance, i.e. a real, rational value p/q. In this case, the radius of convergence tends to 0 as ω tends to the resonance, and what is quite remarkable is that if ω is sufficiently near to the resonance, so as to isolate its contribution from the neighbouring ones, then exactly $2q$ lines of singularities appear.

A similar situation occurs also for the dissipative standard map as shown in [11], where it is remarked the analogy between the effect of the imaginary part of the rotation number and the introduction of dissipation in the system: in both cases small denominators disappear.

Later we will explain the nature of these lines of singularities which appear in the Padé approximants, and relate it to *branch points* and *cuts* for u, regarded as an analytic function of ε, and we will explicitly compute such singularities when ω tends to the first two resonances (0/1 and 1/2).

2. Scaling of the radius of convergence

An important object to study is therefore the radius of convergence of the series (3):

$$\rho(\omega) = \inf_{\theta \in \mathbf{T}} \left(\limsup_{n \to \infty} |u_n(\theta)|^{1/n} \right)^{-1} .$$

It clearly provides a lower bound to the threshold $\varepsilon_c(\omega)$ for the breakdown of the KAM invariant curve with rotation number ω, and for the standard map they are generally believed to be the same (as numerical evidence also suggests: see [12, 13] for details; but see [14] for a different viewpoint). The function $\rho(\omega)$ is a very complicated one, since it vanishes on all rationals and is everywhere discontinuous on the real axis; on the other hand it is non-zero and harmonic off the real axis.

The radius of convergence $\rho(\omega)$ is related to a purely arithmetic function, called *Brjuno's function* by Yoccoz, defined e.g. by the relation:

$$B(\omega) = -\log\omega + \omega B(\omega^{-1}) \qquad \text{for } \omega \in \mathbb{R} \cap [0, \tfrac{1}{2}] \tag{5}$$

with the boundary conditions:

$$B(\omega) = B(\omega + 1) = B(-\omega).$$

We note that $B(\omega)$ depends *only* on the arithmetic properties of ω, in particular on the growth rate of the denominators of the convergents of its continued fraction expansion; it can be easily shown (see [15]) that it is finite if and only if Brjuno's condition:

$$B_1(\omega) = \sum_k \frac{\log q_{k+1}}{q_k} < +\infty \tag{6}$$

is satisfied. In the case of iterated complex analytic maps (the so-called *Siegel's problem*), Yoccoz [15] proved that the absolute value of the difference of the logarithm of the Siegel radius (which is the equivalent, for that case, of our $\rho(\omega)$) with Brjuno's function is bounded for all irrational rotation numbers.

It is quite natural to ask a similar question also for the standard map. In particular, one can look at the quantity:

$$|\log\rho(\omega) + \beta B(\omega)| = \left|\log\frac{\rho(\omega)}{\mathrm{e}^{-\beta B(\omega)}}\right|$$

and ask whether for some $\beta \in \mathbb{R}_+$ it is bounded and continuous as conjectured in [6] (conjecture 3.3). If this proves true, then we can write the critical radius $\rho(\omega)$ as the product of a purely arithmetic function, which captures all of the singular behaviour of $\rho(\omega)$ at resonances, and a bounded strictly positive continuous function $C(\omega)$:

$$\rho(\omega) = C(\omega)\mathrm{e}^{-\beta B(\omega)}. \tag{7}$$

β may be looked upon as a 'critical exponent' which characterizes the 'strength' of the singularities of $u(\theta, \varepsilon, \omega)$ at real, rational ω.

For the semi-standard map, which is a complexified version of the standard map whose Lindstedt series has a simpler structure, it has been proved [16] using the standard majorant series method [17], Brjuno's counting lemma [18] and a minorant series argument, that Brjuno's condition is indeed *necessary and sufficient* for $\rho(\omega)$ to be non-zero. On the numerical side, for the semistandard map it is easy to compute the critical function by the root criterion. In [16] it was also conjectured, on the basis of numerical evidence, that $|\log\rho(\omega) + 2B(\omega)|$ is a bounded continuous function. Recently, Davie [14] proved that $|\log\rho(\omega) + 2B(\omega)| < 100$ using an improvement of Brjuno's counting lemma, but continuity is still an open question.

On the other hand, for the standard map the situation is more difficult also numerically, since the root criterion fails to relax to the radius of convergence at reasonable orders. On the analytical side, Davie [14] proved that $\log\rho(\omega) + 2B(\omega)$ is bounded above, thus establishing that Brjuno's condition is necessary for the Lindstedt series to converge. This result, together with the ones announced in [19, 10], proves the first of the conjectures made in [16], i.e. that Brjuno's condition is *necessary and sufficient* also in the standard map case. Moreover, Forni [20] proved the necessity of violating Brjuno's condition for the analytic destruction of invariant curves of twist maps. It is also interesting to note that Brjuno's condition guarantees the convergence of an approximate renormalization scheme [21], and that (7) would explain the so-called 'modular smoothing' technique [22, 23] as shown in [16].

We calculate $\rho(\omega)$ close to a resonance by means of Padé approximants: the coefficients u_n of the expansion (3) are computed and we locate the singularity nearest to the origin by calculating some high order Padé approximants and determining the zeros of the denominators.

The interpolation (7) of the radius of convergence by Brjuno's function implies a scaling property as the rotation number tends to a resonance. In fact, let ω_n be a sequence of noble numbers which tends to p/q as $n \to \infty$; to be definite, we may take e.g. ω_n to be of the form $[a_1, a_2, \ldots, a_k, n, 1^\infty]$, whith $[a_1, a_2, \ldots, a_k, \infty] = p/q$. Then, by (5) and elementary properties of the continued fraction expansion (see [6] for details), we have that $\exp(-B(\omega_n))$ tends to 0 as $1/n^{1/q}$ for $n \to \infty$; on the other hand $|\omega_n - p/q| \sim O(1/n)$, so that:

$$\rho(\omega_n) \sim \exp(-\beta B(\omega_n)) \sim \left|\omega_n - \frac{p}{q}\right|^{\frac{\beta}{q}}. \tag{8}$$

In this work we evaluate the scaling exponent for the radius of convergence as ω tends to a resonance *both* through real, diophantine numbers and through a path in the *complex* ω plane. In all cases we obtain a strong numerical evidence for the scaling law:

$$\rho\left(\frac{p}{q} + \eta\right) \sim |\eta|^{\frac{2}{q}}$$

independently on the way η tends to 0.

