Dynamics and time series: theory and applications

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Lecture 8, Nov 30 , 2011

Inverse problem

Inverse problem: given $\varepsilon>0$ and a target (fractal) set $\mathcal{T}$ can one find an i.f.s $\mathcal{F}$ such that the corresponding attractor $\mathcal{A}$ is ε-close to $\mathcal{T}$ w.r.t. the Hausdorff distance h?

Collage Theorem (Barnsley 1985) Let $\varepsilon>0$ and let $\mathcal{T}\in\mathcal{H}(X)$ be given. If the i.f.s. $\mathcal{F} = \{w_1, \ldots, w_N\}$ is such that

$$h(\cup_{1\leq i\leq N} w_i(\mathcal{T}) , \mathcal{T}) < \varepsilon$$

then

$$h(\mathcal{T}, \mathcal{A}) < \varepsilon / (1-s)$$

where s is the Lipschitz constant of $\mathcal{F}$

Fractal image compression ?

The Collage Theorem tells us that to find an i.f.s. whose attractor "looks like" a give set one must find a set of contracting maps such that the union (collage) of the images of the given set under these maps is near (w.r.t. Hausdorff metric) to the original set.

The collage theorem sometimes allows incredible compression rates of images (of course with loss). It can be especially useful when the information contained in details is not considered very very important

Fractal image compression !

The topselling multimedia encyclopedia Encarta, published by Microsoft Corporation, includes on one CDROM seven thousand color photographs which may be viewed interactively on a computer screen. The images are diverse; they are of buildings, musical instruments, people's faces, baseball bats, ferns, etc. What most users do not know is that all of these photographs are based on fractals and that they represent a (seemingly magical) practical success of mathematics.

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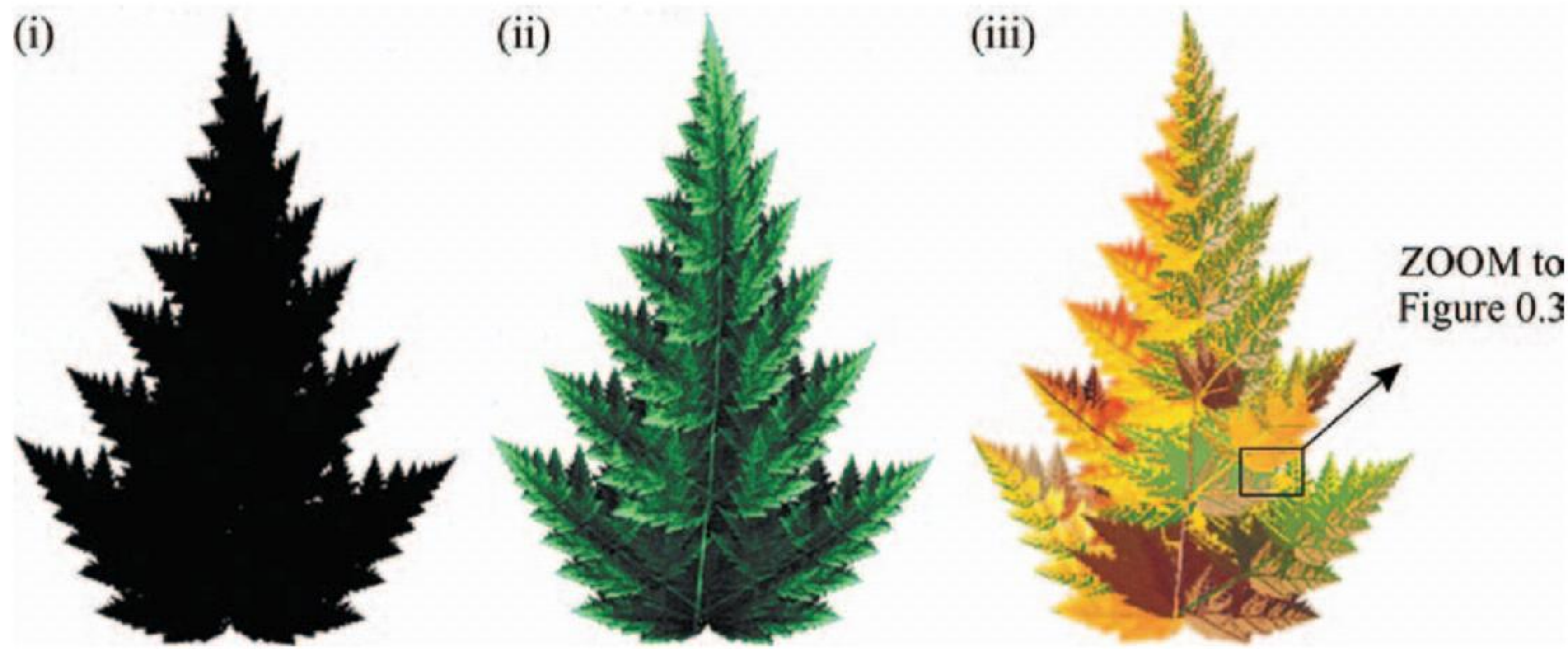
Fractal Image Compression by Michael F. Barnsley

e.g: Barnsley's fern: can be encoded with 160 bytes= 4*10*4

4 maps 10 parameters (each parameter using 4 bytes)

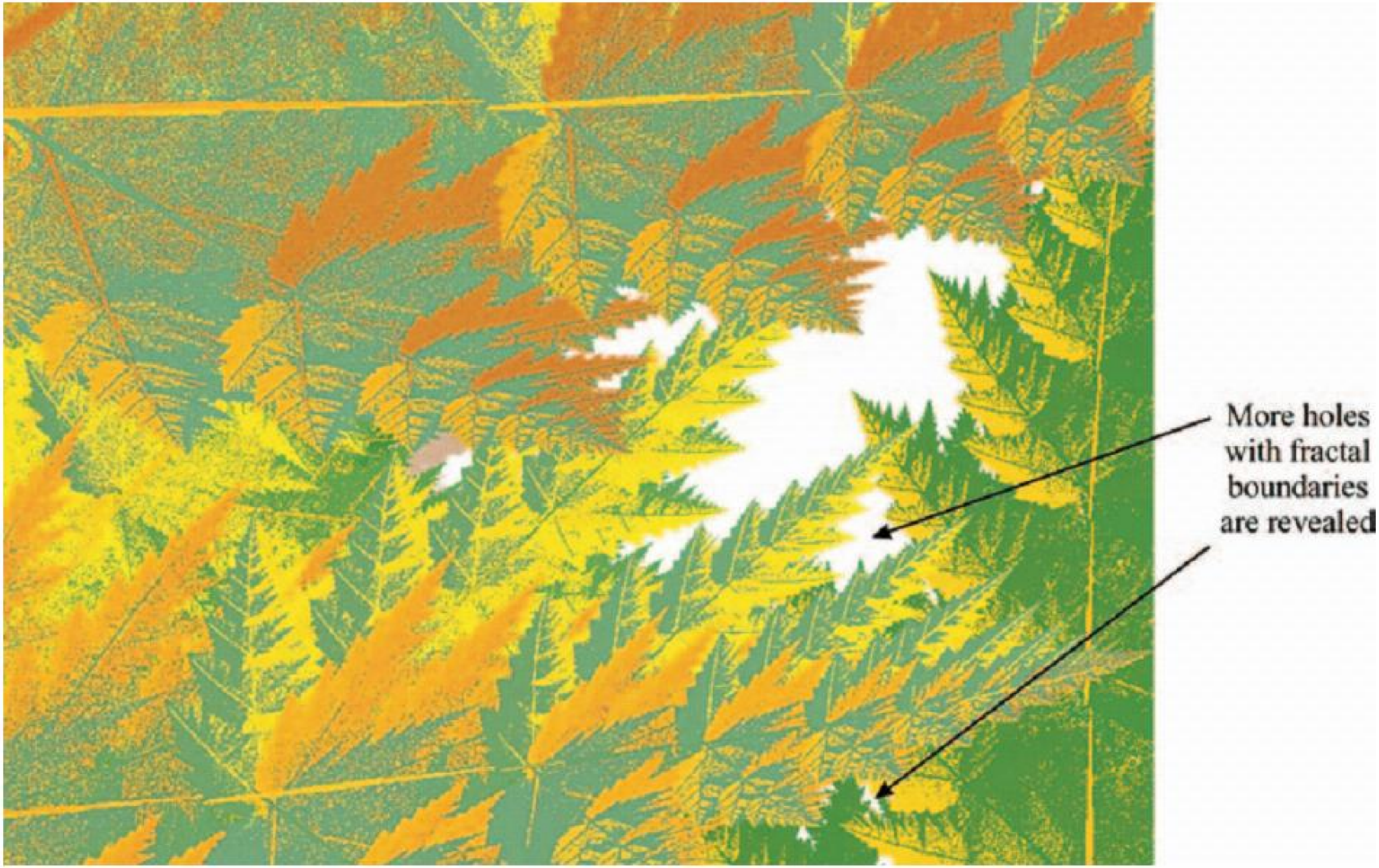


the measure attractor and (iii) the

$$f_n(x, y) = \left(\frac{a_n x + b_n y + c_n}{g_n x + h_n y + j_n}, \frac{d_n x + e_n y + k_n}{g_n x + h_n y + j_n} \right)$$

n	a_n	b_n	c_n	d_n	e_n	k_n	g_n	h_n	j_n	p_n
1	19.05	0.72	1.86	−0.15	16.9	−0.28	5.63	2.01	20.0	$\frac{60}{100}$
2	0.2	4.4	7.5	−0.3	−4.4	−10.4	0.2	8.8	15.4	$\frac{1}{100}$
3	96.5	35.2	5.8	−131.4	−6.5	19.1	134.8	30.7	7.5	$\frac{20}{100}$
4	−32.5	5.81	−2.9	122.9	−0.1	−19.9	−128.1	−24.3	−5.8	$\frac{19}{100}$

From M. Barnsely
SUPERFRACTALS
Cambridge
University Press
2006



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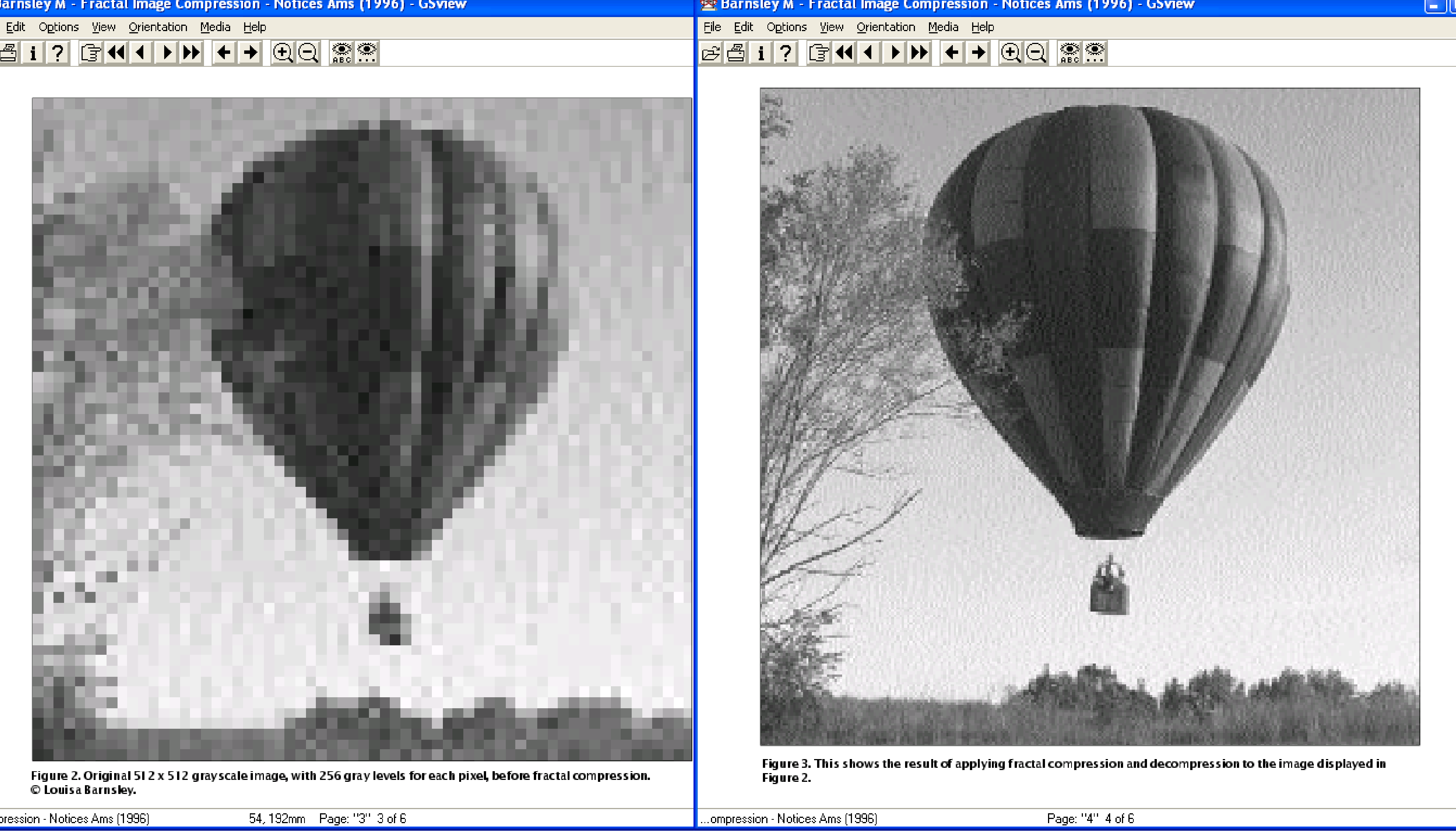

Figure 2. Original 512 x 512 grayscale image, with 256 gray levels for each pixel, before fractal compression. © Louisa Barnsley.

Figure 3. This shows the result of applying fractal compression and decompression to the image displayed in Figure 2.

LEFT: the original digital image of Balloon, 512 pixels by 512 pixels, with 256 gray levels at each pixel. RIGHT: shows the same image after fractal compression. The fractal transform file is approximately one fifth the size of the original.

