Waiting times, recurrence times, ergodicity and quasiperiodic dynamics

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Dynamical Systems

X: a space (measure, topological, manifold)

$T : X \to X$ a map
(continuous, measure-preserving, differentiable, ...).

To study asymptotic behaviour of $T^n(x)$.

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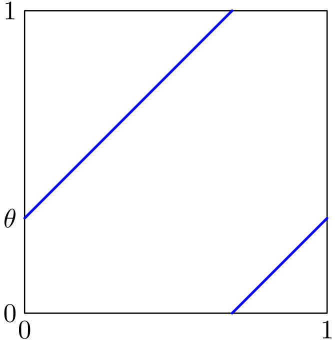
Assume that there is a measure μ on X which is T-invariant.
(time invariant, stationary).
($\mu(E) = \mu(T^{-1}E)$ for all measurable $E \subset X$)

measure μ : volume, area, length, probability .....

An irrational rotation

$$T : [0,1) \to [0,1), \quad T(x) = x + \theta \pmod 1.$$



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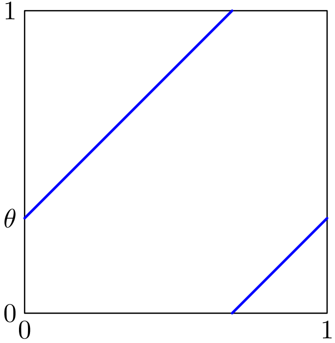

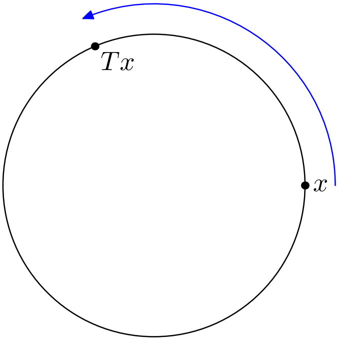

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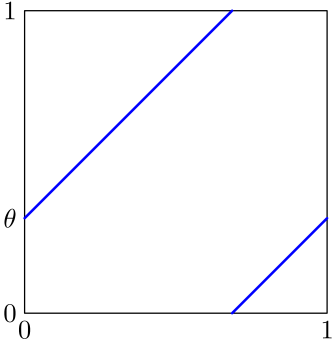
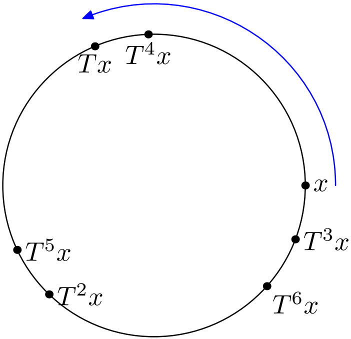
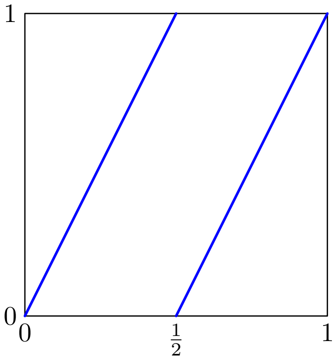




$x \mapsto 2x$ map

$X = [0, 1)$ with Lebesgue measure,
$T : x \mapsto 2x \pmod 1$.



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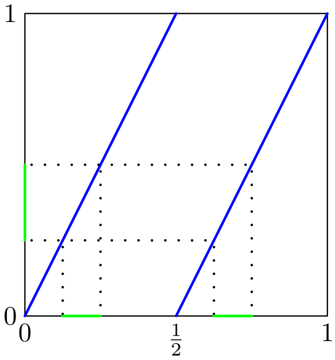


Shift spaces

$X = \prod_{n=1}^{\infty} \mathcal{A}$, $\mathcal{A}$ is a finite set.
$T : (x_1 x_2 x_3 \dots) \mapsto (x_2 x_3 x_4 \dots)$ left-shift
μ: an invariant(stationary) measure