In figure 1 we show the plot of $\rho(\omega)$ versus $\operatorname{Im}\omega$ for $\operatorname{Re}\omega$ equal to 0, 1/2, 1/3, 1/5, 2/5 (so that η tends to 0 through purely imaginary values, i.e. ω tends to the resonance through a vertical straight line in the complex plane). In figures 2 and 3 we show the plot of $\rho(\omega)$ versus $|\eta|$ for $p/q = 1/3$ and for $\eta \to 0$ through curves tangent to the real axis at the origin: respectively a parabola and a curve with an *infinite order of contact* with the real axis, namely $\operatorname{Im}\Delta\omega = 10^{-1/(\operatorname{Re}\Delta\omega)^2}$. In figure 4 we show instead the plot of $\rho(\omega)$ as ω tends to a resonance through real, diophantine ω's: the cases shown correspond to the sequences $\omega_n = [n, 1^\infty] \to 0$, $\omega = [2, n, 1^\infty] \to 1/2$ and $\omega = [n^\infty] \to 0$. For these real sequences, we also compare with Brjuno's function: figure 5 shows the plot of $\rho(\omega)$ versus $\exp(-B(\omega))$. Values of $\Delta\omega$ of the order of 10^{-5} have been reached.

Tables 1 and 2 summarize the numerical data. In table 1 we show the results of the least squares fit to the scaling law $\rho(\omega) = \hat{C}(\omega)|\omega - \omega_0|^{\hat{\beta}/q}$ as $\omega \to \omega_0$ through real and complex sequences (data corresponding to figures 1–4). The values of $\hat{\beta}$ obtained by the fit all agree with $\hat{\beta} = 2$ within 2% and the limit value $\hat{C}(\omega_0)$ does not seem to depend (within the accuracy of the fit) on the choice of the sequence converging to ω_0, suggesting the continuity of the function $\hat{C}(\omega)$. In table 2 we show the result of the least squares fit to the scaling law $\rho(\omega) = C(\omega)e^{-\beta B(\omega)}$ as $\omega \to \omega_0$ through real sequences (figure 5). Again we find $\beta = 2$ and evidence for the continuity of $C(\omega)$. Note that, for these real sequences, using $\hat{\beta} = \beta = 2$ and the functional equation (5), one can show that $\hat{C} = C$ and this is verified by our numerical data.

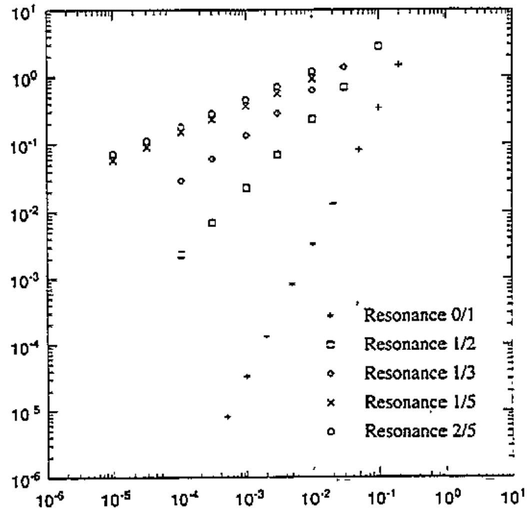

Figure 1. Plot of $\rho(p/q + \mathrm{i}\Delta\omega)$ versus $\Delta\omega$, with $p/q = 0/1, 1/2, 1/3, 1/5, 2/5$.

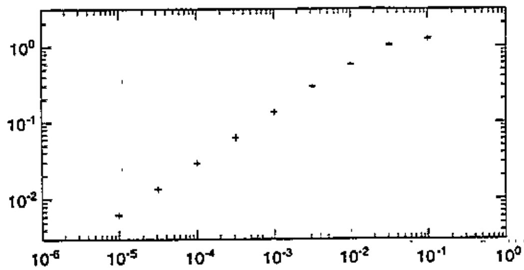

Figure 2. Plot of $\rho(1/3+\Delta\omega)$ versus $\Delta\omega$ as $\Delta\omega$ tends to 0 along the curve $\mathrm{Im}(\Delta\omega) = \mathrm{Re}(\Delta\omega)^2$.

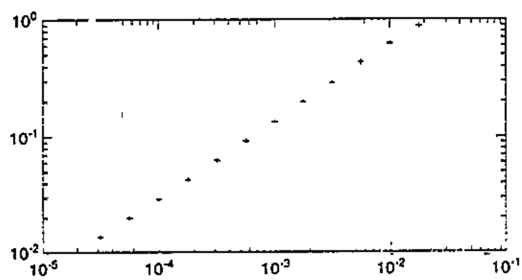

Figure 3. Plot of $\rho(1/3 + \Delta\omega)$ versus $\Delta\omega$ as $\Delta\omega$ tends to 0 along the curve $\mathrm{Im}(\Delta\omega) = 10^{-1/\mathrm{Re}(\Delta\omega)^2}$.

3. Resonances 0/1 and 1/2

We note that not only the radius of convergence of the Lindstedt series scales as $\omega \to p/q$, but the locations of all the poles and zeros of the Padé approximants scales as well, with the

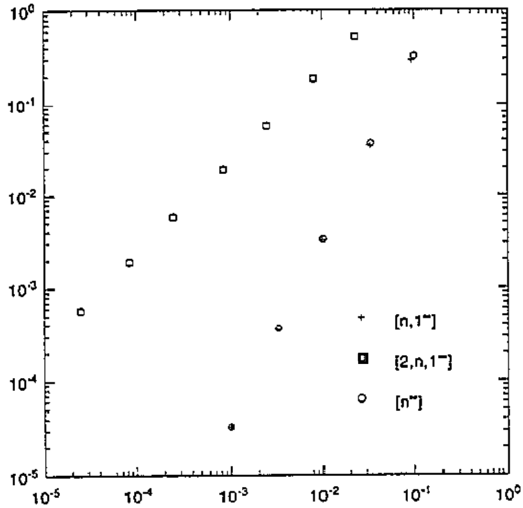

Figure 4. Plot of $\rho(\omega_n)$ versus $\omega_\infty - \omega_n$, with $\omega_n = [n, 1^\infty, [2, n, 1^\infty], [n^\infty]$.