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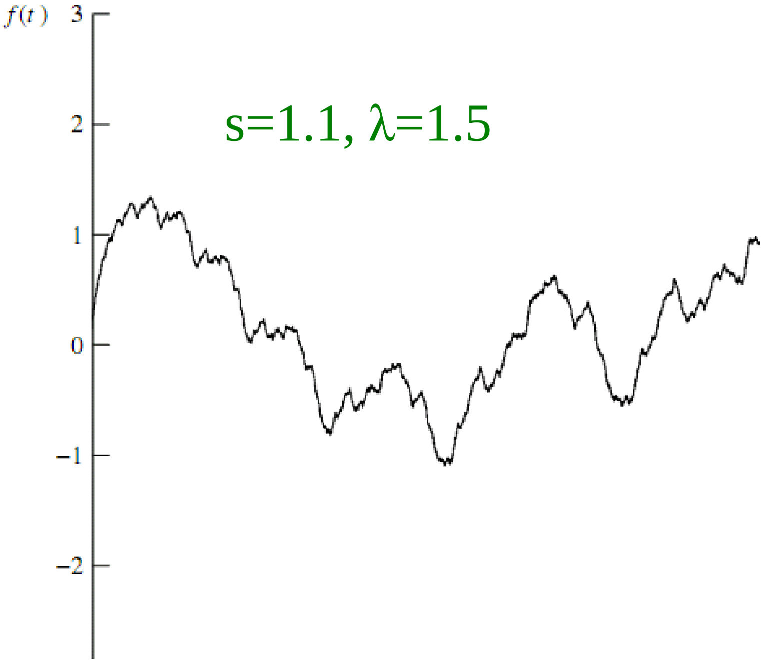
Fractal graphs of functions

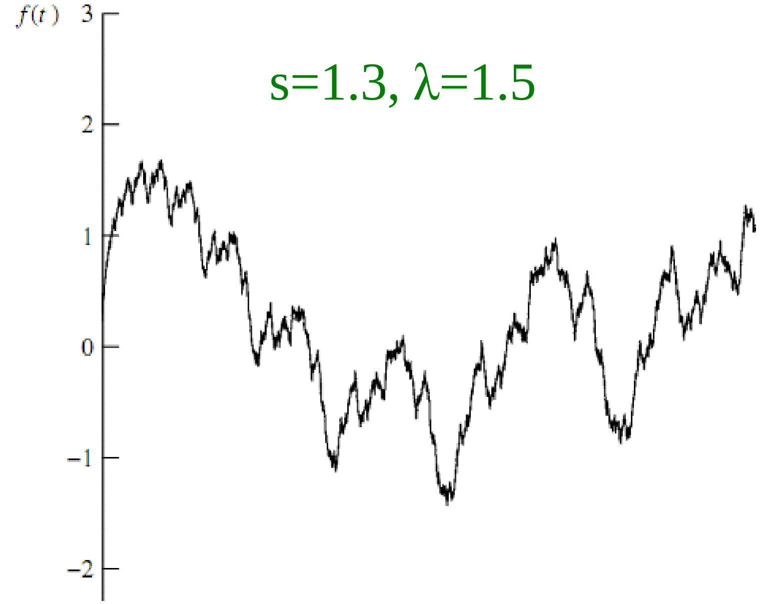
Many interesting fractals, both of theoretical and practical importance, occur as graphs of functions. Indeed many time series have fractal features, at least when recorded over fairly long time spans: examples include wind speed, levels of reservoirs, population data and some financial time series market (the famous Mandelbrot cotton graphs)

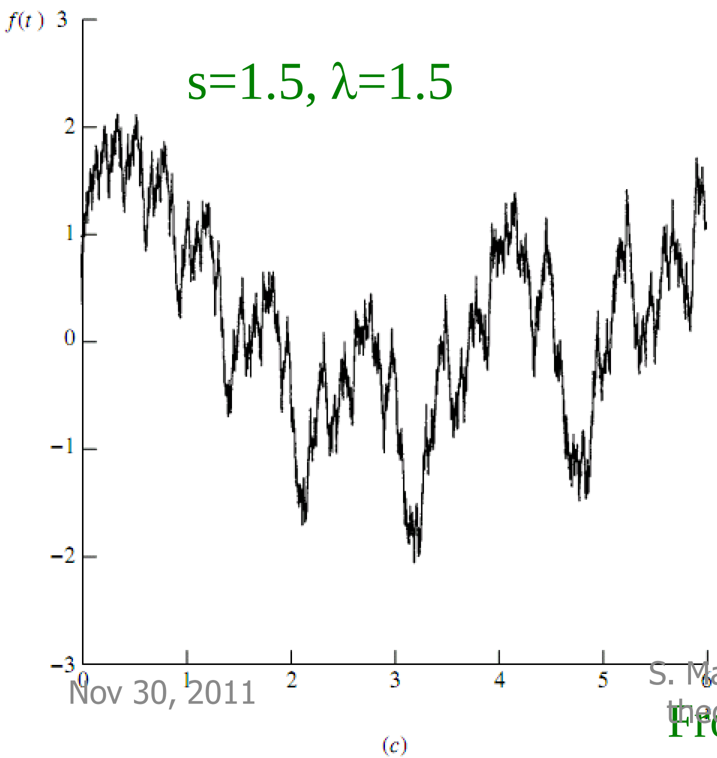
Weierstrass nowhere differentiable continuous function:

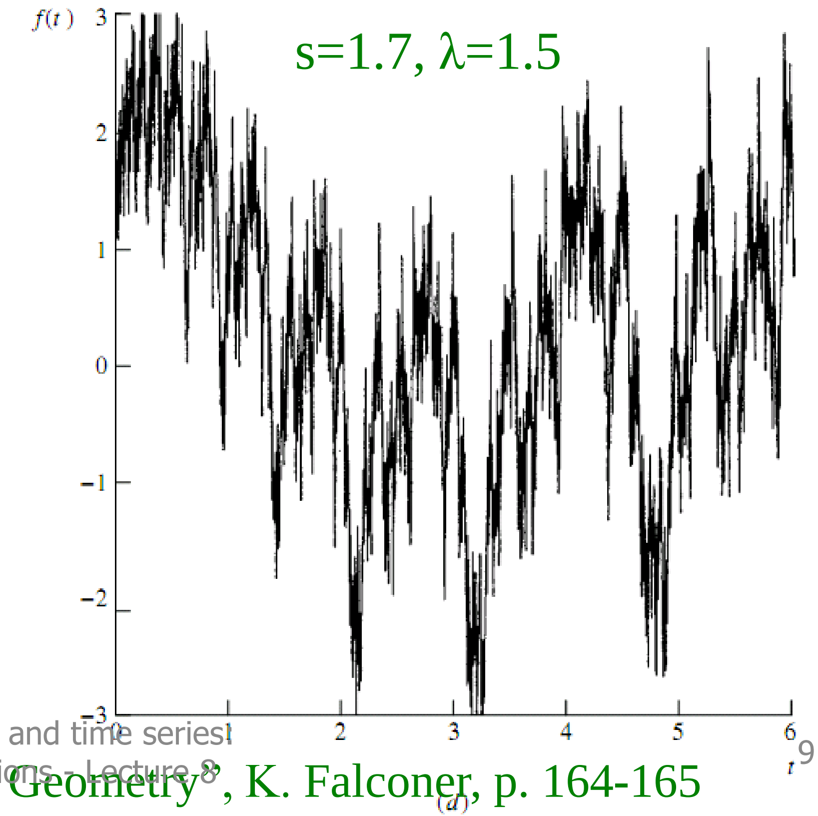
$$f(t)=\Sigma_{1\leq k\leq\infty}\ \lambda^{(s-2)k} \sin(\lambda^k\ t) \qquad 1<s<2,\ \lambda>2$$

The graph of f has box dimension s for λ large enough.









From “Fractal Geometry”, K. Falconer, p. 164-165

Fractal graphs and i.f.s.

(from K. Falconer, Fractal Geometry, Wile (2003)

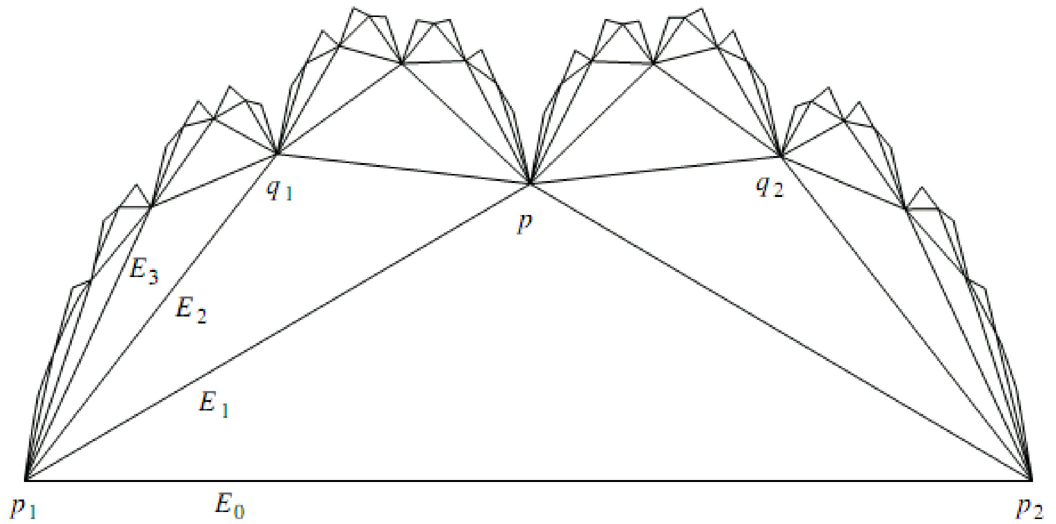


Figure 11.3 Stages in the construction of a self-affine curve F. The affine transformations S_1 and S_2 map the generating triangle p_1pp_2 onto the triangles p_1q_1p and pq_2p_2, respectively, and transform vertical lines to vertical lines. The rising sequence of polygonal curves $E_0, E_1, \ldots$ are given by $E_{k+1} = S_1(E_k) \cup S_2(E_k)$ and provide increasingly good approximations to F (shown in figure 11.4(a) for this case)

$$S_i(t, x) = (t/m + (i-1)/m,\ a_it + c_ix + b_i).$$

Thus the S_i transform vertical lines to vertical lines, with the vertical strip $0 \leqslant t \leqslant 1$ mapped onto the strip $(i-1)/m \leqslant t \leqslant i/m$. We suppose that

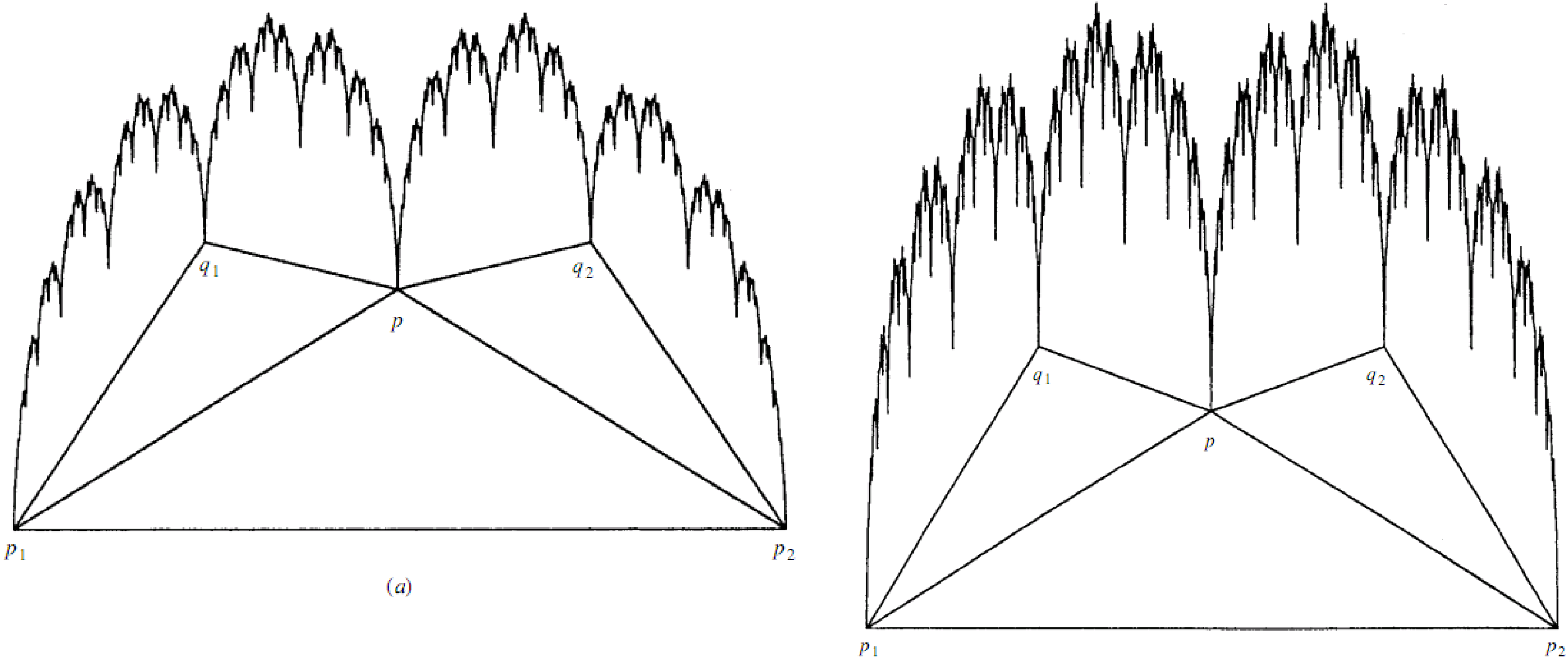
$$1/m < c_i < 1 \qquad (11.9)$$

so that contraction in the t direction is stronger than in the x direction.

Let $p_1 = (0, b_1/(1-c_1))$ and $p_m = (1, (a_m + b_m)/(1-c_m))$ be the fixed points of S_1 and S_m. We assume that the matrix entries have been chosen so that

$$S_i(p_m) = S_{i+1}(p_1) \qquad (1 \leqslant i \leqslant m-1) \qquad (11.10)$$

so that the segments $[S_i(p_1), S_i(p_m)]$ join up to form a polygonal curve E_1. To



Self-affine curves defined by the two affine transformations that map the triangle p_1pp_2 onto p_1q_1p and pq_2p_2 respectively. In (a) the vertical contraction of both transformations is 0.7 giving dim graph f = 1.49, and in (b) the vertical contraction of both transformations is 0.8, giving dim graph f = 1.68

from K. Falconer, Fractal Geometry, Wiley (2003)

Probabilistic i.f.s.