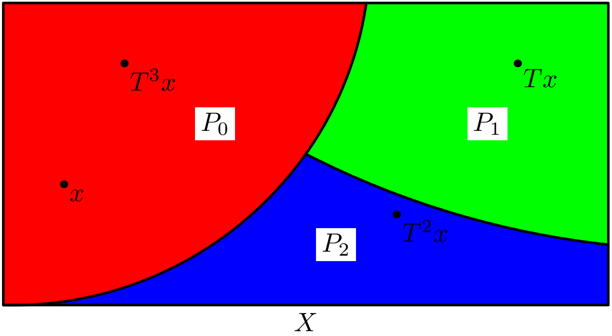
Fair coin tossing: (i.i.d. process)
$X = \prod_{n=1}^{\infty} \{H, T\}$, e.g., $HTHHHTHTTTT \cdots \in X$.

$$\mu(x_n = a_n \,|\, x_1^{n-1} = a_1^{n-1}) = \mu(x_n = a_n) = \mu(x_1 = a_n)$$

for all $n \geq 1$, and $a_1^n \in \mathcal{A}^n$.
μ is a product measure on $\mathcal{A}^{\mathbb{N}}$.

(i.i.d. process $\Rightarrow$ Chaotic or Random system)

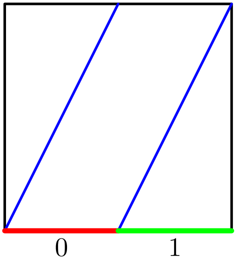
$\mathcal{P}$: a partition of X.
$(x_0, x_1, x_2, \dots)$: $\mathcal{P}$ name of x if $T^i x \in P_{x_i}$, $i = 0, 1, \dots$.



$\mathcal{P} = \{P_0, P_1, P_2\}$. $\mathcal{P}$ name of x is 0120 ....

$$(X, T) \Longleftrightarrow (\{0,1,2\}^{\mathbb{Z}}, \sigma), \quad x \leftrightarrow 0120\dots$$

$X = [0, 1)$, $T : x \mapsto 2x \pmod 1$.



(X, μ, T) is isomorphic to $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift (coin tossing).

$$x = (x_1 x_2 x_3 \ldots)_{(2)}, \quad x \leftrightarrow x_1 x_2 x_3 \ldots$$

since $x_i \in 0, 1$ if $T^{i-1}x \in P_0, P_1$ respectively.

Sequence of the powers of 2

2	2048	2097152	2147483648	2199023255552
4	4096	4194304	4294967296	4398046511104
8	8192	8388608	8589934592	8796093022208
16	16384	16777216	17179869184	17592186044416
32	32768	33554432	34359738368	35184372088832
64	65536	67108864	68719476736	70368744177664
128	131072	134217728	137438953472	140737488355328
256	262144	268435456	274877906944	281474976710656
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Last digits : $2 \longrightarrow 4 \longrightarrow 8 \longrightarrow 6 \longrightarrow \ldots$

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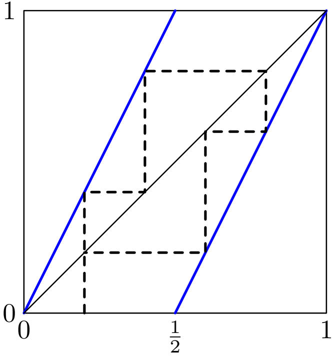
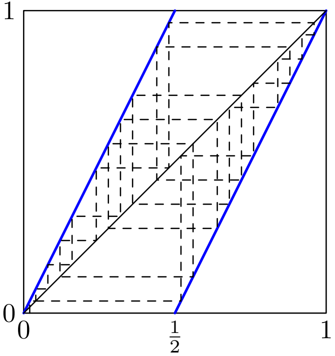
Last 2 digits :
04 08 16 32 64 28 56 12 24 48 96 92 84 68 36 72 44 88 76 52


1
0
0
$\frac{1}{2}$
1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$$\log x_n = \log x_{n-1} + \log 2, \quad \log_{10} 2 = 0.3010\cdots \approx \frac{3}{10}.$$