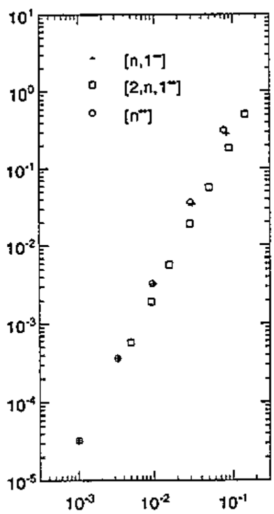

Figure 5. Plot of $\rho(\omega_n)$ versus $\exp(-B(\omega_n))$, with $\omega_n = [n, 1^\infty, [2, n, 1^\infty], [n^\infty]$.

same exponent and with a remarkable accuracy. This suggests that by rescaling ε it may be possible to obtain a 'scaling limit' for u. In particular, we conjecture that the following limit exists:

$$u^{(p/q)}(\theta, \varepsilon) = \lim_{\eta \to 0} u(\theta, \varepsilon\eta^{2/q}, 2\pi p/q + \eta) \tag{9}$$

and is independent on the way $\eta \to 0$ from e.g. the upper complex half plane.

In the case of the first two resonances ($p/q = 0/1$ and $p/q = 1/2$), we can *prove* that this limit exists and actually compute it; we state this result in the following two propositions. Note that our propositions *prove that $\beta = 2$ is necessary for the validity of* (7). We begin with the elementary case of the resonance (0/1).

Table 1.

ω_0	$\hat{\beta}$	$\hat{C}(\omega_0)$	Comments
0	2.01	34.9	(a)
1/2	2.04	26.3	(a)
1/3	2.03	14.0	(a)
1/5	2.00	5.9	(a)
2/5	2.02	7.43	(a)
1/3	2.00	13.5	(b) (c)
1/3	2.00	13.4	(d) (e)
0	2.00	32.3	(f)
1/2	2.00	23.0	(g)
0	2.00	32.4	(h)

(a) ω tends to ω_0 through a vertical straight line in the complex plane.
(b) ω tends to ω_0 through an arc of parabola.
(c) The three rightmost data points have been removed since too far from the asymptotic regime.
(d) ω tends to ω_0 through a curve with an infinite order of contact to the real axis.
(e) The two rightmost data points have been removed since too far from the asymptotic regime.
(f) ω tends to 0 through the sequence of real, diophantine rotation numbers $\omega_n = [n, 1^\infty]$.
(g) ω tends to 1/2 through the sequence of real, diophantine rotation numbers $\omega_n = [2, n, 1^\infty]$.
(h) ω tend to 0 through the sequence of real, diophantine rotation numbers $\omega_n = [n^\infty]$.

Table 2.

ω_0	β	$C(\omega_0)$	Comments
0	2.00	32.2	(a)
1/2	2.00	23.0	(b)
0	2.00	32.4	(c) (d)

(a) ω tends to 0 through the sequence of real, diophantine rotation numbers $\omega_n = [n, 1^\infty]$.
(b) ω tends to 1/2 through the sequence of real, diophantine rotation numbers $\omega_n = [2, n, 1^\infty]$.
(c) ω tends to 0 through the sequence of real, diophantine rotation numbers $\omega_n = [n^\infty]$.
(d) The rightmost data point has been removed since too far from the asymptotic regime.

Proposition 1. *Let $\mathcal{X}_\alpha$ be the space of 2π-periodic functions $f(\theta)$ analytic and bounded in a neighbourhood of the complex strip:*

$$S_\alpha = \{\theta \in \mathbb{C} \mid |Im\,\theta| < \alpha\}$$

with the norm:

$$\|f\|_\alpha = \sum_{k\in\mathbb{Z}} |\hat{f}_k| e^{|k|\alpha}$$

where $\hat{f}_k$ are the Fourier coefficients of f, and let C_γ be the closed complex cone defined by:

$$C_\gamma = \{\eta \in \mathbb{C} \mid Im\,\eta > 0, |Re\,\eta| \leqslant \gamma\, Im\,\eta\} \qquad \gamma > 0.$$

Then if $\eta \in C_\gamma$ the limit:

$$u^{(0/1)}(\theta, \varepsilon) = \lim_{\eta\to 0} u(\theta, \varepsilon\eta^2, \eta)$$

exists in $\|\cdot\|_\alpha$ *norm, uniformly with respect to* $\varepsilon \in \mathcal{D}_r = \{\varepsilon \in \mathbb{C} | |\varepsilon| < r\}$, *where:*

$$r = \frac{1}{16} e^{-\alpha} \frac{1}{1+\gamma^2}.$$

The function $u^{(0/1)}(\theta, \varepsilon)$ *is then analytic in* $\mathcal{S}_\alpha \times \mathcal{D}_r$ *and satisfies the differential equation:*

$$u^{(0/1)}_{\theta\theta}(\theta, \varepsilon) = \varepsilon \sin(\theta + u^{(0/1)}(\theta, \varepsilon))$$

with boundary conditions $u^{(0/1)}(0, \varepsilon) = u^{(0/1)}(2\pi, \varepsilon) = 0$.

Proof. Let:

$$\tilde{u}(\theta, \varepsilon, \eta) = \sum_{n=1}^{\infty} \eta^{2n} u_n(\theta, \eta) \varepsilon^n = \sum_{n=1}^{\infty} \tilde{u}_n(\theta, \eta) \varepsilon^n.$$

We will first prove that $\tilde{u}_n(\theta, \eta)$ has a finite limit as $\eta \to 0$, so that the Lindstedt series has a finite limit order by order in perturbation theory. Then, by taking $\eta \in \mathcal{C}_\gamma$ we will prove that:

$$\|\tilde{u}_n(\theta, \eta)\|_\alpha \leqslant r^{-n}$$

for some positive constant r which depends on γ, so that the limit series actually converges to an *analytic* function in $\mathcal{S}_\alpha \times \mathcal{D}_r$. Finally, we show, by direct calculation, that this limit satisfies the given differential equation.