$\mathcal{F} = \{w_1, \ldots, w_N\}$, $w_i : X \to X$ contraction of constant s_i, $0 \le s_i < 1$

$(p_1,\ldots,p_N)$ probability vector $0 \le p_i \le 1$, $p_1+\ldots+p_N = 1$

Iteration: at each step with probability p_i one applies w_i

i.f.s.: k iterates of a point $\to N^k$ points $\mathcal{W} : \mathcal{H}(X) \to X$

$\mathcal{W}(E) = U_1\, w_i(E)$

Probabilistic i.f.s.: k iterates of a point $\to$ k points

Theorem: each probabilistic i.f.s. has a unique Borel probability invariant measure μ with support $= \mathcal{A}$

Invariance: $\mu(E) = \Sigma_{1\le i\le N}\, p_i \mu(w_i^{-1}(E))$ for all Borel sets E, equivalently

$\int_X g(x)d\mu(x) = \Sigma_{1\le i\le N}\, p_i \int_X g(w_i(x))d\mu(x)$ for all continuous functions g

Probabilistic i.f.s.

If $\mathcal{M}$ denotes the space of Borel probability measures on X endowed with the metric

$$d(\nu_1,\nu_2)=\sup\{|\int_X g(x)d\nu_1(x)-\int_X g(x)d\nu_2(x)|, \text{ g Lipschitz, } Lip(g) \le 1\}$$

Then a probabilistic i.f.s. acts on measures as follows

$$L_{p,w}\,\nu= \Sigma\, p_i \nu\circ w_i^{-1}$$

And by duality acts con continuos functions $g:X\rightarrow \mathbf{R}$

$$\int_X g(x)d(L_{p,w}\,\nu)(x)= \Sigma_{1\le i\le N}\, p_i \int_X g(w_i(x))d\nu(x)$$

It is easy to verify that

$$d(L_{p,w}\,\nu_1 , L_{p,w}\,\nu_2) \le s\, d(\nu_1,\nu_2)$$

from which the previous theorem follows

Multifractal analysis of measures

Local dimension (local Hölder exponent) of a measure μ at a point x:

$\dim_{loc} \mu(x) = \lim_{r \to 0} \log \mu(B(x,r))/\log r$ (when the limit exists)

$\alpha > 0$, $E_\alpha = \{x \in X, \dim_{loc} \mu(x) = \alpha\}$

For certain measures μ the sets E_α may be non-empty over a range of values of α: multifractal measures

multifractal spectrum (singularity spectrum) of the multifractal measure μ: is the function $\alpha \to f(\alpha) = \dim E_\alpha$

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With equal probabilities, the Random Algorithm for the IFS with these rules

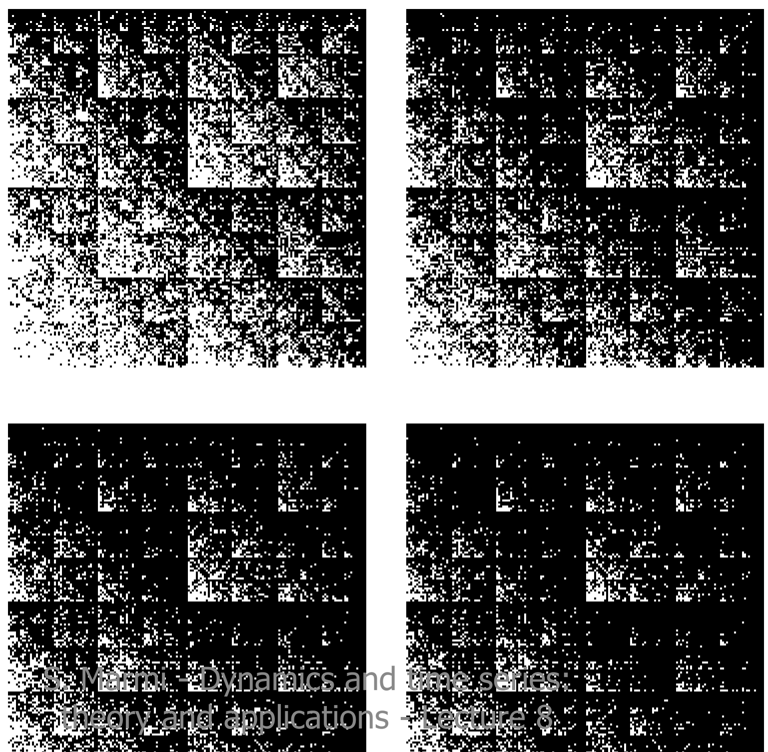
$T_3(x, y) = (x/2, y/2) + (0, 1/2)$	$T_4(x, y) = (x/2, y/2) + (1/2, 1/2)$
$T_1(x, y) = (x/2, y/2)$	$T_2(x, y) = (x/2, y/2) + (1/2, 0)$

fills in the unit square uniformly.

The pictures below were generated with these probabilities

$$p_1 = 0.1,\ p_2 = p_3 = p_4 = 0.3.$$

Successive pictures show increments of 25000 points. With enough patience, the whole square will fill in, but some regions fill in more quickly than others.



Multifractals

Variable Probability Histograms

The probabilities of applying each transformation represent the fraction of the total number of iterates in the region determined by the transformation.

With the IFS and probabilities of the last example, in a typical picture about 0.1 of the points will lie in the square with address 1, and about 0.3 of the points will lie in each of the squares with address 2, 3, and 4.

Arguing in the same way, about 0.01 = 0.1*0.1 of the points will lie in the square with address 11, about 0.03 = 0.1*0.3 of the points will lie in the square with address 12, and so on.

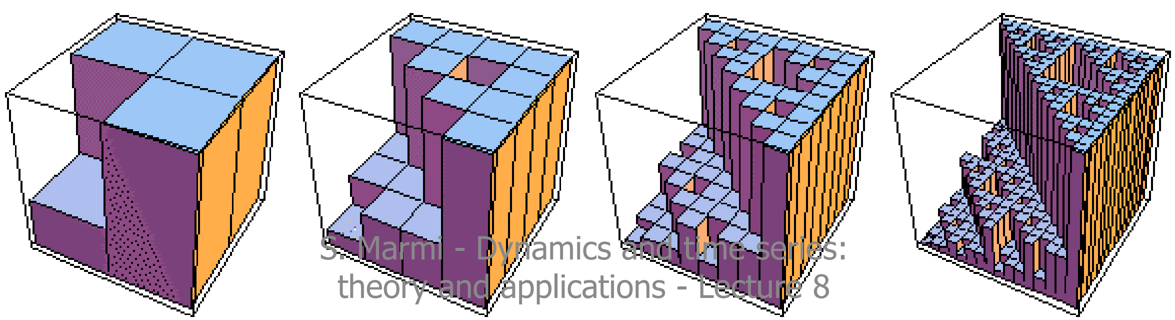
.3	.3
.1	.3

.09	.09	.09	.09
.03	.09	.03	.09
.03	.03	.09	.09
.01	.03	.03	.09

Higher iterates are easier to understand visually.

Here we show the first four generations, with the height of the box in a region representing the fraction of the points in that region.

All the pictures have been adjusted to have the same height, whereas square 4 has 0.3 of the points, square 44 has 0.09 of the points, square 444 has 0.027 of the points, and so on.



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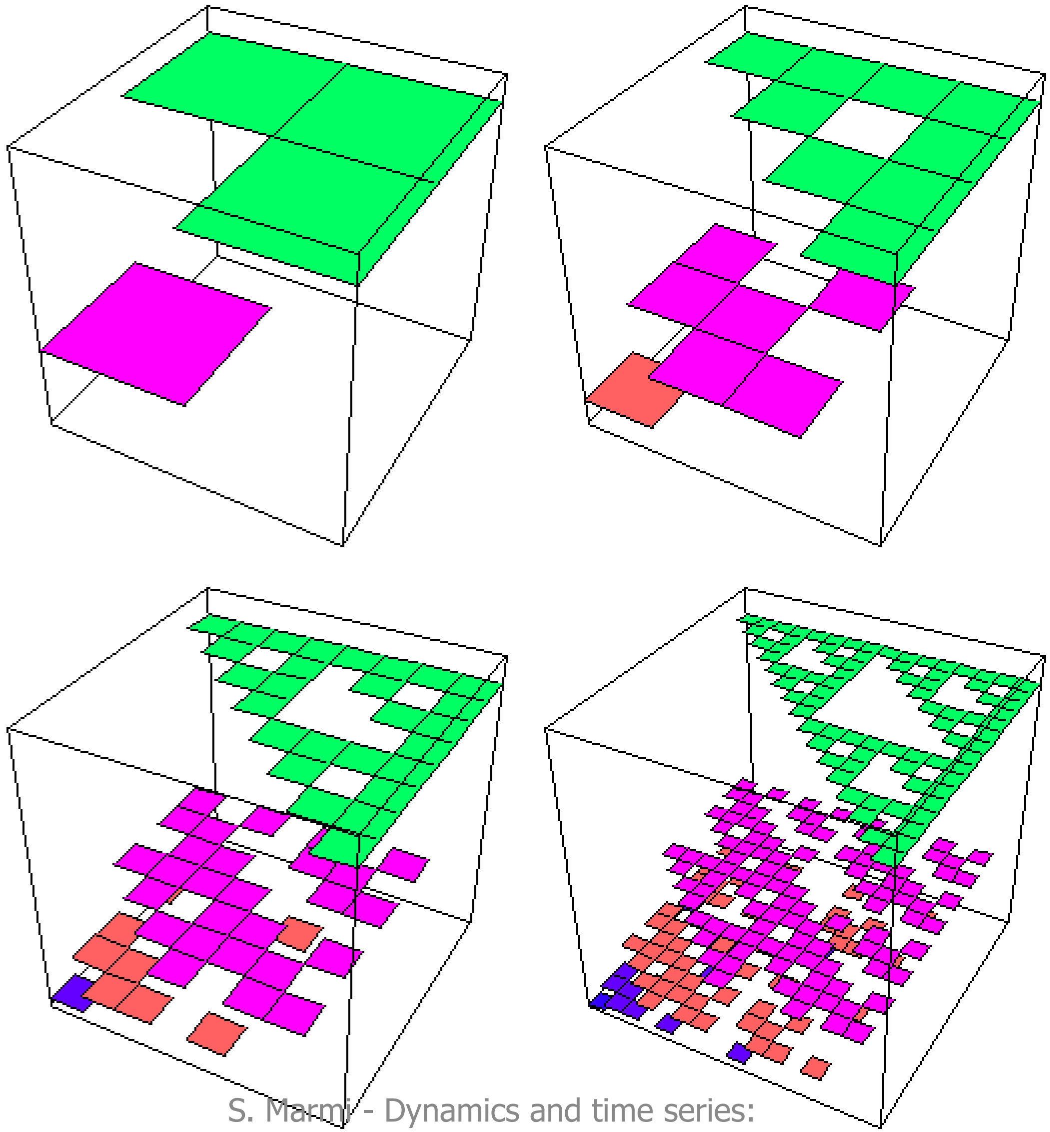
So again the height represents the fraction of the points landing in that region.



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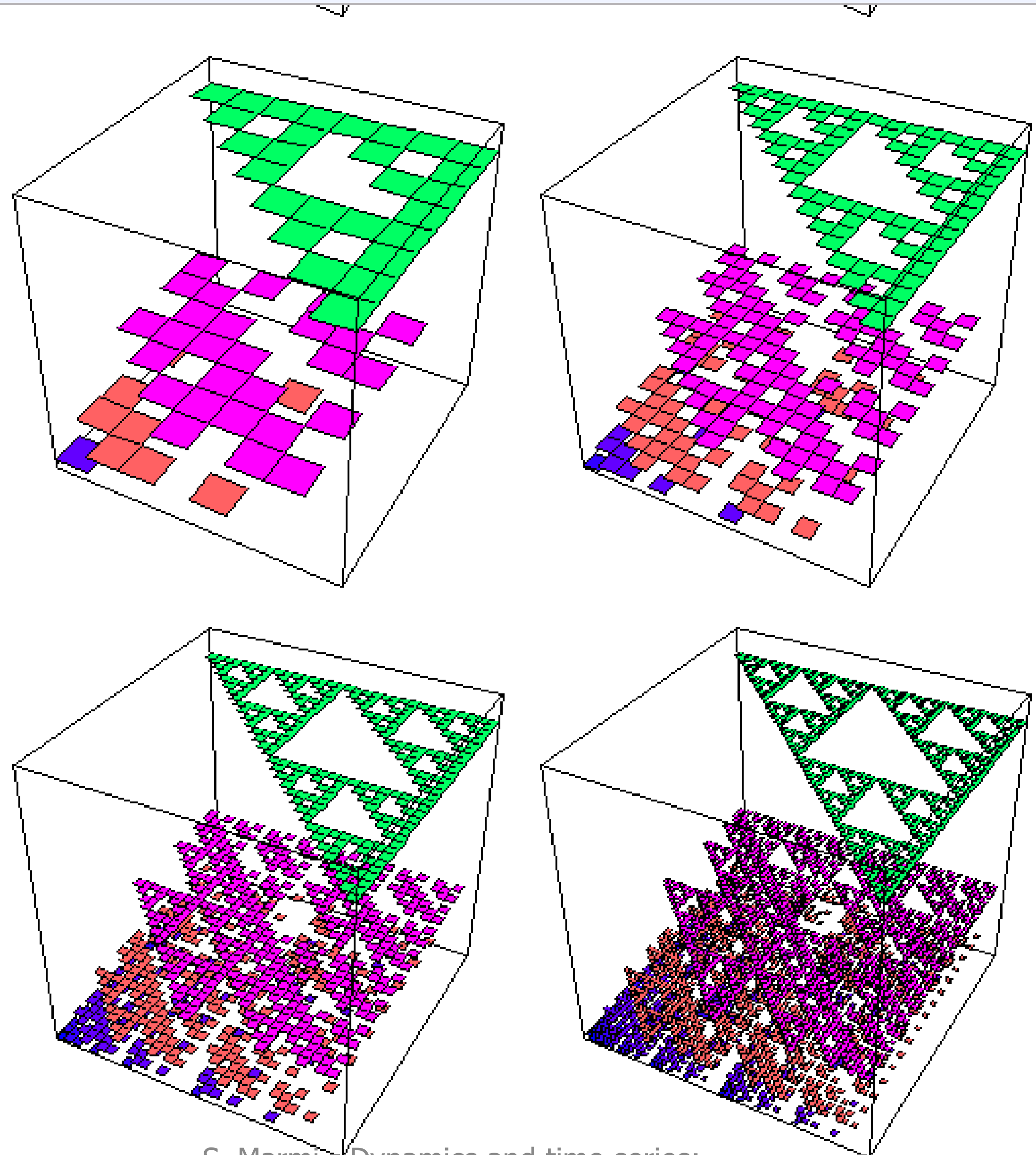