An irrational rotation

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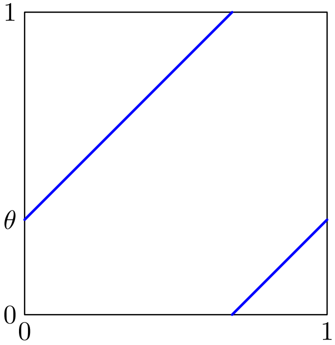
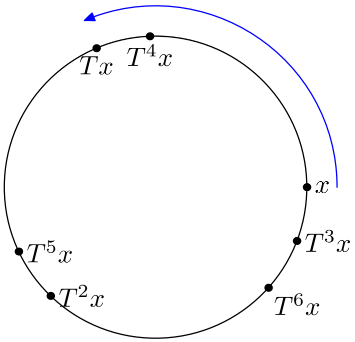
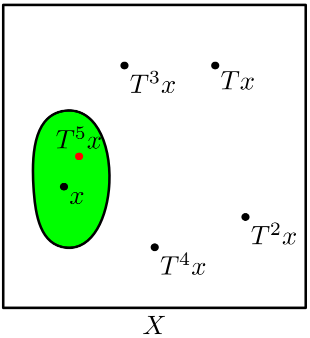




Quasi-periodic : $\exists n_i$ s.t. $|T^{n_i} - id| \to 0$.

The Poincaré Recurrence Theorem



Under suitable assumptions a typical trajectory of the system comes back infinitely many times in any neighborhood of its starting point.

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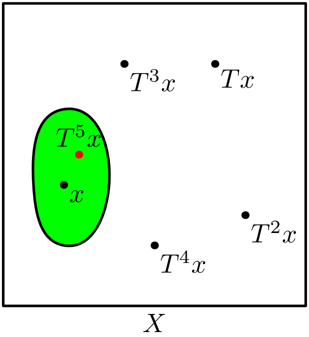

Under suitable assumptions a typical trajectory of the system comes back infinitely many times in any neighborhood of its starting point.

How many iterations of an orbit is necessary to come back within a distance r from the starting point?

The quantitative recurrence theory investigates this kind of questions.

Define $\tau_r(x)$ to be the first return time of x into the ball $B(x,r)$ centered in x and with radius r.

$$\tau_r(x) = \min\{j \geq 1 : T^j(x) \in B(x,r)\}.$$

Questions:

- Distribution of τ_r. $\Pr(\tau_r(x) > s)$?
- Asymptotic limits of $\dfrac{\log \tau_r(x)}{-\log r}$

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Questions:

- Distribution of τ_r. $\Pr(\tau_r(x) > s)$?
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Define $\tau_r(x, y)$ to be the hitting time or waiting time of x into the ball $B(y, r)$ centered in y and with radius r.

Let $X = \{0,1\}^{\mathbb{N}}$ and σ be a left-shift map.
Define R_n to be the first return time of the initial n-block, i.e.,

$$R_n(x) = \min\{j \geq 1 : x_1 \dots x_n = x_{j+1} \dots x_{j+n}\}.$$

$$x = \overbrace{\boxed{1\,0\,1\,0}\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0}^{15}\,\boxed{1\,0\,1\,0}\cdots \Rightarrow R_4(x) = 15.$$

The convergence of $\dfrac{1}{n}\log R_n(x)$ to the entropy h was studied in a relation with data compression algorithm such as the Lempel-Ziv compression algorithm.

Lempel-Ziv data compression algorithm

The Lempel-Ziv data compression algorithm provide a universal way to coding a sequence without knowledge of source.
Parse a source sequence into shortest words that has not appeared so far:

$$1011010100010 \cdots \Rightarrow 1, 0, 11, 01, 010, 00, 10, \ldots$$

For each new word, find a phrase consisting of all but the last bit, and recode the location of the phrase and the last bit as the compressed data.

$$(000, 1)\,(000, 0)\,(001, 1)\,(010, 1)\,(100, 0)\,(010, 0)\,(001, 0)\ldots$$

Theorem (Wyner-Ziv(1989), Ornstein and Weiss(1993))

For ergodic processes with entropy h,

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The meaning of entropy

- Entropy measures the information content or the amount of randomness.
- Entropy measures the maximum compression rate.
- Totally random binary sequence has entropy $\log 2 = 1$. It cannot be compressed further.