From the recurrence (4) it is clear that each coefficient $\hat{u}_{n,k}$ in the Lindstedt series contains exactly n factors ('small denominators') of the type:

$$\Gamma_\omega(h) = \frac{1}{4 \sin^2 \pi h \omega}.$$

Now, if $\omega \equiv \eta$ tends to zero, each $\Gamma_\omega(h)$ diverges as η^{-2} and the factor η^{2n} in $\tilde{u}_n$ exactly compensates for this, so $\tilde{u}_n$ has a finite limit for $\eta \to 0$. We note that $\tilde{u}_n$ contains at least a term which does not vanish identically in the limit, and precisely the term proportional to $\prod_{h=1}^{n} \Gamma_\omega(h)$ (the so-called 'linear tree' of [9, 10]).

Next we have to estimate $\|\tilde{u}_n\|_\alpha$. We use the simple estimates of [6, appendix 2], based on the standard majorant series method: in particular, we have:

$$\|u_n(\theta, \omega)\|_\alpha \leqslant 4^{2n} e^{n\alpha} |\operatorname{Im} \omega|^{-2n}.$$

It then follows immediately that

$$\|\eta^{2n} u_n(\theta, \eta)\| \leqslant 4^{2n} e^{n\alpha} \left| \frac{\eta}{\operatorname{Im} \eta} \right|^{2n} \leqslant r^{-n}$$

uniformly in the cone $\mathcal{C}_\gamma$.

Finally, from the homologic equation satisfied by u we obtain:

$$\frac{1}{\eta^2} D^2_\eta u(\theta, \varepsilon\eta^2, \eta) = \varepsilon \sin(\theta + u(\theta, \varepsilon\eta^2, \eta)).$$

Taking the limit $\eta \to 0$ (shown to exist above) we obtain the differential equation:

$$u^{(0/1)}_{\theta\theta}(\theta, \varepsilon) = \varepsilon \sin(\theta + u^{(0/1)}(\theta, \varepsilon)) \tag{10}$$

with the boundary conditions $u^{(0/1)}(0, \varepsilon) = u^{(0/1)}(2\pi, \varepsilon) = 0$ (note that by symmetry we also have $u^{(0/1)}(\pi, \varepsilon) = 0$). □

In a similar way we prove the following:

Proposition 2. *Under the same hypothesis of the above proposition, the limit:*

$$u^{(1/2)}(\theta, \varepsilon) = \lim_{\eta \to 0} u(\theta, \varepsilon\eta, \pi + \eta)$$

exists and satisfies the differential equation:

$$u_{\theta\theta}^{(1/2)}(\theta, \varepsilon) = -\tfrac{1}{8}\varepsilon^2 \sin 2(\theta + u^{(1/2)}(\theta, \varepsilon))$$

with boundary conditions $u^{(1/2)}(0, \varepsilon) = u^{(1/2)}(2\pi, \varepsilon) = 0$.

Proof. The proof follows the same lines as the preceding one. In the case of the resonance $(1/2)$, as η tends to zero, only those $\Gamma_{\pi+\eta}(h)$ with even h diverge (always as η^{-2}); therefore we have to know *how many* divergent small denominators there are actually in each $u_n(\theta)$: we will prove that this number is $\lfloor n/2 \rfloor$. To this end, we first recall that the Fourier expansion in θ of $u_n(\theta)$ contains only frequencies of the same parity of n; then we use induction on n and equation (4) to prove our statement above. In fact, clearly $u_1(\theta)$ does not contain any divergent small denominator, and $u_2(\theta)$ contains exactly one such small denominator ($\Gamma_{\pi+\eta}(2)$). Next, we have that the sum

$$\sum_{n_1+\ldots+n_h=n} u_{n_1}(\theta) \cdots u_{n_h}(\theta)$$

in (4) contains

$$\lfloor n_1/2 \rfloor + \ldots + \lfloor n_h/2 \rfloor \leqslant \lfloor n/2 \rfloor$$

divergent small denominators, and therefore $D_\omega^2 u_{n+1}(\theta)$ as well. Now, the expression:

$$\frac{\mathrm{d}^h \sin\theta}{\mathrm{d}\theta^h} \sum_{n_1+\ldots+n_h=n} u_{n_1}(\theta) \cdots u_{n_h}(\theta)$$

contains only frequencies in its Fourier expansion in θ of the same parity as $n+1$, for every h in the range $1 \ldots n$, and therefore the operator D_ω^{-2} acting on it will produce a divergent small denominator only if $n+1$ is even. Summarizing, if $n+1$ is even it will contain $\lfloor n/2 \rfloor + 1 = (n+1)/2 = \lfloor (n+1)/2 \rfloor$ divergent small denominators, while if $n+1$ is odd it will contain $\lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor$ such factors, and the inductive statement is proved. We have therefore shown that $u_n(\theta, 2\pi(1/2+\eta)$ diverges at worst as $\eta^{-2\lfloor n/2 \rfloor}$ and each $\eta^n u_n$ has a finite limit.

For what matters the convergence of the limit series, we note that the null space of the operator D_π^2 is the space of 2π*-periodic* functions $f(\theta)$, i.e. functions whose Fourier expansion contains only even frequencies. Now, take η in the cone C_γ and let f be a π-periodic function in $\mathcal{X}_\alpha$. Then:

$$\|D_{\pi+\eta}^{-2} f\|_\alpha \leqslant \frac{c_1}{|\operatorname{Im}\eta|^2} \|f\|_\alpha$$

for some positive constant c_1. On the other hand, if $f \in \mathcal{X}_\alpha$ contains only odd frequencies in its Fourier expansion, it is easy to see that:

$$\|D_{\pi+\eta}^{-2} f\|_\alpha \leqslant c_2 \|f\|_\alpha.$$

We will prove, by induction on n, that:

$$\|u_n\|_\alpha \leqslant 4^n c_3^n \left(\frac{1}{|\operatorname{Im}\eta|^2}\right)^{\lfloor n/2 \rfloor}. \tag{11}$$

This implies that $\sum_{n\geqslant 1}(\eta\varepsilon)^n u_n(\theta, \pi+\eta)$ converges uniformly in η in the cone $\mathcal{C}_\gamma$; this also implies that the limit is a π-periodic function (since all odd order terms contain an extra η factor uncompensated by the small denominators).