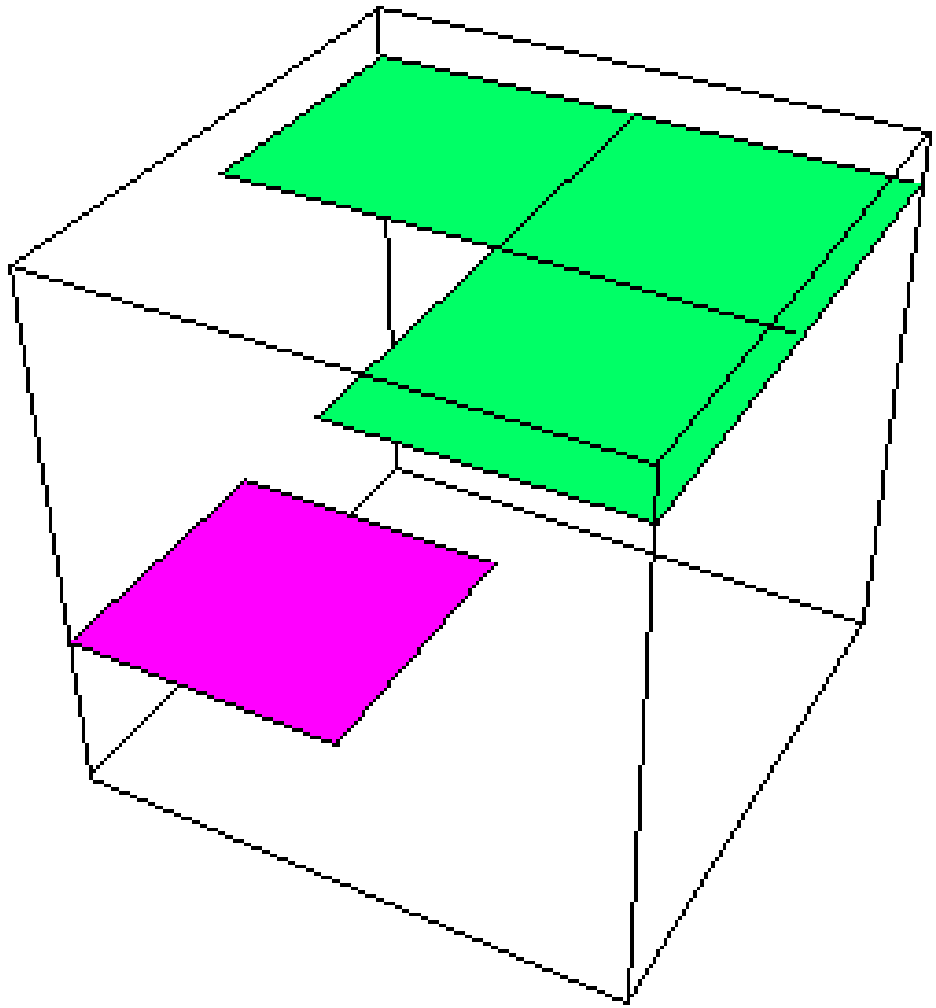

http://classes.yale.edu/fractals/MultiFractals/MFGaskSect/MFGaskSectMv.gif

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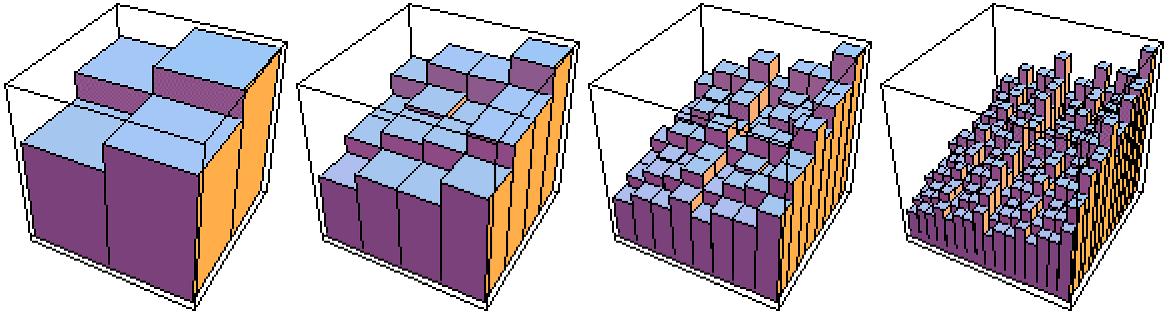
Different Probabilities, Another Example

In this example, we introduce more variability in the probabilities:

$$p_1 = 0.2,\ p_2 = 0.25,\ p_3 = 0.25, \text{ and } p_4 = 0.3.$$

Among other things, the number of values of the probabilities of regions increases more rapidly.



Smaller regions have smaller probabilities; if these graphs weren't rescalled vertically they would appear to become closer and closer to a flat surface of heig
0. Click here for an animation of the first four iterates, all drawn to the same vertical scale.

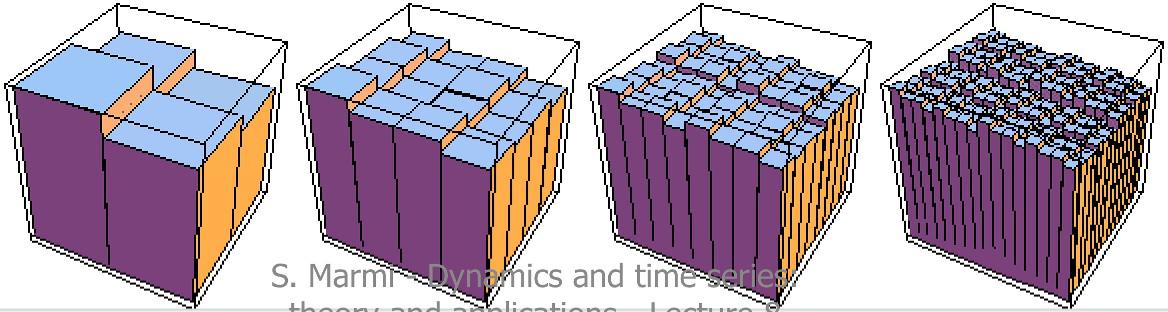
For each region we expect that

$$\text{prob scales as (side length)}^{\text{some power}}$$

So instead of letting the height of the graph represent the probability of the region, now we assign height Log(prob)/Log(side length) to the region.

Because the probability measures the fraction of the points that occupy a region, we think of this ratio as a dimension.

Being viewed at the resolution of the side length of the region, this is a coarse Holder exponent; it is also called the coarse dimension.



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Multifractals

Local Holder Exponents

Taking limits as the side length of the regions go to zero, the coarse Holder exponent can be refined to the local Holder exponent (or *roughness*) at (x, y) is

$$d_{loc}(x,y) = \lim_{n \to \text{infinity}} \text{Log}(\text{Prob}(i_1 \ldots i_n))/\text{Log}(2^{-n})$$

where $\text{Prob}(i_1 \ldots i_n)$ is the probability $pr(i_1) * \ldots * pr(i_n)$, if (x,y) lies in the square with address $i_1 \ldots i_n$.

The value for a square of finite length address is called the coarse Holder exponent. So the local Holder exponent of a point (x, y) is the limit as N -> infinity of the coarse Holder exponents of the length N address squares containing (x, y).

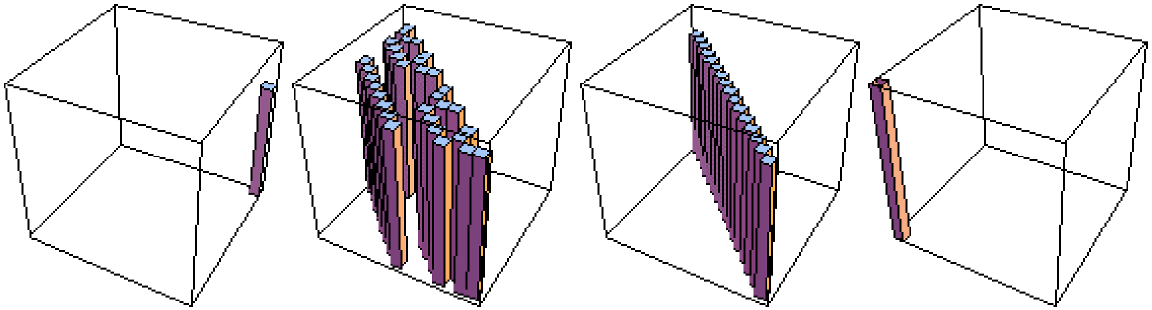
Now define

$$E_{alpha} = \{(x, y): d_{loc}(x, y) = alpha\},$$

the collection of all points of the fractal having local Holder exponent alpha.

As alpha takes on all values of the local Holder exponent, we decompose the fractal into these sets E_{alpha}.

Here are examples, E_{alpha} (alpha = column height) for the lowest value of alpha (on the left), two intermediate values, and the highest value.



Click here for an animation scanning through all the values of alpha, from lowest to highest, resolved to boxes have side length $1/2^4$.

Because each local Holder exponent alpha is the exponent for a power law, a multifractal is a process exhibiting scaling for a range of different power laws.

The multifractal structure is revealed by plotting $\dim(E_{alpha})$ as a function of alpha.

(In general, a dimension more subtle than the box-counting dimension must be used. We ignore this complication here.)

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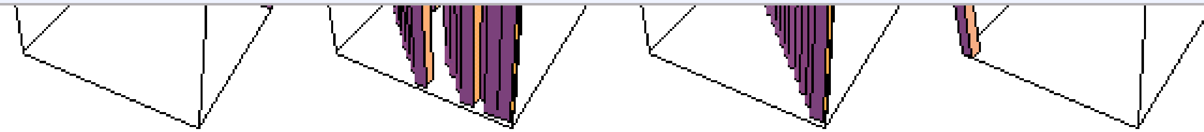

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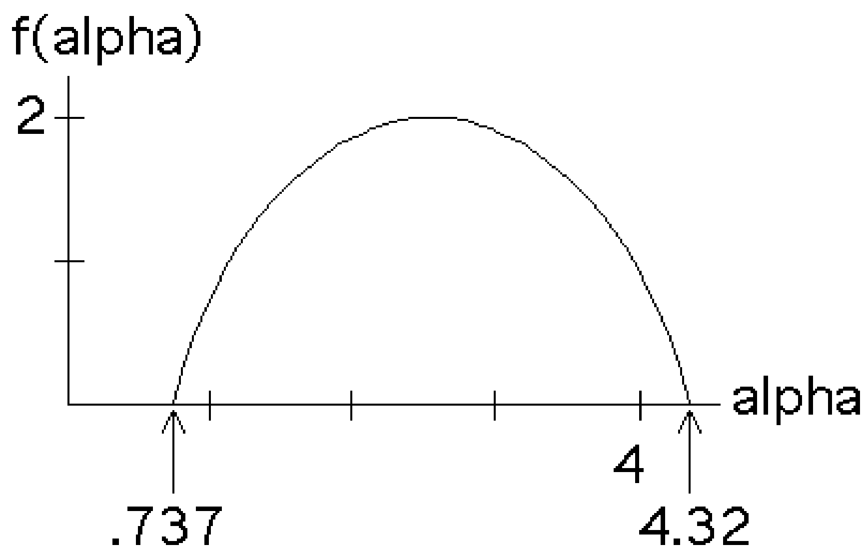
The multifractal structure is revealed by plotting $\dim(E_{alpha})$ as a function of alpha.

(In general, a dimension more subtle than the box-counting dimension must be used. We ignore this complication here.)

This graph is called the f(alpha) curve.

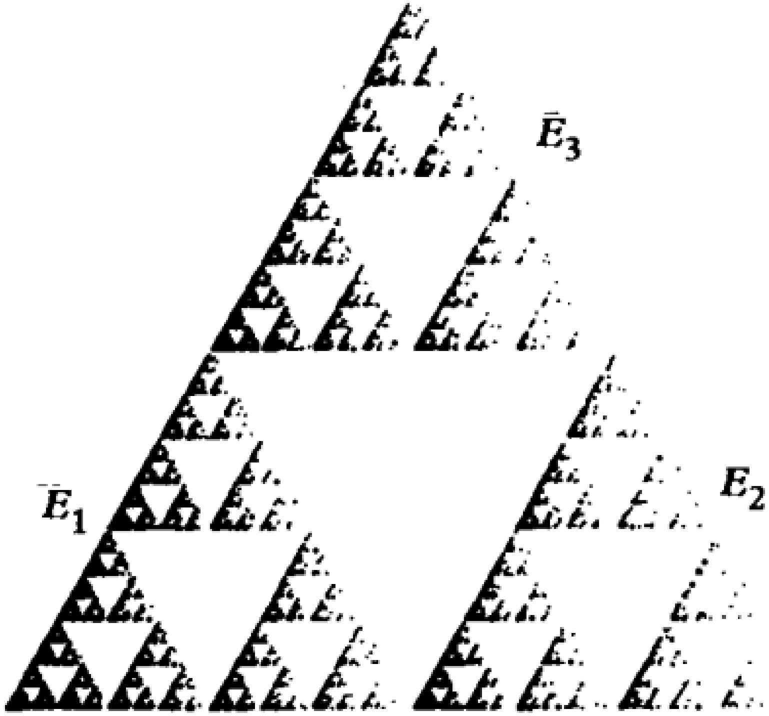
Here is the f(alpha) curve for the example with $p_1 = 0.2$, $p_2 = p_3 = 0.25$, and $p_4 = 0.3$.

At least in this example, sets E_{alpha} for the lowest and highest values of alpha reduce to points in the limit, hence have dimension f(alpha) = 0. This is represented in the left and right endpoints of the curve lying on the x-axis.