The Shannon-McMillan-Brieman theorem states that

$$\lim_{n\to\infty} -\frac{1}{n}\log P_n(x) = h \quad \text{a.e.},$$

where $P_n(x)$ is the probability of $x_1 x_2 \dots x_n$.

If the entropy h is positive,

$$\lim_{n\to\infty} \frac{\log R_n(x)}{-\log P_n(x)} = 1 \quad \text{a.e.}$$

For many hyperbolic (chaotic) systems

$$\lim_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = d_\mu(x),$$

where d_μ is the local dimension of μ at x.
(Saussol, Troubetzkoy and Vaienti (2002), Barreira and Saussol (2001, 2002), G.H. Choe (2003), C. Kim and D. H. Kim (2004))

What happens, if $h = 0$, which implies that $\log R_n$ and $\log P_n$ do not increases linearly.

Diophantine approximation

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$$|T^q x - x| < \delta$$

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Diophantine approximation:

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

An irrational number θ, $0 < \theta < 1$, is said to be of type η if

$$\eta = \sup\{\beta : \liminf_{j\to\infty} j^{\beta}\|j\theta\| = 0\},$$

$\|\cdot\|$ is the distance to the nearest integer ($\|t\| = \min_{n\in\mathbb{Z}} |t-n|$).

- Note that every irrational number is of type $\eta \geq 1$. The set of irrational numbers of type 1 (Called Roth type) has measure 1.
- A number with bounded partial quotients is of type 1.
- There exist numbers of type ∞, called the Liouville numbers. For example $\theta = \sum_{i=1}^{\infty} 10^{-i!}$.

Let $T(x) = x + \theta \pmod 1$ on $[0,1)$ for an irrational θ of type η,

Theorem (Choe-Seo (2001))

For every x

$$\liminf_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = \frac{1}{\eta}, \quad \limsup_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1.$$

Theorem (K-Seo (2003))

For almost every y

$$\limsup_{r\to 0^+} \frac{\log \tau_r(x,y)}{-\log r} = \eta, \qquad \liminf_{r\to 0^+} \frac{\log \tau_r(x,y)}{-\log r} = 1.$$

Fibonacci sequence

0

1

10

101

10110

10110101

1011010110110

....

1011010110110101101011011010110101101101011010110110 ...

Let $\sigma : A^* \to A^*$ be a substitution ($A^* = \cup_{n \geq 0} A^n$)

$\sigma(0) = 1,\ \sigma(1) = 10,\ \sigma(ab) = \sigma(a)\sigma(b)$

Sturmian sequence

- Let $u = u_0 u_1 u_2 \ldots$ be an infinite sequence. Let $p_u(n)$ be the complexity function which count the number of different words of length n occurring in u.

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  $p(4) = 5$

Sturmian sequence (continued)

$u = u_0 u_1 u_2 \dots$ is Sturmian
if and only if u is an infinite $\mathcal{P}$-naming of an irrational rotation, i.e., there is an irrational slope θ and a starting point $s \in [0, 1)$ such that

$$u_n = \begin{cases} 0, & \text{if } \{n\theta + s\} \in [0, 1 - \theta), \\ 1, & \text{if } \{n\theta + s\} \in [1 - \theta, 1). \end{cases}$$

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Theorem (K-K.K. Park (2007))

$$\liminf_{n\to\infty} \frac{\log R_n(u)}{\log n} = \frac{1}{\eta}, \ \limsup_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1, \textit{ almost every } s.$$

Moreover, if $\eta > 1$*, then for every* s $\dfrac{\log R_n(u)}{\log n}$ *does not converge.*

Morse sequence (or Prouhet-Thue-Morse sequence)

$$\sigma(1) = 10,\ \sigma(0) = 01,\ \sigma(ab) = \sigma(a)\sigma(b)$$

$$0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto \ldots\ldots\ldots \mapsto$$
$$0110100110010110100101100110100110010110 \ldots$$

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u_n is the number of 1's (mod 2) in the binary expansion of n.

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The complexity of the Morse sequence is

$$\limsup \frac{p_u(n)}{n} = \frac{10}{3}, \quad \liminf \frac{p_u(n)}{n} = 3.$$

Automatic sequence

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- The Morse sequence is 2-automatic.