In fact, (11) holds for $n = 1$ and 2 by direct calculation of u_1 and u_2. Next, we consider separately the cases of n even and n odd and using the fact that a term of a given order in ε contains only frequencies in θ of the same parity as the order we obtain:

$$\|u_n\|_\alpha \leqslant \begin{cases} \dfrac{c_1 e^\alpha}{|\operatorname{Im}\eta|^2} \displaystyle\sum_{l=1}^{n} \sum_{n_1+\ldots+n_l=n} \|u_{n_1}\|_\alpha \cdots \|u_{n_l}\|_\alpha & \text{if } n \text{ is odd} \\ c_2 e^\alpha \displaystyle\sum_{l=1}^{n} \sum_{n_1+\ldots+n_l=n} \|u_{n_1}\|_\alpha \cdots \|u_{n_l}\|_\alpha & \text{if } n \text{ is even.} \end{cases}$$

If we now take $c_3 = e^\alpha \max(c_1, c_2)$ and count the factors $1/|\operatorname{Im}\eta|^2$, by using standard majorant series estimates we obtain (11). It immediately follows that

$$\|\eta^n u_n\|_\alpha \leqslant r^{-n}$$

uniformly in the cone $\mathcal{C}_\gamma$, where $r = (1/4c_3)(1+\gamma^2)^{1/2}$.

Finally, we determine the differential equation satisfied by the limit $u^{(1/2)}(\theta, \varepsilon)$. This *case is handled by twice iterating the map and reducing the resonance (1/2) to a resonance* (1/1) which is the same as (0/1). Specifically, we use the formula:

$$D_{2\eta}^2 f(\theta) = D_{\pi+\eta}^2[f(\theta+(\pi+\eta)) + 2f(\theta) + f(\theta-(\pi+\eta))] \tag{12}$$

valid for 2π periodic functions f and easily proved by direct calculations. Applying (12) to $u(\theta, \varepsilon\eta, \pi+\eta)$ with $\eta \in \mathcal{C}_\gamma$ and using:

$$D_{\pi+\eta}^2 u(\theta, \varepsilon\eta, \pi+\eta) = \varepsilon\eta \sin(\theta + u(\theta, \varepsilon\eta, \pi+\eta)) \tag{13}$$

we obtain:

$$\begin{aligned} D_{2\eta}^2 u(\theta, \varepsilon\eta, \pi+\eta) &= D_{\pi+\eta}^2(u_+ + 2u_0 + u_-) \\ &= \varepsilon\eta[\sin(\theta+(\pi+\eta)+u_+) + 2\sin(\theta+u_0) + \sin(\theta-(\pi+\eta)+u_-)] \\ &= -2\varepsilon\eta(\mathrm{A}) \end{aligned} \tag{14}$$

where:

$$\begin{aligned} (\mathrm{A}) = \sin\left(\frac{\eta+u_+-u_0}{2}\right)\cos\left(\theta+\frac{\eta+u_++u_0}{2}\right) \\ - \sin\left(\frac{\eta+u_0-u_-}{2}\right)\cos\left(\theta+\frac{-\eta+u_0+u_-}{2}\right) \end{aligned}$$

and we used the shorthand:

$$u_\pm = u(\theta \pm (\pi + \eta), \varepsilon\eta, \pi + \eta)$$

$$u_0 = u(\theta, \varepsilon\eta, \pi + \eta).$$

Now, the fact that $u^{(1/2)}$ is π-periodic implies that:

$$\cos\left(\theta + \frac{\eta + u_+ + u_0}{2}\right), \cos\left(\theta + \frac{-\eta + u_0 + u_-}{2}\right) \to \cos(\theta + u^{(1/2)}(\theta, \varepsilon))$$

$$\frac{u_+ - u_0}{2}, \frac{u_0 - u_-}{2} \to 0$$

as $\eta \to 0$, so:

$$\begin{aligned}\lim_{\eta\to 0}\frac{1}{\eta}(\mathrm{A}) = \cos(\theta + u^{(1/2)}(\theta, \varepsilon))\\ \times \lim_{\eta\to 0}\frac{1}{\eta}\left[\sin\left(\frac{\eta + u_+ - u_0}{2}\right) - \sin\left(\frac{\eta + u_0 - u_-}{2}\right)\right]\\ = \cos(\theta + u^{(1/2)}(\theta, \varepsilon)) \lim_{\eta\to 0}\frac{u_+ - 2u_0 + u_-}{\eta}\\ = \tfrac{1}{4}\varepsilon \sin 2(\theta + u^{(1/2)}(\theta, \varepsilon))\end{aligned}$$

where in the last step we used again the homologic equation (13).

Dividing (14) by $4\eta^2$ and taking the limit $\eta \to 0$ we obtain the differential equation:

$$u^{(1/2)}_{\theta\theta}(\theta, \varepsilon) = -\tfrac{1}{8}\varepsilon^2 \sin 2(\theta + u^{(1/2)}(\theta, \varepsilon)) \qquad (15)$$

with the boundary conditions $u^{(1/2)}(0, \varepsilon) = u^{(1/2)}(2\pi, \varepsilon) = 0$ (and again we also have $u^{(1/2)}(\pi, \varepsilon) = 0$). □

4. Complex analytic structure at resonances

We now solve these differential equations, compute the singularities in the complex θ and ε planes and compare the result with the numerical result from Padé approximants.