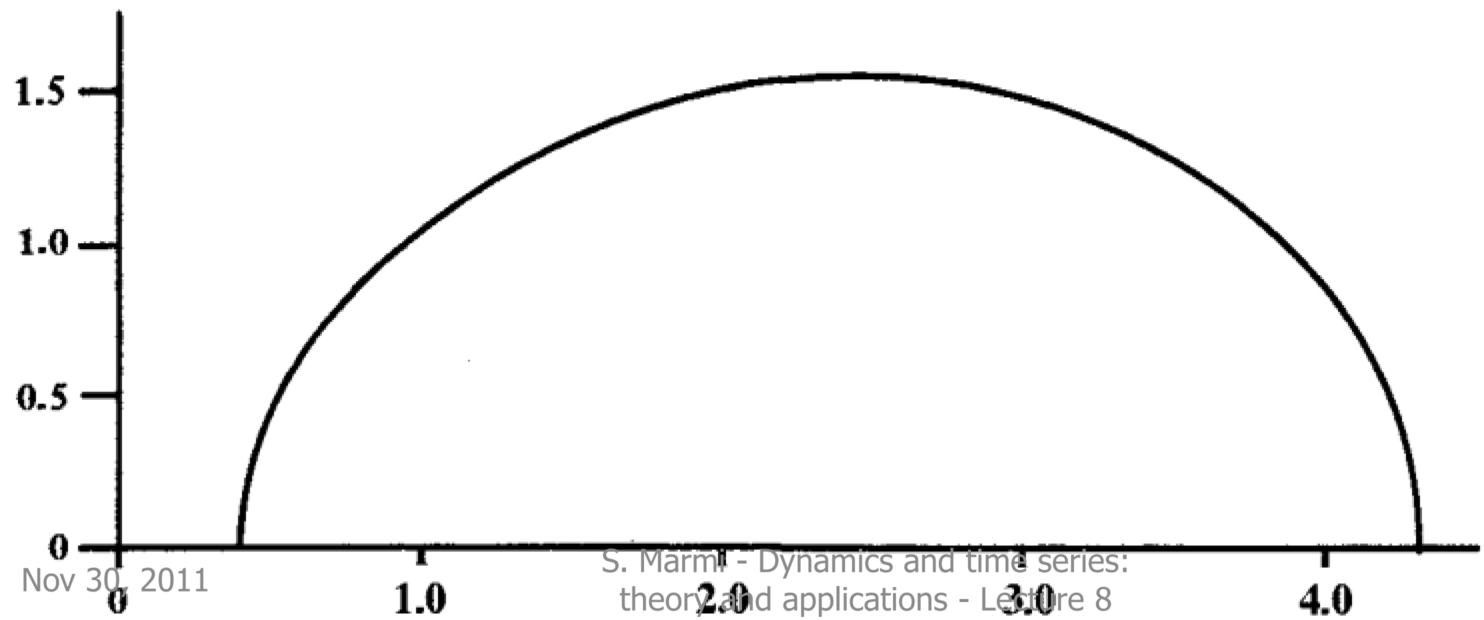
This result is derived under more general conditions in a later section.

Return to Multifractals.



K. Falconer, Techniques in Fractal geometry

P=(0.8,0.05,0.15)



The Legendre transform of f(α)

$\mathcal{F} = \{w_1, \ldots, w_N\}$, $w_i : X \rightarrow X$ contraction of constant s_i, $0 \leq s_i < 1$

$(p_1,\ldots,p_N)$ probability vector $0 \leq p_i \leq 1$, $p_1+\ldots+p_N = 1$

The dimension d of the attractor $\mathcal{A}$ is the solution of the equation

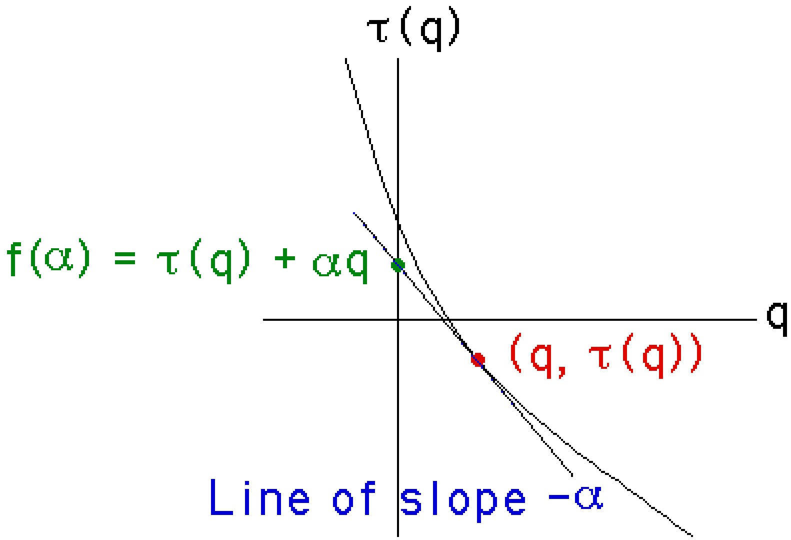
$$s_1^d + s_2^d + \ldots + s_N^d = 1$$

The singularity spectrum $\alpha \rightarrow f(\alpha)$ of a probabilistic i.f.s. is the Legendre transform of the function $q \rightarrow \tau(q)$ obtained solving the functional equation

$$p_1^q\, s_1^{\tau(q)} + p_2^q\, s_2^{\tau(q)} + \ldots + p_N^q\, s_N^{\tau(q)} = 1$$

Defining f(α)

For each point (q, τ(q)) say the slope of the tangent line is -α. That is, α = -dτ/dq.



This tangent line passes through the point (q, τ(q)) and the point (0, y). Consequently,

$$-\alpha = (y - \tau(q))/(0 - q)$$

Solving for y,

$$y = q\cdot\alpha + \tau(q)$$

Call this y-value f(α):

$$f(\alpha) = q\cdot\alpha + \tau(q)$$

Return to Multifractals from IFS.

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The singularity spectrum α→f(α) of a probabilistic i.f.s. is the Legendre transform of the function q→τ(q)

Fundamentals of investing

Investment returns are strongly related to their risk level

Usually and loosely risk is quantified using volatility (standard deviation)

U.S. Treasury bills /bonds (short/long term bonds 1month-1year / 2-30 years): very safe (until now…) and very low/medium yield. Most of the price uncertainty for longer term bonds comes from the effect of inflation
T.I.P. : inflation indexed bonds which guarantee a positive real return
Stocks: risky but higher returns (on the long run…). Companies sell shares of stock to raise capital: they ``go public'' by agreeing to sell a certain number of shares on an exchange. Each share represents a given fraction of the ownership of the company.
Certain stocks pay *dividends*, cash payments reflecting profits returned to shareholders. Other stocks reinvest all returns back into the business.
In principle, what people will pay for a stock reflects the health of its current business, future prospects, and expected returns. But the current *price* of a stock is completely determined by what people are willing to pay for it. If there were no differences of opinion as to the value of a stock, there would be no trading.

Analisi dei rendimenti degli indici S&P500, Lehman Long Term Government Bonds, MSCI Europe Australasia Far East, FTSE North American Real Estate Investment Trusts e Goldman Sachs Commodities Index dal 1973 al 2007. Tratto da "The case for multi-asset investing. Combining asset classes to enhance risk/return potential", Jennison Dryden-Prudential Investment disponibile online al link : http://www.jennisondryden.com/view/upload?docURL=/WDocs/45FB1E842986A540852573E2006BA8C8/$File/JD2065MultipleClass.pdf&docType=pdf

Periodo	*S&P 500*	*Lehman Long-Term*	*MSCI EAFE*	*FTSE NAREIT*	*Goldman Sachs3*	*Portafoglio classico:*	*Portafoglio AA: 20% S&P500 20% Bonds*
1973-2007	**total return**	**Govern-ment Bond**	**total return**	**Equity Index**	**Commo-dities Index**	**50% S&P500**	**20% EAFE 20% NAREIT**
Rendimento annuale medio	10.97%	8.90%	11.09%	13.16%	10.92%	10.31%	12.22%
deviazione standard	17.23%	11.49%	21.58%	21.58%	24.46%	11.67%	9.36%
anno migliore	37.43%	42.08%	69.94%	47.59%	74.96%	34.17%	29.91%
anno peggiore	-26.74%	-8.73%	-23.19%	-21.40%	-35.75%	-11.55%	-9.35%
% anni positivi	71%	80%	74%	80%	74%	80%	89%

Tavola 1

Rivista Aiaf n. 68 (Ottobre 08) 47

Financial markets

An *exchange* is a place where buyers and sellers trade *securities* such as stocks, bonds, options, futures, and commodities.

Each stock is typically traded on a particular exchange. Each exchange has different rules about the qualifications of companies which can be listed on it.Exchanges also differ in the *rules* by which they match buyers to sellers. The exact trading rules and mechanisms can have a significant impact on the price one gets for a given security.

The strength of an exchange's rules and their enforcement impacts the *confidence* of investors and their willingness to invest.
Exchanges provide *liquidity*, the ability to buy and sell securities quickly, inexpensively, and at fair market value.
In general, the more trading that occurs in a security, the greater its liquidity.

Bonds, Commodities and Currencies

Bond markets trade bonds (``loans") made to governments and companies. Bond prices vary according to the *term* (length of time) of the loan, the interest rate and payment schedule, the financial strength of the borrowing party, and the returns available from other investments.
Commodities are types of goods which can be defined so that they are largely indistinguishable in terms of quality (e.g. orange juice, gold, cotton, pork bellies). Commodities markets exist to trade such products, from before they are produced to the moment of shipping. Agricultural futures sell the right to buy a certain amount of a commodity at a particular price at a particular point in the future. The existence of agricultural futures gives suppliers and consumers ways to protect themselves from unexpected changes in prices.
The prices of agricultural commodities are affected by changes in supply and demand resulting from weather, political, and economic forces.
Currency Markets: The largest financial markets by volume trade different types of currency, such as dollars, Euros, and Yen.
The *spot price* gives the cost of buying a good now, while *futures* permit one to buy the right to buy or sell goods at fixed prices at some future date.
Typically, each seller has a buy and sell price for a given currency, and makes their money from the *spread* between these two prices.
Ideally the demand for buying equals selling, or else the prices must change.
Currency markets are used to (a) acquire funds for international trade, (b) hedge against risks of currency fluctuations, (c) speculate on future events.

Stock prices and indices

Stock indices are typically weighted averages of the prices of the component stocks. Usually the weights are proportional to the market capitalization= (price of a share)*(number of existing shares) of the stock.

The same formulae as before are used to calculate returns from index levels. Very often dividends are excluded from the index.

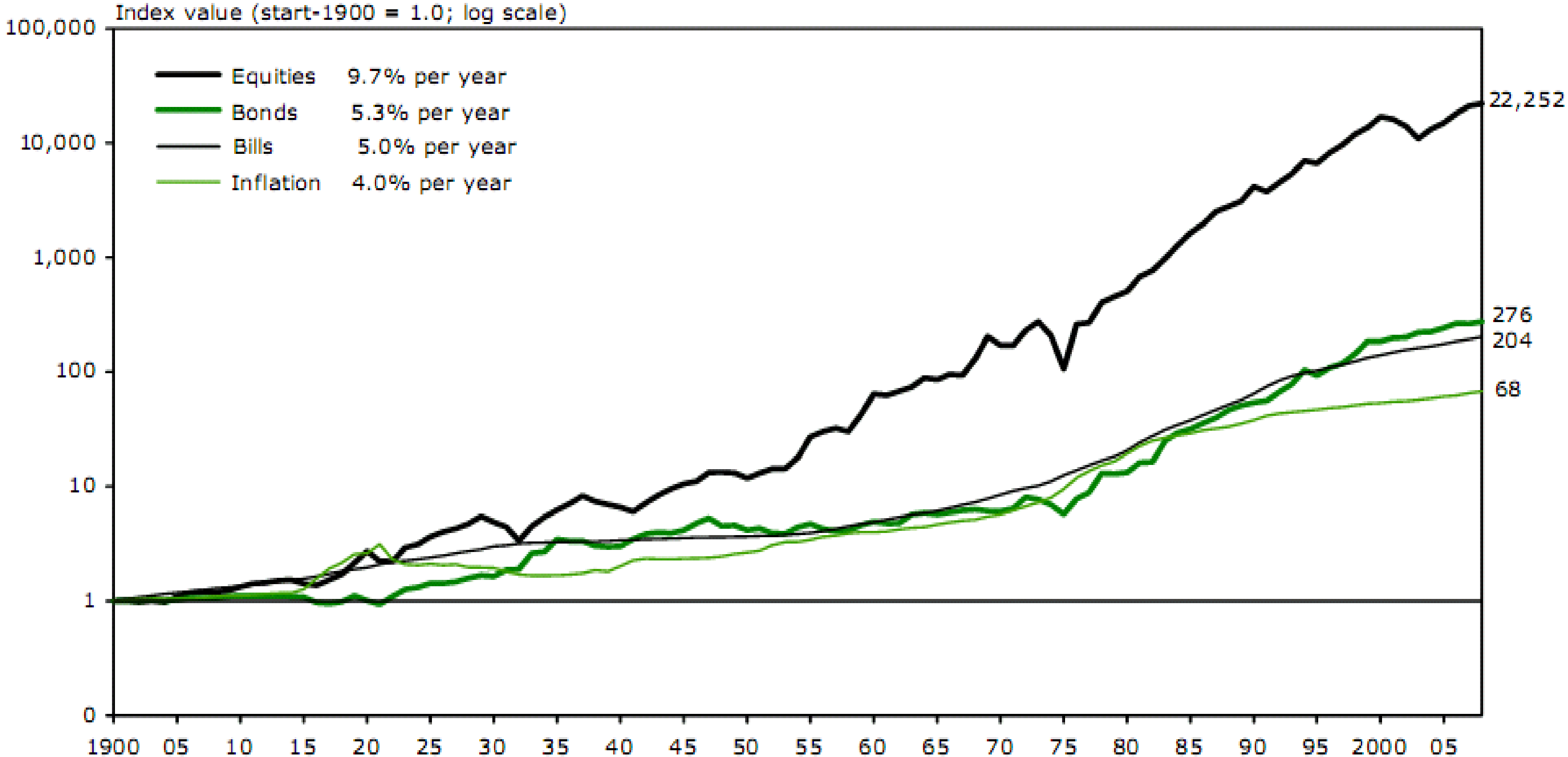
Dow Jones Industrial Average: 30 U.S. stocks (corresponding to 30 leading companies), price weighted