Automatic sequence

- u is called k-automatic if it is generated by a k-automaton.
- An infinite sequence is k-automatic if and only if it is the image under a coding of a fixed point of a k-uniform morphism σ.
- The Morse sequence is 2-automatic.

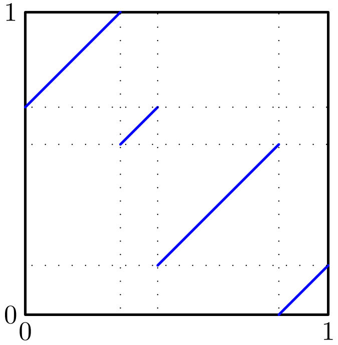
Theorem

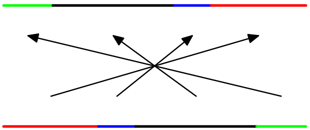
Let u be a non-eventually periodic k-automatic infinite sequence. Then we have

$$\lim_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1.$$

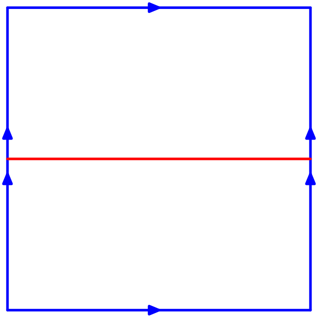
An interval exchange map

Generalization of the irrational rotation



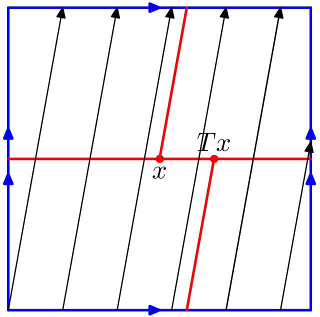


Comparing the interval exchange map and the irrational rotation



torus

Comparing the interval exchange map and the irrational rotation



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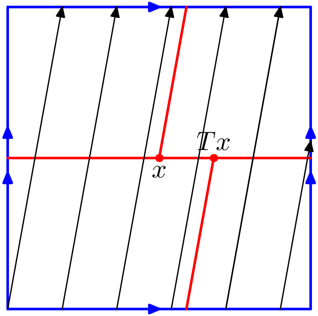
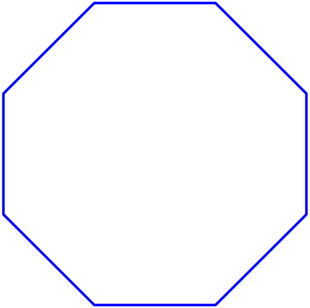
Comparing the interval exchange map and the irrational rotation

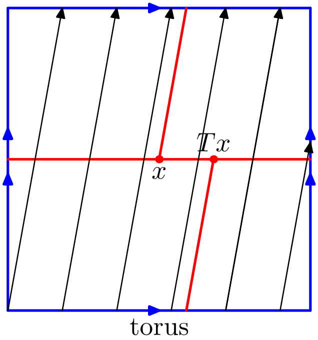


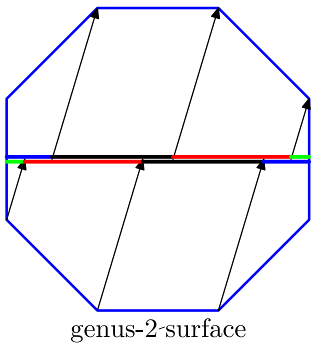
torus



genus-2-surface

Comparing the interval exchange map and the irrational rotation





Properties of the interval exchange map

- Kean (1975) : If the length data are rationally independent, then the i.e.m. is minimal (i.e., all orbits are dense)

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$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = 1, \quad \lim \frac{\log R_n(x)}{\log n} = 1, \text{ a.e.}$$

Another definition of "Roth type" for i.e.m.

Infinite invariant measure systems

- Such systems are used for models of statistically anomalous phenomena such as intermittency and anomalous diffusion and they do have interesting statistical behavior.