We note that both can be put in the same form:

$$x''(t) = \lambda \sin x(t) \qquad (16)$$

with the boundary conditions $x(0) = 0$, $x(2\pi) = 2\pi$, by letting:

$$x(t) = \theta + u^{(0/1)}(\theta, \varepsilon) \qquad t = \theta,\ \lambda = \varepsilon$$

and:

$$x(t) = 2(\theta + u^{(1/2)}(\theta, \varepsilon)) \qquad t = 2\theta,\ \lambda = -\frac{\varepsilon^2}{16}.$$

Equation (16) is the pendulum equation; qualitative analysis shows easily that a unique solution with the given boundary conditions exists for all *real* values of λ. This solution, as is well known (see e.g. [24]), can be written in term of Jacobian elliptic functions:

$$x(t) = \pi + 2\,\mathrm{am}\left(\frac{K}{\pi}(t-\pi), k\right) \tag{17}$$

where $K(k)$ is the quarter-period function:

$$K(k) = \int_0^{\pi/2} \mathrm{d}\phi \frac{1}{\sqrt{1-k^2\sin^2\phi}}.$$

The value of the modulus k is fixed by the boundary conditions to satisfy:

$$kK(k) = \pi\sqrt{\lambda}$$

and therefore is uniquely determined by λ (see e.g. [25, 26] for details and a background on elliptic functions).

The singularities of the solution of (17) in the complex t plane are immediately found: in fact, the function $\mathrm{am}(u,k)$ has branch points of infinite order at the points $u = \mathrm{i}K' + 2nK + 2m\mathrm{i}K'$, $m, n \in \mathbb{Z}$, where $K' = K(k')$ and k', the *complementary modulus*, satisfies $k^2 + k'^2 = 1$; the singularities closest to the real t axis for the function $x(t)$ are therefore:

$$\pm\,\mathrm{i}\pi\frac{K'}{K} + \pi.$$

Using the standard notation:

$$\tau = \mathrm{i}\frac{K'}{K} \qquad q = \mathrm{e}^{\pi\mathrm{i}\tau}$$

we see that the singularities are at $\theta = \pm\pi\tau + \pi$ or, in the plane of the complex variable $\mathrm{e}^{\mathrm{i}\theta}$, at $-q$, $-1/q$.

For the function $u^{(0/1)}(\theta,\varepsilon)$ this means that the analyticity strip in the complex θ plane is limited by two branch points at $\theta = \pi(1\pm\tau)$, while for $u^{(1/2)}(\theta,\varepsilon)$ we have four branch points at $\theta = (1\pm\tau)\pi/2$ and $\theta = (3\pm\tau)\pi/2$ (taking into account the 2π-periodicity in θ).

The analytic structure in λ for $x(t)$ is harder to derive. We write the Fourier expansion of $x(t)$ (from the Fourier expansion of the Jacobi amplitude function $\mathrm{am}(u,k)$):

$$x(t) = t + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{n(1+q^{2n})}\sin nt = t + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cos n\pi\tau}\sin nt. \tag{18}$$

This series converges for $t \in \mathbb{R}$ when $|q| < 1$ (that is when $\mathrm{Im}\,\tau > 0$), and depends analytically on q inside the unit circle (depends analytically on τ in the upper complex half plane). On the other hand, $x(t)$) depends on λ through the dependence of q (and τ) on λ, therefore it will have singularities in λ for those values of λ which are singular for $q(\lambda)$. We cannot make explicit the dependence of q on λ, but the inverse is easy to derive; in fact, from the theory of elliptic and theta functions we have that:

$$\sqrt{\frac{2kK}{\pi}} = \vartheta_2(0,q) = 2q^{1/4}\sum_{n=0}^{\infty} q^{n^2+n}$$

where $\vartheta_2(z, q)$ is a theta function (we use the conventions of [25] for naming the theta functions, and we refer to this reference as well as to [26] for a background on theta functions); but in our case $kK = \pi\sqrt{\lambda}$ so that:

$$\lambda = \frac{1}{4}\vartheta_2^4(0, q) = 4q\left(\sum_{n=0}^{\infty} q^{n^2+n}\right)^4. \tag{19}$$

The singularities of $q(\lambda)$ will occur at the critical values of $\vartheta_2^4(0, q)/4$. We do not have an exact result for those values so we resort to estimates.

Let $f(q)$ be the function defined in (19):

$$f(q) = q\left(\sum_{n=0}^{\infty} q^{n^2+n}\right)^4.$$

This series converges for $|q| < 1$. We will show that the critical points closest to the origin are $q = \pm i\bar{q}$ with $|\bar{q}| < 1/2$, and calculate numerically this value. We will also show that in the disk $\mathcal{D}_{1/2} = \{|q| < 1/2\}$ there are no other critical values of $f(q)$.

In fact, we have:

$$f'(q) = \left(\sum_{n=0}^{\infty} q^{n^2+n}\right)^3 \left(\sum_{n=0}^{\infty}(2n+1)^2 q^{n^2+n}\right).$$

We will show that the first factor does not vanish inside $\mathcal{D}_{\infty/\in}$ and that the second factor has exactly two zeros inside the same disk.

We have:

$$\left|\sum_{n=0}^{\infty} q^{n^2+n}\right| \geqslant 1 - \sum_{n=1}^{\infty} |q|^{n^2+n} \geqslant 1 - |q|^2 \sum_{n=0}^{\infty} |q|^{4n} = 1 - \frac{|q|^2}{1-|q|^4} \geqslant \frac{11}{15}$$

in $\mathcal{D}_{1/2}$ so that the first factor does not vanish.

Next, we see that the series in the second factor (which converges absolutely inside the unit circle) is dominated by the term $n = 1$ in $\mathcal{D}_{1/2}$:

$$|9q^2| > \left|1 + \sum_{n=2}^{\infty}(2n+1)^2 q^{n^2+n}\right|. \tag{20}$$

In fact $|9q^2| = 9/4$ so it is enough to show that e.g. $|\sum_{n=2}^{\infty}(2n+1)^2 q^{n^2+n}| \leqslant 1$ to prove (20). Letting $|q| = 1/2$ we have:

$$\left|\sum_{n=2}^{\infty}(2n+1)^2 q^{n^2+n}\right| \leqslant |q|^6 \sum_{n=0}^{\infty}(2n+5)^2 |q|^{6(n-2)} < \frac{1}{2}$$

for $|q| = 1/2$ and using the Rouché theorem we conclude that this factor has exactly two zeros in $\mathcal{D}_{1/2}$. These zeros must be either both real or both purely imaginary: in fact, if they had both non-vanishing real and imaginary parts, there would be *four* zeros in $\mathcal{D}_{1/2}$ (because f has real Taylor coefficients and because it is an even function). Since for real values of q the series has positive terms it cannot vanish, so they must be purely imaginary. Numerics gives $\bar{q} \approx 0.328\,107$ and therefore a critical value for lambda $\lambda_c \approx \pm 0.827\,524\mathrm{i}$.