S&P500: 500 U.S. stocks, capitalization weighted

Stoxx 600: 600 european stocks, capitalization weighted

Stocks, bonds, bills and inflation in the UK from 1900 to 2007

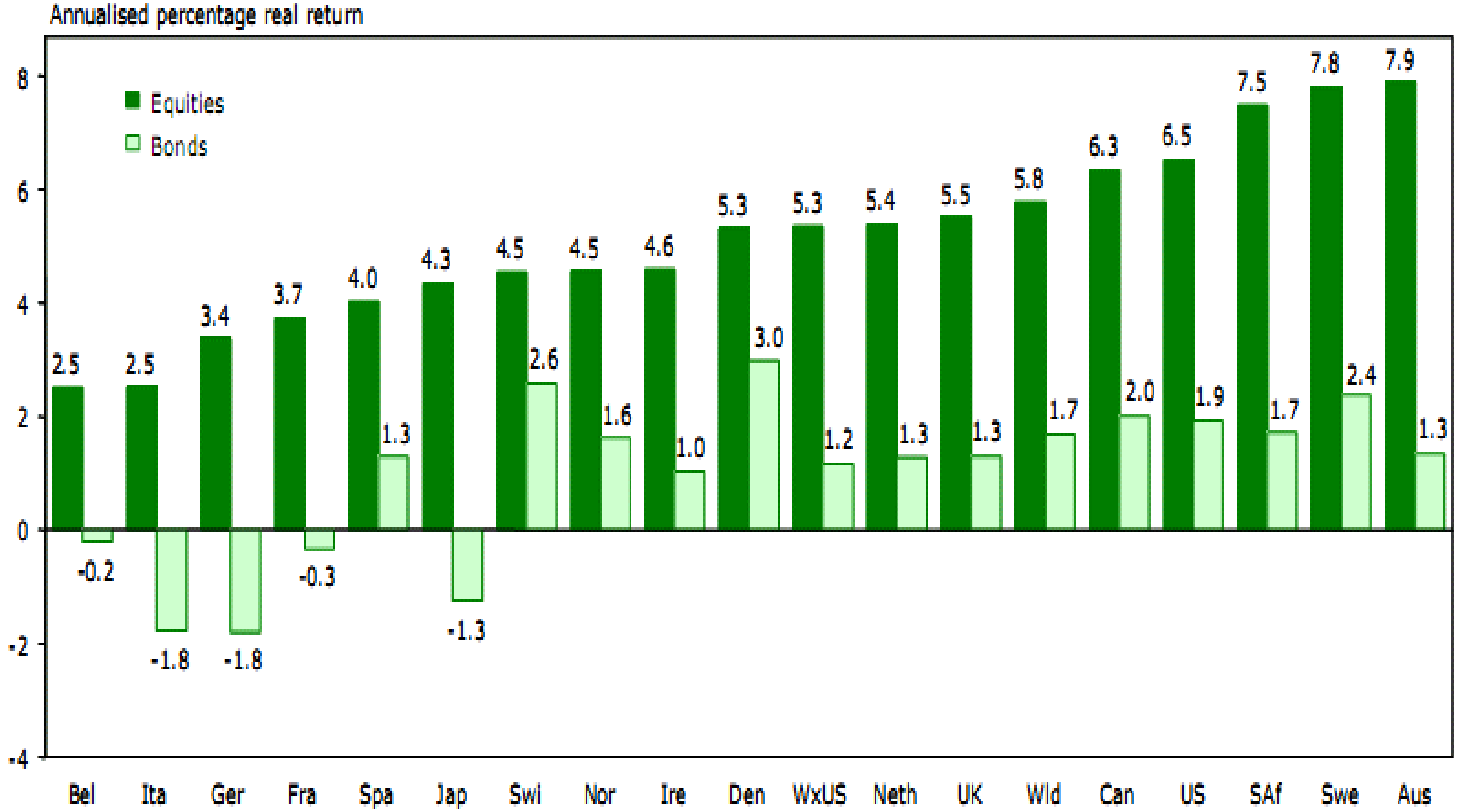
Figure 4: Cumulative returns on UK asset classes in nominal terms, 1900–2007



Source: ABN AMRO/LBS Global Investment Returns Yearbook 2008, chart 12

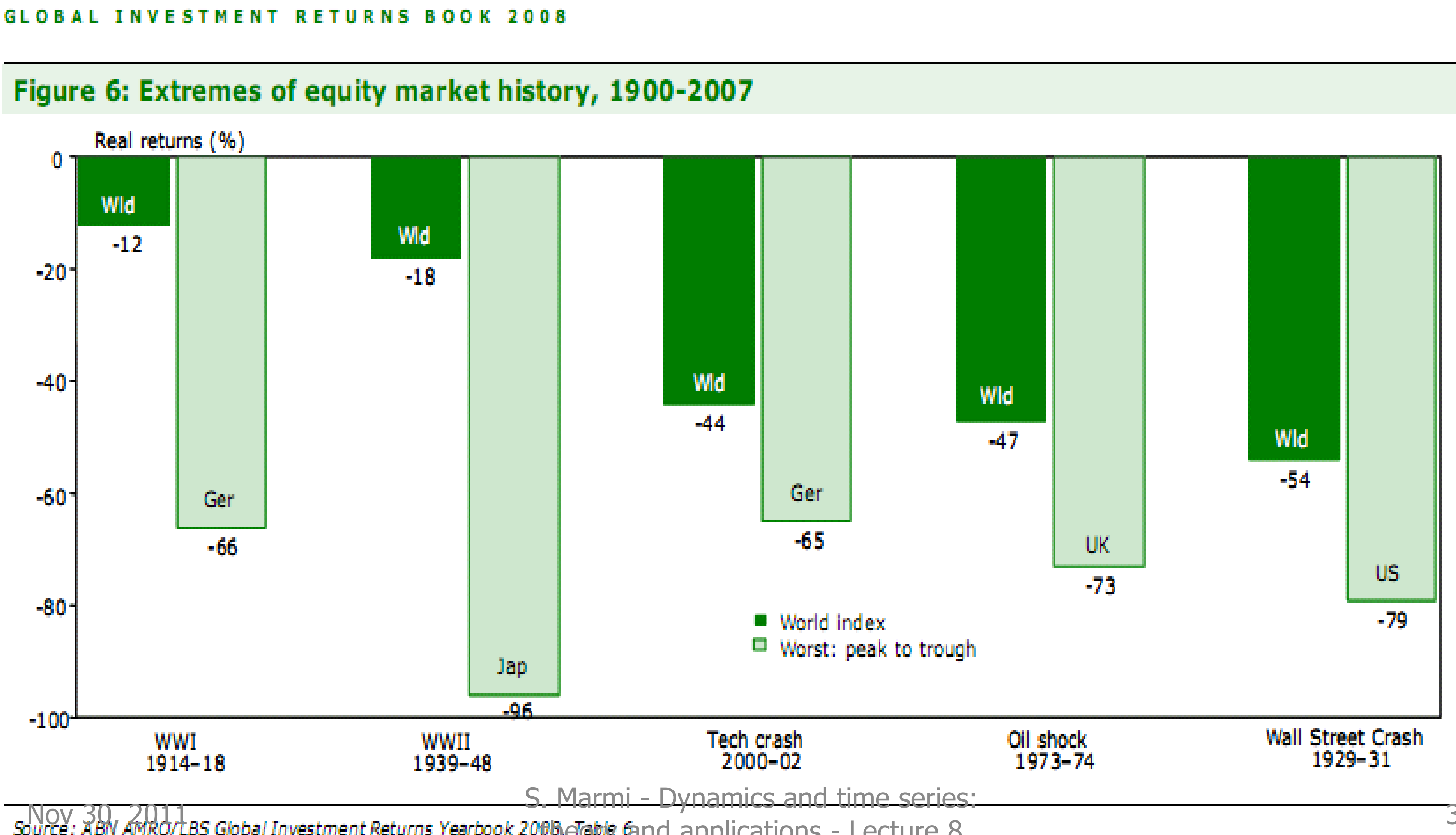
Annualized real (after inflation) returns of bonds and stocks: 1900-2007

Figure 5: Real returns on equities versus bonds internationally, 1900—2007



Source: ABN AMRO/LBS Global Investment Returns Yearbook 2008, chart 14

Stock market crashes (before 2008)

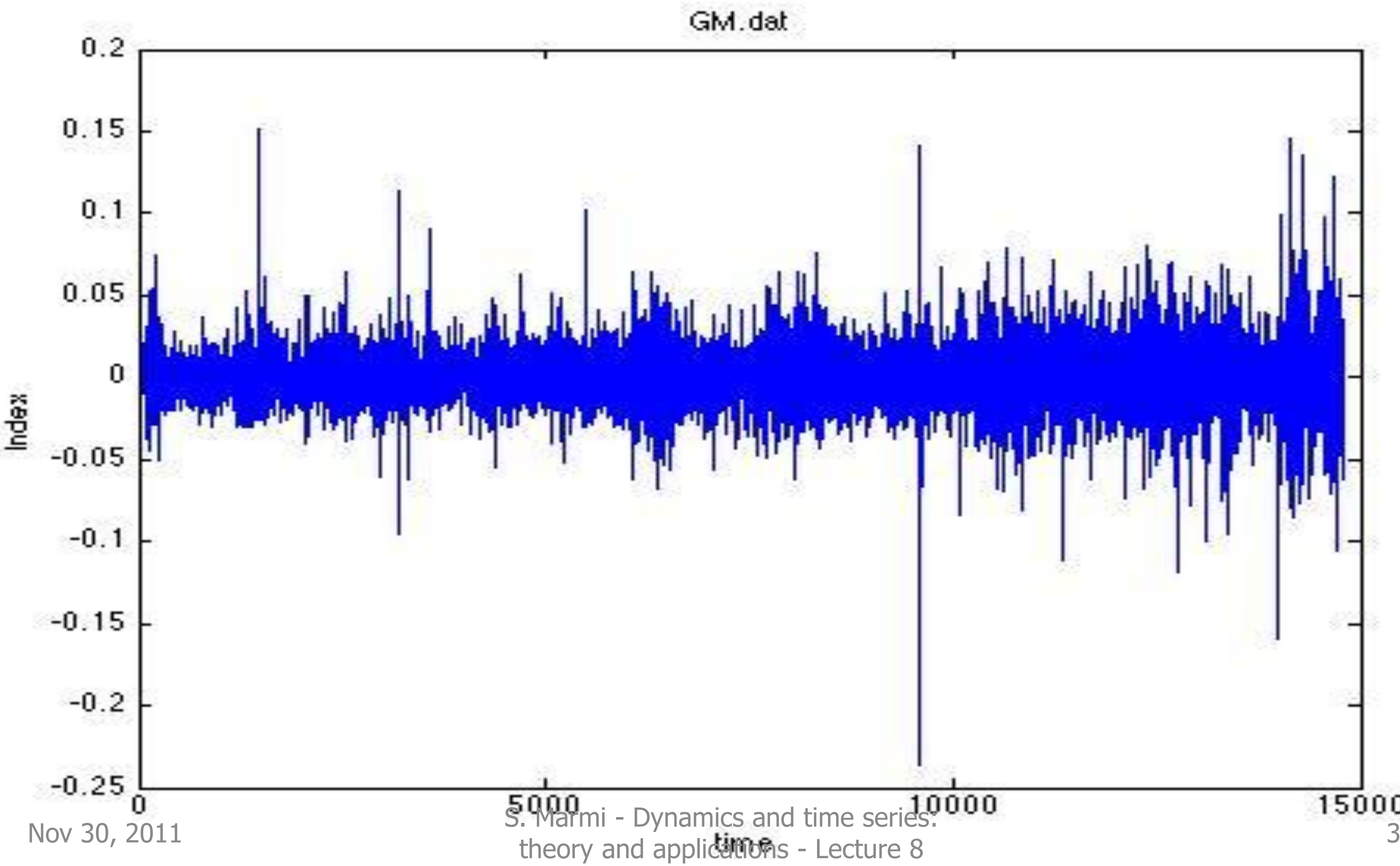


Volatility of stocks

During the period 1900-2007, UK's standard deviation of 19.8% places it alongside the US (20.0%) at the lower end of the risk spectrum. The highest volatility markets were Germany (32.3%), Japan (29.8%), and Italy (28.9%), reflecting the impact of wars and inflation.

Chicago Board Options Exchange Volatility Index, a popular measure of the implied volatility of S&P500 index options. A high value corresponds to a more volatile market and therefore more costly options, which can be used to defray risk from volatility. If investors see high risks of a change in prices, they require a greater premium to insure against such a change by selling options. Often referred to as the *fear index*, it represents one measure of the market's expectation of volatility over the next 30 day period.

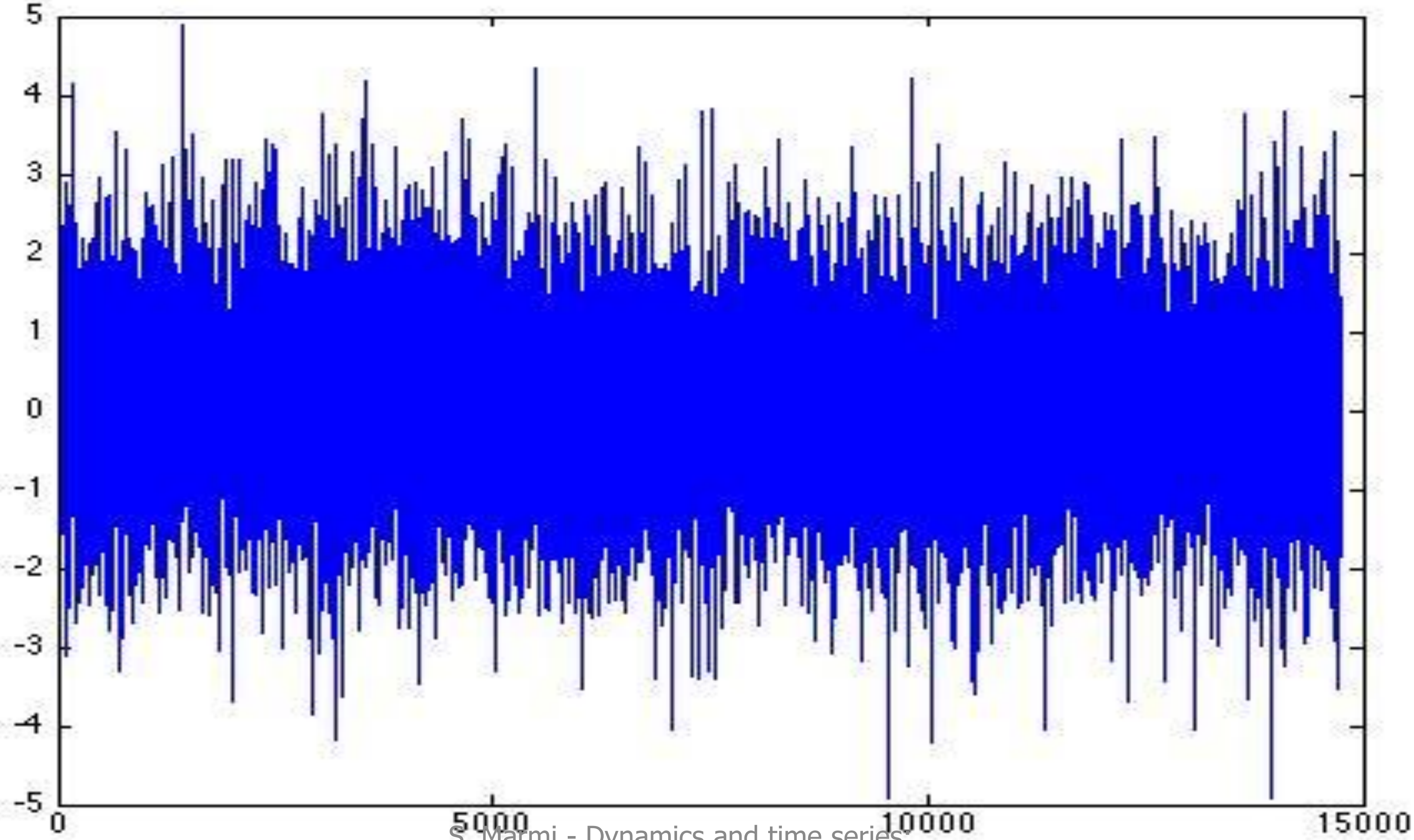
Daily returns of General Motors (1950-2008)