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- Many classical theorems of finite measure preserving systems from ergodic theory can be extended to the infinite measure preserving case.
- The Hopf ratio ergodic theorem: Let T be conservative and ergodic and $f, g \in L^1$ such that $\int g d\mu \neq 0$, then

$$\frac{\sum_{k=0}^{n-1} f(T^k(x))}{\sum_{k=0}^{n-1} g(T^k(x))} \to \frac{\int f d\mu}{\int g d\mu}, \qquad \text{a.e.}$$

Entropy for infinite invariant measure systems

Let T be a conservative, ergodic measure preserving transformation on a σ-finite space $(X, \mathcal{A}, \mu)$. Then the entropy of T can be defined as

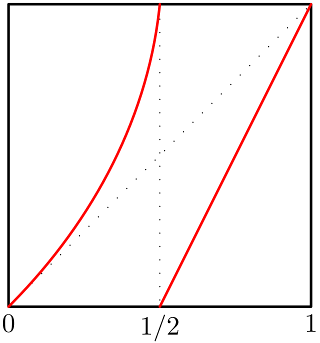
$$h_\mu(T) = \mu(Y) h_{\mu_Y}(T_Y)$$

where $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$ and T_Y is the induced map of Y ($T_Y(x) = T^{R_Y(x)}$ where

$$R_Y(x) = \min\{n \geq 1 : T^n(x) \in Y\}$$

when $x \in Y$) and μ_Y is the induced measure ($\mu_Y(E) = \frac{\mu(E \cap Y)}{\mu(Y)}$) which is invariant and ergodic under T_Y.

The Manneville-Pomeau map



$$T(x) = \begin{cases} x + 2^{z-1}x^z, & 0 \le x < 1/2, \\ 2x - 1, & 1/2 \le x < 1. \end{cases}$$

have an indifferent "slowly repulsive" fixed point at the origin. When $z \in [2, \infty)$ this forces the natural invariant measure for this map to be infinite and absolutely continuous with respect to Lesbegue.

It is not hard to see that $\mathcal{P} = \{[0, 1/2), [1/2, 1)\}$ is a generating partition and the entropy $h_\mu(T)$ is positive and finite.

Shannon-McMillan-Breiman Theorem

For every $f \in L^1(\mu)$, with $\int f \neq 0$

$$\frac{-\log(\mu(P_n(x)))}{S_n(f,x)} \to \frac{h_\mu(T)}{\int f d\mu} \quad \text{a.e.} \quad \text{as } n \to \infty.$$

Here $S_n(f,x)$ is the partial sums of f along the orbit of x:

$$S_n(f,x) = \sum_{k \in [0,n-1]} f(T^k(x)).$$

("information content" growing as a sublinear power law as time increases)

$(X, T, \mathcal{A}, \mu)$: a measure preserving system.
Let ξ be a partition of X and A be an atom of ξ.
Let $S_n(A, x)$ be the number of $T^i x \in A$ for $0 \leq i \leq n-1$, i.e.,

$$S_n(A, x) = S_n(1_A, x) = \sum_{i=0}^{n-1} 1_A(T^i(x)).$$

Define

$$R_n(x) = \min\{j \geq 1 \mid \xi_n(x) = \xi_n(T^j x)\}$$

considering a fixed set $A \in \mathcal{A}$ we also define $\bar{R}_n(x)$ by

$$\bar{R}_n(x) = \min\{S_j(A, x) \geq 1 \mid \xi_n(x) = \xi_n(T^j x)\}.$$

Note that

$$\bar{R}_n(x) = S_{R_n(x)}(A, x).$$

Lemma

Let T be a conservative, ergodic measure preserving transformation (c.e.m.p.t.) on the σ-finite space $(X, \mathcal{B}, \mu)$ and let ξ be a finite generating partition (mod μ). Assume that there is a subset A which is a union of atoms in ξ with $0 < \mu(A) < \infty$ and $H(\xi_A) < \infty$. For almost every $x \in A$

$$\lim_{n\to\infty} \frac{\log \bar{R}_n(x)}{S_n(A,x)} = \frac{h_\mu(T)}{\mu(A)}.$$