This value of λ corresponds to a branch point singularity for $x(t)$ as a function of λ, and *its location in the complex plane does not depend on* t, except for the trivial cases in which t is a multiple of π.

We can now locate in the complex θ and ε planes the singularities of $u^{(0/1)}$ and $u^{(1/2)}$ and compare with the results from Padé approximants for $u(\theta, \varepsilon, \omega)$ with $\omega \approx 0, \pi$.

For $u^{(0/1)}$, the singularity in the complex θ plane are at $\mathrm{Re}\,\theta = \pi$, that is on the real, negative axis in the $e^{i\theta}$ plane. The exact position of the singularity depends on ε. We generate the coefficients of the Lindstedt series very close to the resonance (0/1) and compute the Padé approximants on the plane of the complex variable $\zeta = e^{i\theta}$ at a value of ε given by (19) and compare the result from Padé approximants calculations for ω *close* to a resonance with the *exact* result *at the resonance*. For example, we take $q = 0.1$ and $\eta = 10^{-3}$, which gives a value of $\varepsilon \approx 16.4326 \times 10^{-6}$; computing poles and zeros of the Padé approximant in ζ [18/18] (we kept the order very low so to avoid any potential problem with the accuracy of the calculations) we found a singularity at $\zeta \approx -10.1828$, that is with an error of less than 2% on the theoretical value $\zeta = -1/q = -10$; in figure 6 we plotted the poles and the zeros of the above-mentioned Padé approximant, removing the ghost pole-zero pairs detected by computing the residue of each pole; note the line of poles alternating with zeros, typical of branch points (we also removed the reciprocal poles generated by the singularity at $\zeta = -q$).

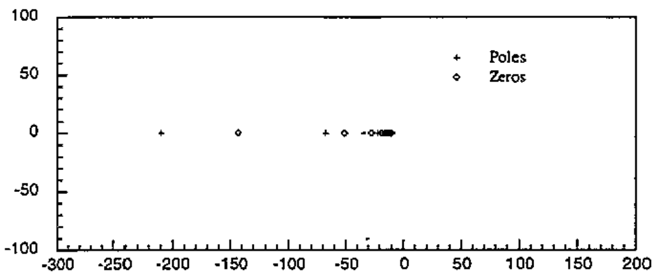

Figure 6. Plot of the poles and zeros of the Padé approximant [18/18] in the complex $\zeta = e^{i\theta}$ plane; $\varepsilon = 16.4326 \times 10^{-6}$ and $\omega = i10^{-3}$.

For what matters the singularities in the complex ε plane, we conjecture that the singularity closer to the origin is the one generated by the critical value λ_c computed above. We cannot rule out in full mathematical rigour the existence of other critical values of λ *smaller* than λ_c, but a numerical survey of the complex q plane in (19) suggests that our conjecture is indeed correct. We obtain, at $\omega = 10^{-3}i$, a pair of singularities at $\varepsilon = \pm 32.6693 \times 10^{-6}i$, while the Padé approximant [30/30] gives $\pm 32.9 \times 10^{-6}i$ with a numerically negligible real part (with an error of less than 1%). In figure 7 we show the poles and zeros of this Padé approximant: again, note the line of alternating poles and zeros, suggesting a branch point as the singularity at the edge of the line.

For $u^{(1/2)}$, the singularities in the complex ζ plane are at $\zeta = \pm i\sqrt{q}, \pm i/\sqrt{q}$; using Padé approximants, taking $q = 0.1$ and $\omega = 1/2 + i \times 10^{-3}$, we find a value for the singularity outside the unit disk the values $\zeta \approx \pm i3.264$ with a negligible real part from the approximant [18/18], to be compared with the analytically computed value of $\pm i\sqrt{10} \approx \pm i3.162$, so that the numerically computed value is within an error of about 3.3% from the analytical one computed at the resonance. For what matters the singularities in the complex ε plane, the calculations carried out so far give four branch points at $\varepsilon = 4\sqrt[4]{-1}\sqrt{\lambda_c} \cdot \Delta\omega$; from the

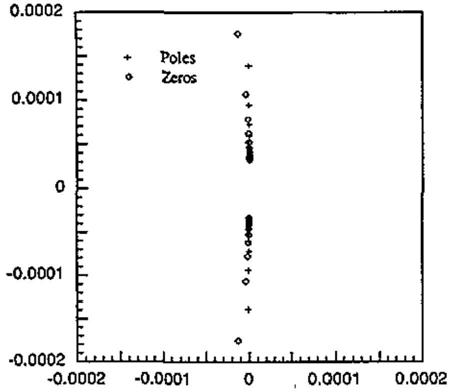

Figure 7. Plot of the poles and zeros of the Padé approximant [30/30] in the complex ε plane; $\theta = 1$ and $\omega = \mathrm{i}10^{-3}$.

Padé approximant [30/30] computed at $\omega = 1/2 + \mathrm{i}10^{-4}$ we find (cf figure 6 of [1]) four lines of alternating poles and zeros, radially in the directions of the four fourth roots of -1, with the closest poles at $|\varepsilon| \approx 0.002\,31$, to be compared with the value of $|\varepsilon| \approx 0.002\,29$ coming from the analytic calculations: the error is again less than 1%. The same structure of poles and zeros for the Padé approximants as in the case of the resonance $0/1$ shows up, again confirming the idea that the picture we have for the rescaled conjugating function at the resonance holds for ω close to it, in the scaling regime.