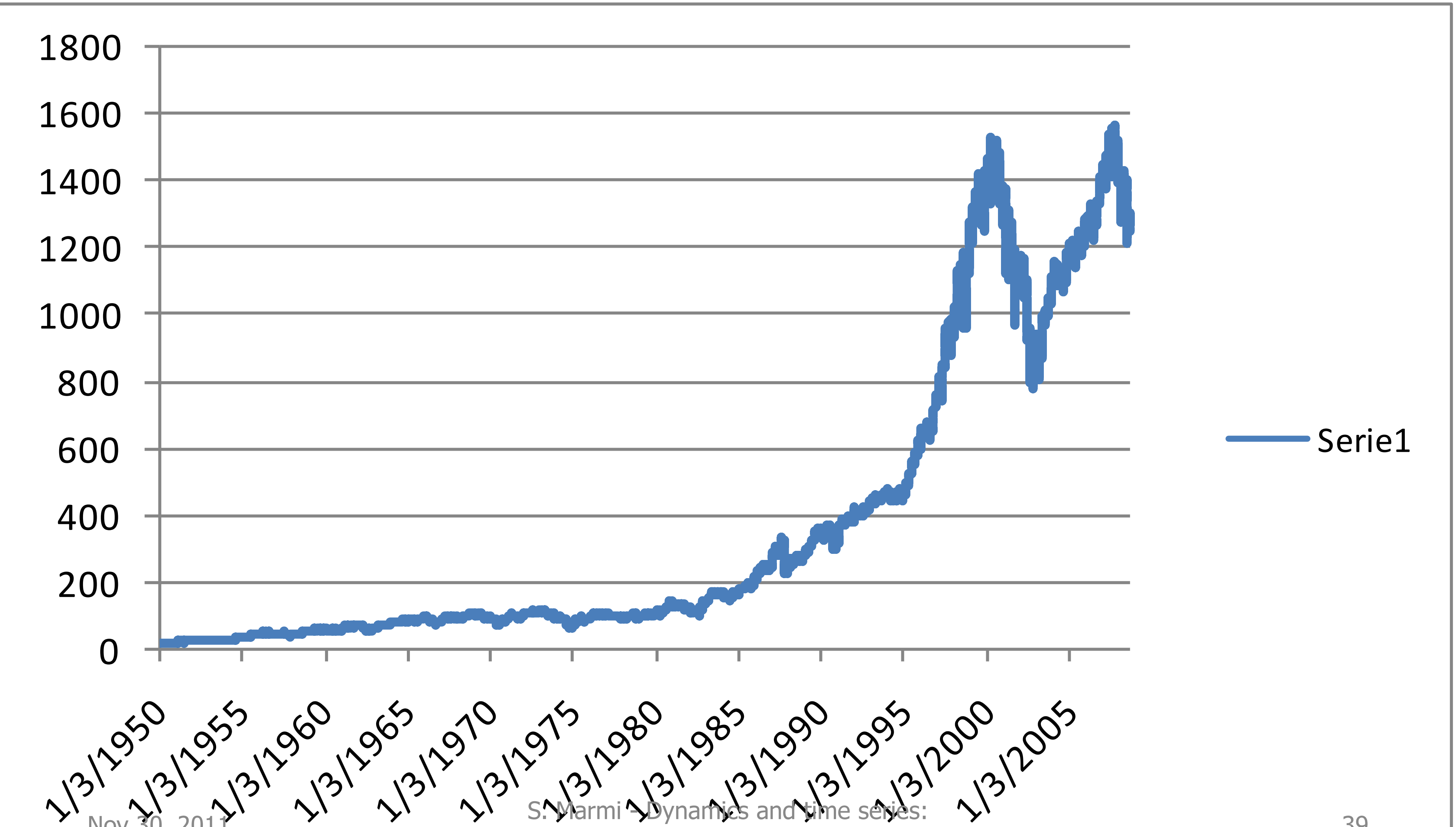
Volatility clustering

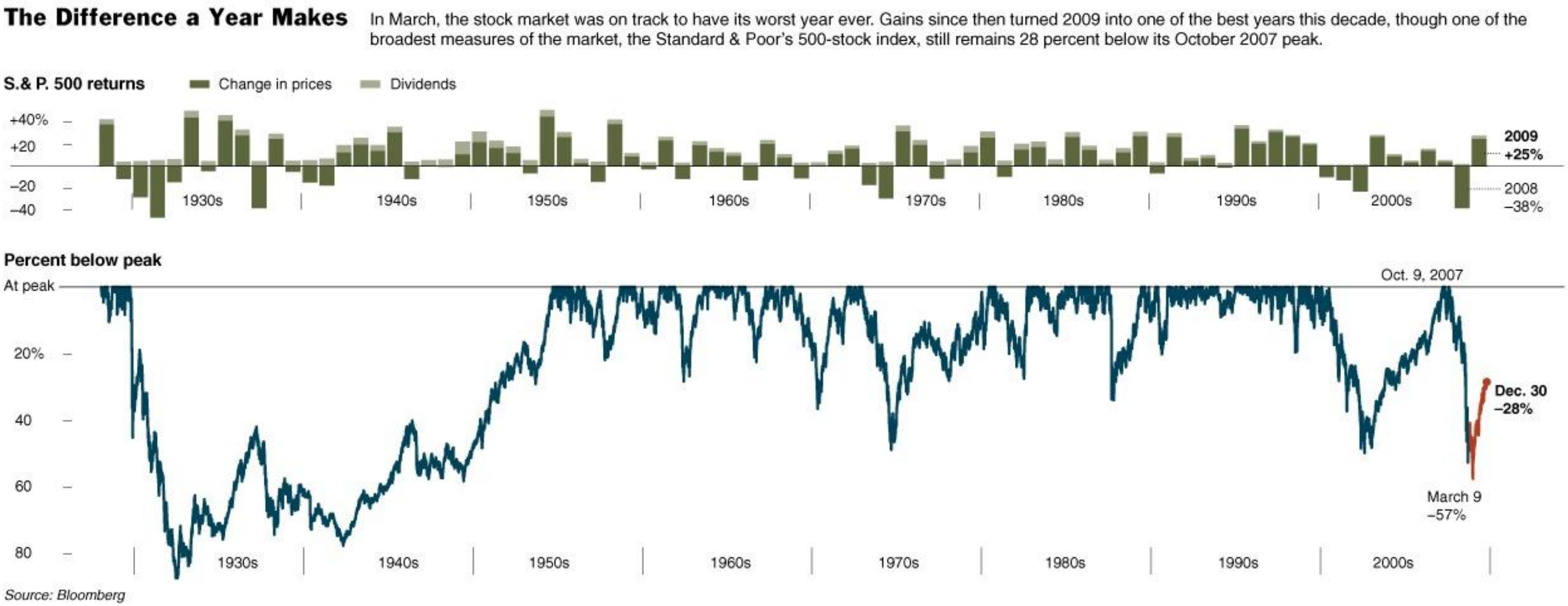
Time series plots of returns display an important feature that is usually called volatility clustering. This empirical phenomenon was first observed by Mandelbrot (1963), who said of prices that “large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes.”Volatility clustering describes the general tendency for markets to have some periods of high volatility and other periods of low volatility. High volatility produces more dispersion in returns than low volatility, so that returns are more spread out when volatility is higher. A high volatility cluster will contain several large positive returns and several large negative returns, but there will be few, if any, large returns in a low volatility cluster.

Daily returns of GM after normalization by short-term (25 days) volatility



S&P500 1950-early 2008





http://www.nytimes.com/2009/12/31/business/31stox.html?th&emc=th
http://alfaobeta.blogspot.com

Speculation and hedging

Speculators are investors who deliberately assume the risk of a loss, in return for the uncertain possibility of a reward. They bet on future events. For example, they will buy a stock because they think it will go up.

Hedgers are investors who trade so as to reduce their exposure to risk. For example, they will both buy and short a stock simultaneously.

The economic benefit of speculation

The well known speculator Victor Niederhoffer, describes the benefits of speculation: "Let's consider some of the principles that explain the causes of shortages and surpluses and the role of speculators. When a harvest is too small to satisfy consumption at its normal rate, speculators come in, hoping to profit from the scarcity by buying. Their purchases raise the price, thereby checking consumption so that the smaller supply will last longer. Producers encouraged by the high price further lessen the shortage by growing or importing to reduce the shortage. On the other side, when the price is higher than the speculators think the facts warrant, they sell. This reduces prices, encouraging consumption and exports and helping to reduce the surplus."

Another service provided by speculators to a market is that by risking their own capital in the hope of profit, they add liquidity to the market and make it easier for others to offset risk, including those who may be classified as hedgers and arbitrageurs.

Arbitrage

Arbitrage is a trading strategy which takes advantage of two or more securities being inconsistently priced relative to each other. In financial and economics theory arbitrage is the practice of taking advantage of a price differential between two or more markets or assets: striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the prices. When used by academics, an arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state; in simple terms, a risk-free profit.
Advanced arbitrage techniques involve sophisticated mathematical analysis and rapid trading.

More arbitrage and market efficiency

The classical joke on arbitrage and market efficiency: A finance professor and a normal person go on a walk and the normal person sees a 100$ bill lying on the street. When the normal person wants to pick it up, the finance professor says:
'Don't try to do that! It is absolutely impossible that there is a 100$ bill lying on the street. Indeed, if it were lying on the street, somebody else would already have picked it up before you' (end of joke).
How about financial markets? There it is already much more reasonable to assume that there are no 100 bills lying around waiting to be picked up. We shall call such opportunities of picking up money that is 'lying around' arbitrage possi- bilities. Let us illustrate this with an easy example.

Stock Returns

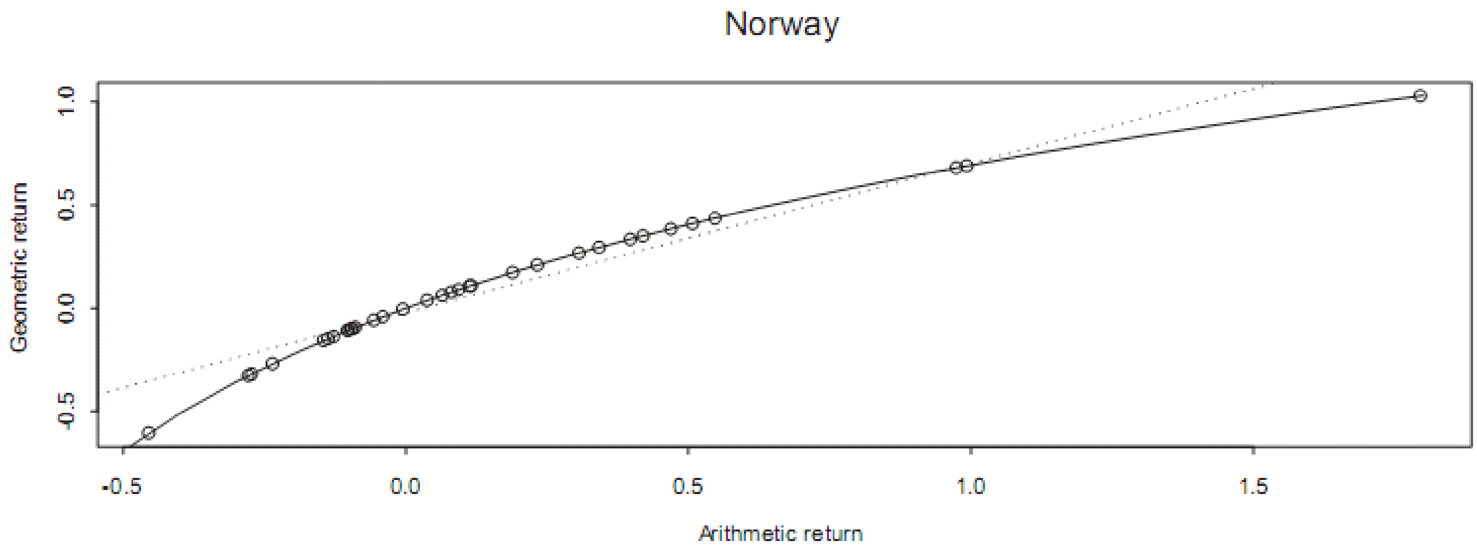
Let p_t be a representative price for a stock in period t (final transaction price or final quotation during the period). Assume that the buyer pays the seller immediately for stock bought .
Let d_t be the present value of dividends, per share, distributed to those people who own stock during period t . On almost all days there are no dividend payments $\rightarrow d_t = 0$. Sometimes dividend payments are simply ignored, so then $d_t = 0$ for all days t .
Three price change quantities appear in empirical research:

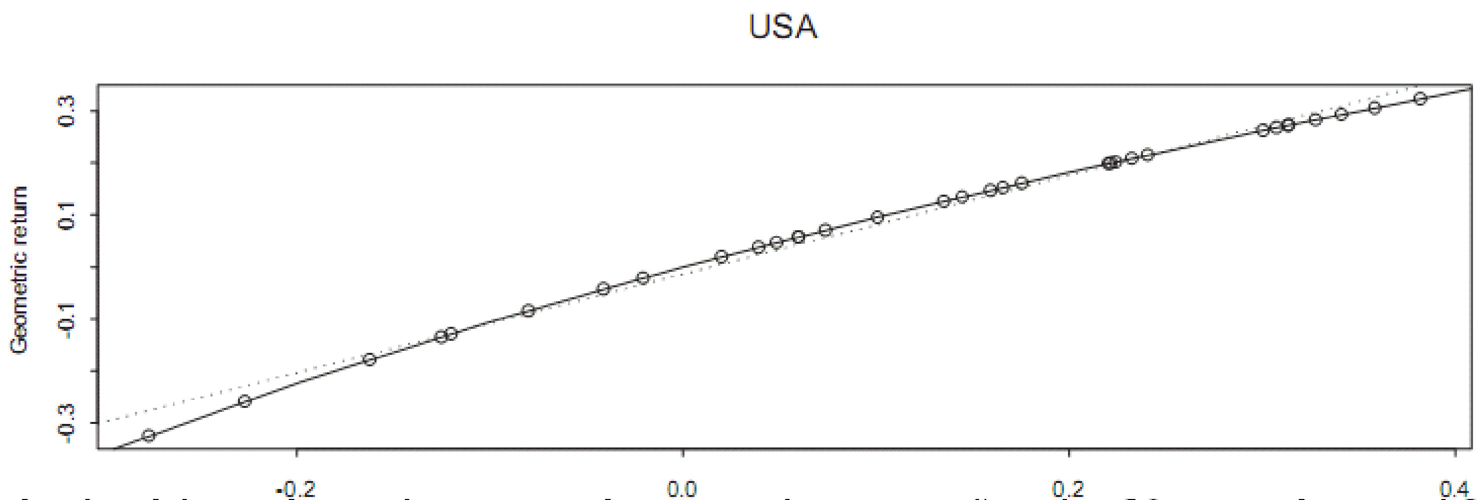
$r^*_t = p_t + d_t - p_{t-1}$

$r'_t = (p_t + d_t - p_{t-1})/\, p_{t-1}$, simple net return (arithmetic)

$r_t = \log(p_t + d_t\,) - \log p_{t-1}$. log returns (geometric)

The return measures r_t and r'_t are very similar numbers, since
$1 + r'_t = \exp(r_t) = 1 + r_t + ½\, r_t^2 + \dots$
and very rarely are daily returns outside the range from −10% to 10%. It is common to assume that single-period geometric returns follow a normal distribution.