Let ξ_A be the induced partition on A,

$$\xi_A = \cup_{k\geq 1}\{V \cap \{R_A = k\} : V \in A \cap \xi_k\}.$$

Theorem (Galatolo-K-Park (2006))

Let T be a c.e.m.p.t. on the σ-finite space $(X, \mathcal{B}, \mu)$ and let $\xi \subset \mathcal{B}$ be a finite generating partition (mod μ). Assume that there is a subset A which is a union of atoms in ξ with $0 < \mu(A) < \infty$ and $H(\xi_A) < \infty$. Then for any $f \in L^1(\mu)$ with $\int f d\mu \neq 0$,

$$\limsup_{n\to\infty} \frac{\log R_n(x)}{S_n(f,x)} = \frac{h_\mu(T)}{\alpha \int f d\mu} \quad a.e.,$$

where

$$\alpha = \sup_{0<\mu(B)<\infty, B\in\mathcal{B}} \left(\sup\{\beta : \int_B (R_B)^\beta d\mu < \infty\} \right).$$

Moreover, if $\alpha = 0$, then the limsup goes to infinity.

Darling-Kac set

A set A is called a Darling-Kac set, if $\exists\{a_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \hat{T}^k 1_A = \mu(A), \quad \text{almost uniformly on } A.$$

A function f is slowly varying at ∞ if $\frac{f(xy)}{f(x)} \to 1$ as $x \to \infty, \forall y > 0$. Suppose that T has a Darling-Kac set and $a_n(T) = n^\alpha L(n)$, where $L(n)$ is a slowly varying. The Darling-Kac Theorem states

$$\frac{S_n(x)}{a_n(T)} \to Y_\alpha, \text{ in distribution},$$

Y_α : the normalized Mittag-Leffler distribution of order α.

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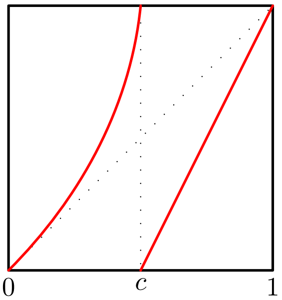
$$\lim_{n\to\infty} \frac{\log R_n(x)}{S_n(f,x)} = \frac{h_\mu(T)}{\alpha \int f d\mu} \quad a.e.$$

Moreover, if $\alpha = 0$, then the limit goes to infinity.

A map $T : [0,1] \to [0,1]$ is a Manneville-Pomeau map (MP map) with exponent z if it satisfies the following conditions:

- there is $c \in (0,1)$ such that, if $I_0 = [0,c]$ and $I_1 = (c,1]$, then $T\big|_{(0,c)}$ and $T\big|_{(c,1)}$ extend to C^1 diffeomorphisms, $T(I_0) = [0,1]$, $T(I_1) = (0,1]$ and $T(0) = 0$;



- there is $\lambda > 1$ such that $T' \geq \lambda$ on I_1, whereas $T' > 1$ on $(0,c]$ and $T'(0) = 1$;
- the map T has the following behaviour when $x \to 0^+$

$$T(x) = x + rx^z + o(x^z)$$

for some constant $r > 0$ and $z > 1$.

- When $z \geq 2$ these maps have an infinite, absolutely continuous invariant measure μ with positive density and the entropy can be calculated as $h_\mu(T) = \int_{[0,1]} \log(T')d\mu$.

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- If $z > 2$, we have a behavior of the return time sequence

$$a_n(T) = n^{1/(z-1)} L(n),$$

where $L(n)$ is a slowly varying function.
- Setting $S_n(x) = \sum_{i \leq n} 1_{I_1}(T^i(x))$, we have

$$\lim_{n\to\infty} \frac{\log R_n(x)}{S_n(x)} = \frac{h_\mu(T)}{\mu(I_1)}(z-1).$$

Theorem (Galatolo-K-Park (2006))

Let (X, T, ξ) satisfy (1)-(3) and μ be the absolutely continuous invariant measure then

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = \begin{cases} 1 & \text{if } z \leq 2 \\ z-1 & \text{if } z > 2 \end{cases}$$

for almost all points x
(recall that $\tau_r(x)$ is the first return time of x in the ball $B(x, r)$).