We are not able, as yet, to compute $u^{(p/q)}$ for resonances higher than $(1/2)$; the combinatorics we used to show the compensation of the small denominators at each order in the Lindstedt series when we rescale ε with $|\Delta\omega|^{2/q}$, in fact, works only for the resonance $(1/2)$. The numerical results of section 2, though, suggest that the same picture holds, leading us to conjecture the following:

Conjecture. *For each rational number $r \in \mathbb{Q} \cap (0, 1]$, which we write in form of irreducible fraction as p/q, the following limit:*

$$u^{(r)}(\theta, \varepsilon) = \lim_{\substack{\eta \to 0 \\ Im\,\eta > 0}} u(\theta, \varepsilon\eta^{2/q}, 2\pi r + \eta)$$

exists and defines an analytic function of (θ, ε). Moreover, there is a constant C_r such that $u^{(r)}(\theta, \varepsilon)$ satisfies the differential equation:

$$u^{(r)}_{\theta\theta}(\theta, \varepsilon) = C_r \varepsilon^q \sin q(\theta + u^{(r)}(\theta, \varepsilon))$$

with boundary conditions $u^{(r)}(0, \varepsilon) = u^{(r)}(2\pi, \varepsilon) = 0$, whose solution in term of Jacobi's amplitude function is:

$$u^{(r)}(\theta, \varepsilon) = -\frac{2}{q}\left(\frac{q\theta - \pi}{2} - am\left(\frac{q\theta - \pi}{2}\frac{K}{\pi}, k\right)\right)$$

with the value of the modulus k given by the equation:

$$kK(k) = \pi\left(\frac{C_r \varepsilon^q}{q}\right)^{1/2}.$$

We examine now the consequences of this conjecture. First, we see that it predicts $2q$ branch points in the complex θ plane, in pairs with the same real part and opposite imaginary parts, and the real parts are given by $\pi(2k+1)/q$, $k=0,\ldots,q-1$. Next, we see that it predicts $2q$ branch points in the complex ε plane, whose location is at points proportional to the $2q$th roots of -1. This qualitative picture fully explains the phenomenology observed in [1], i.e. what do singularities matter in the complex ε plane; singularities in the complex θ plane near resonances have been studied extensively, from the numerical point of view, in [27], and our conjecture above fully explains their observations.

The exact positions of the singularities depend on the value of the costant C_r, which can be computed, assuming the validity of our conjecture, from the qth order of the perturbation theory for the Lindstedt series for u close to the resonance p/q. Since the third order of the perturbation theory is quite easy to compute by hands, we can derive at once the value of the constant $C_{1/3}$ and therefore check the agreement between the results from Padé approximants and the results derived from our conjecture; we obtain $C_{1/3} = -1/24$. If we take $q = 0.1$ and $\eta = 10^{-4}$, we have $\varepsilon \approx 22.7866 \times 10^{-3}$; for this value of q, the branch points outside the unit circle are at a distance from the origin in the ζ plane equal to $\sqrt[3]{10} \approx 2.15$, while the Padé approximant [30/30] gives 2.19: the agreement is within 2%. In the complex ε plane our conjecture predicts, for $\omega = 1/3 + \mathrm{i}10^{-4}$, a radius of convergence of the Lindstedt series equal to 0.0287, while the Padé approximant [30/30] gives a value of 0.0291: the agreement is within less than 1.5%.

5. Conclusions

We want to state here some comments about previous work and some problems which deserves further study, both analytical and numerical.

In [6] it was conjectured that there exists a value of $\beta \in \mathbb{R}_+$ such that (7) is verified with a bounded, continuous $C(\omega)$. Numerical evidence using Greene's residue criterion was given supporting the conjecture. To compute in this way the radius of convergence when the rotation number is close enough to a resonance, where the important cancellations between $\log\rho(\omega)$ and $B(\omega)$ occur, is extremely difficult. Therefore the question of determining the correct value of β was left open.

Contrary to the case of Greene's residues criterion, the computational effort involved in estimating $\rho(\omega)$ by means of Padé approximants does not depend crucially on the arithmetic properties of ω. In particular, it is possible to compute the radius of convergence for rotation numbers very close to a resonance (whithin a distance of the order of 10^{-4}–10^{-5}, to be compared with 10^{-2}, which is the minimum distance from a resonance for which the Greene's method is still possible, see [6]). Moreover, we can let ω tend to a resonance also from the upper complex half plane, where the Lindstedt series (3) depends analytically on ω. As a result with our method we are able to plunge deeply into the asymptotic regime and the conjecture, with a value of β equal to 2, is confirmed. We also note that, when the rotation number is complex and close to a resonance, the behaviour of the series is no longer dominated by small denominators, which are compensated by the scaling factor multiplying ε in order to keep the radius of convergence fixed.

We also want to stress that the method of Padé approximants provides reasonably accurate *quantitative* predictions for the critical radius and for some critical exponents: in the cases where it was possible to compare with exact results, the errors have been of about a few percent, sometimes less than 1%, even using low-order Padé approximants and VAX double-precision (G-floating) arithmetic (that is about 13 digits). A similar *quantitative* agreement of the results obtained by Padé approximants and results obtained by other methods, traditionally considered more accurate, has been found in [28, 11].

Note also that there are general theorems on the convergence of Padé approximants to functions with branch points [29]. In particular one can prove [30] the convergence in capacity of diagonal (i.e. $[N/N]$, as the ones we used) Padé approximants to functions with an even number of branch points of square root type on the Mittag–Leffler star (i.e. the complex plane minus the union of lines emanating from the branch points and going radially to infinity).

In the case of the semi-standard map most of the calculations are simpler and we are actually able to prove that the conjugating function $u(\theta, \varepsilon, \omega)$ has a limit when $\omega \to p/q$ and ε is suitably rescaled for *all* resonances p/q. Further work on this topic is in progress [31].

To compute such a limit and to derive its analyticity properties is a first step toward a *perturbative* renormalization approach to the problem of the break-up of invariant curves for area-preserving maps of the cylinder. In fact, the renormalization group picture of the transition to chaos in Hamiltonian systems [32, 33] is usually non-perturbative and based on a dynamics in phase space, with the renormalization group acting as a flow on a space of Hamiltonians. This leaves open many questions which are best formulated in the language of perturbation theory, in particular questions related to analyticity properties of invariant tori and conjugating transformations. As in quantum field theory the non-perturbative renormalization group theory is complemented by renormalized perturbation theory, we believe that an approach to renormalization in Hamiltonian systems based on perturbation theory could explain many numerically observed phenomena and clarify the analytic properties of invariant tori. We refer to [34] for further details and references (and for some attempts to generalize these results to other maps).

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