Historical arithmetic and geometric annual returns for the Norwegian and U.S. stock market (1970-2002). The historical annual volatilities in the two markets are very different: 18% for the U.S. market and 44% for the Norwegian market. From "Statistical modelling of financial time series: An introduction" K. Aas, X. Dimakos (2004) http://www.nr.no/files/samba/bff/SAMBA0804.pdf

Statistical distribution of returns

In real world data analysis, no
only are the true mean and
standard deviations unknown
but the type of distribution tha
generated the observed return
(if any) is also unknown.

A simple test for normality is orovided by the studentized range SR: given a random variable x_i one defines

$$SR = (\max x_i - \min x_i)/\sigma$$

It depends heavily on the extreme observations

Fractiles SR(p, T) of the Distribution of the Studentized Range in Samples of Size T from a Normal Population

SIZE OF SAMPLE T	LOWER PERCENTAGE POINTS (p)					UPPER PERCENTAGE POINTS (p)					SIZE OF SAMPLE T
	.005	.01	.025	.050	.10	.90	.95	.975	.99	.995	
						1.997	1.999	2.000	2.000	2.000	3
						2.409	2.429	2.439	2.445	2.447	4
						2.712	2.753	2.782	2.803	2.813	5
						2.949	3.012	3.056	3.095	3.115	6
						3.143	3.222	3.282	3.338	3.369	7
						3.308	3.399	3.471	3.543	3.585	8
						3.449	3.552	3.634	3.720	3.772	9
10	2.47	2.51	2.59	2.67	2.77	3.57	3.685	3.777	3.875	3.935	10
11	2.53	2.58	2.66	2.74	2.84	3.68	3.80	3.903	4.012	4.079	11
12	2.59	2.65	2.73	2.80	2.91	3.78	3.91	4.01	4.134	4.208	12
13	2.65	2.70	2.78	2.86	2.97	3.87	4.00	4.11	4.244	4.325	13
14	2.70	2.75	2.83	2.91	3.02	3.95	4.09	4.21	4.34	4.431	14
15	2.75	2.80	2.88	2.96	3.07	4.02	4.17	4.29	4.43	4.53	15
16	2.80	2.85	2.93	3.01	3.13	4.09	4.24	4.37	4.51	4.62	16
17	2.84	2.90	2.98	3.06	3.17	4.15	4.31	4.44	4.59	4.69	17
18	2.88	2.94	3.02	3.10	3.21	4.21	4.38	4.51	4.66	4.77	18
19	2.92	2.98	3.06	3.14	3.25	4.27	4.43	4.57	4.73	4.84	19
20	2.95	3.01	3.10	3.18	3.29	4.32	4.49	4.63	4.79	4.91	20
30	3.22	3.27	3.37	3.46	3.58	4.70	4.89	5.06	5.25	5.39	30
40	3.41	3.46	3.57	3.66	3.79	4.96	5.15	5.34	5.54	5.69	40
50	3.57	3.61	3.72	3.82	3.94	5.15	5.35	5.54	5.77	5.91	50
60	3.69	3.74	3.85	3.95	4.07	5.29	5.50	5.70	5.93	6.09	60
80	3.88	3.93	4.05	4.15	4.27	5.51	5.73	5.93	6.18	6.35	80
100	4.02	4.00	4.20	4.31	4.44	5.68	5.90	6.11	6.36	6.54	100
150	4.30	4.36	4.47	4.59	4.72	5.96	6.18	6.39	6.64	6.84	150
200	4.50	4.56	4.67	4.78	4.90	6.15	6.38	6.59	6.85	7.03	200
500	5.06	5.13	5.25	5.37	5.49	6.72	6.94	7.15	7.42	7.60	500
1000	5.50	5.57	5.68	5.79	5.92	7.11	7.33	7.54	7.80	7.99	1000

Stylized facts and volatility. A bibliography:

R. Cont "Empirical properties of asset returns: stylized facts and statistical issues" Quantitative Finance 1 (2001) 223–236

http://www.proba.jussieu.fr/pageperso/ramacont/papers/empirical.pdf

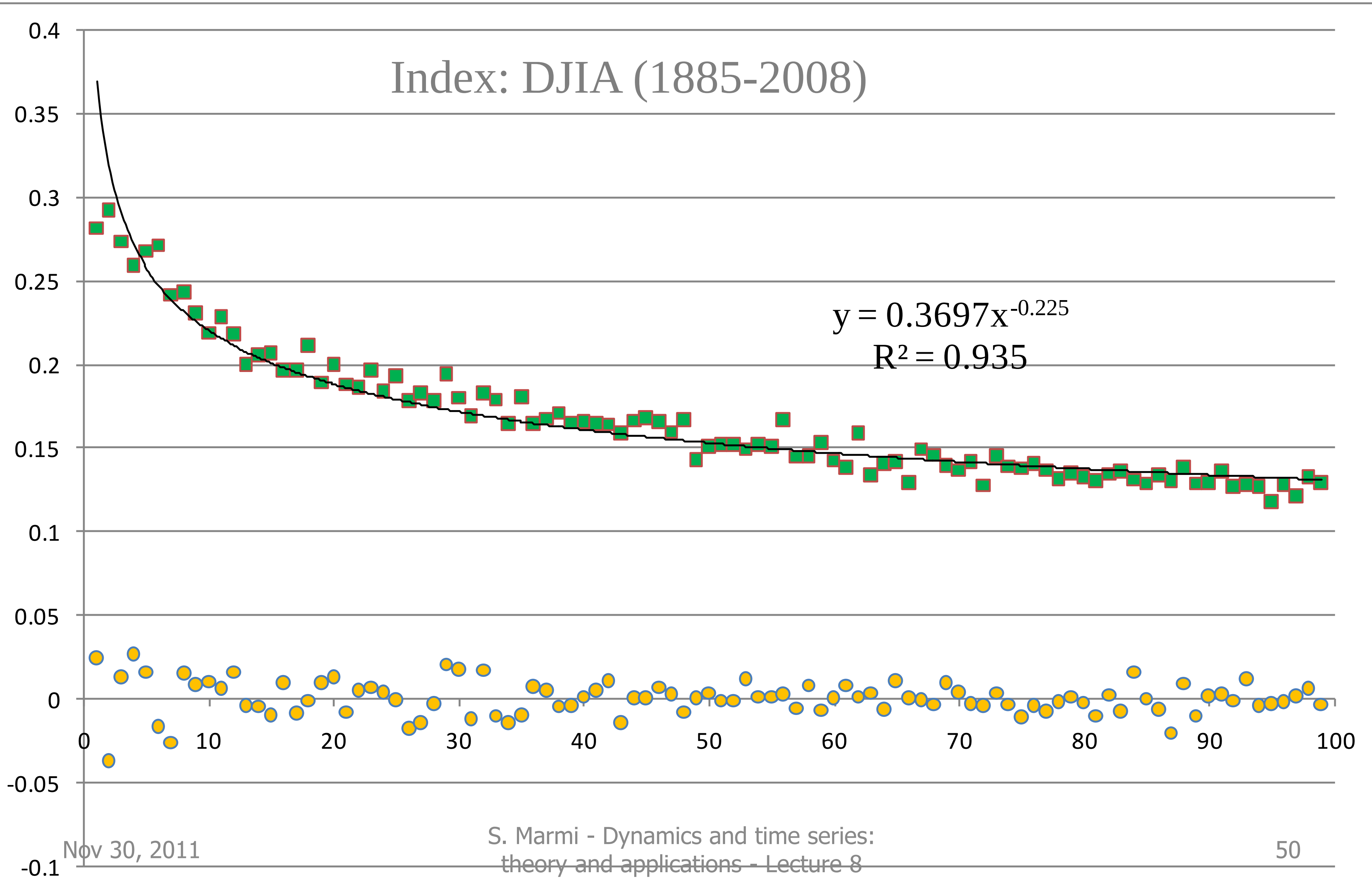
S.J. Taylor "Asset Price Dynamics, Volatility, and Prediction" Princeton University Press (2005). Chapters 2 and 4

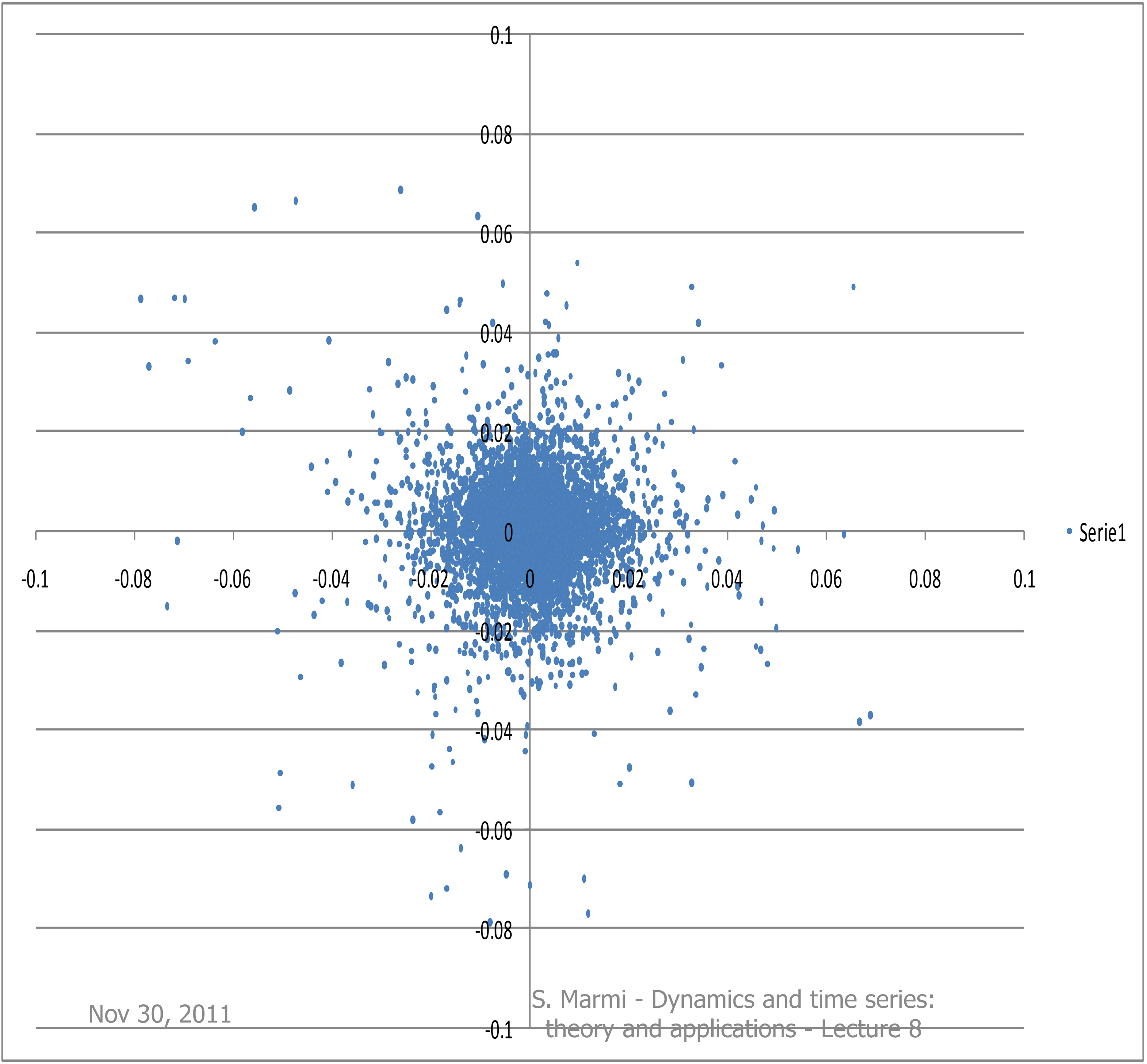
Steven Skiena CSE691 Computational Finance class at Stony Brook: http://www.cs.sunysb.edu/~skiena/691/

Stylized facts (R. Cont, Quantitative Finance (2001))

1. Absence of autocorrelations: (linear) autocorrelations of asset returns are often insignificant, except for very small intraday time scales ($\approx$ 20 minutes) for which microstructure effects come into play.
2. Heavy tails: the (unconditional) distribution of returns seems to display a power-law or Pareto-like tail, with a tail index which is finite, higher than two and less than five for most data sets studied. In particular this excludes stable laws with infinite variance and the normal distribution. However the precise form of the tails is difficult to determine.
3. Gain/loss asymmetry: one observes large drawdowns in stock prices and stock index values but not equally large upward movements

Autocorreletion of daily returns and of their absolute values. The black line is the best power law fit of the absolute values autocorrelations





DJIA: daily return at day i vs. return at day i-1, 5000 days (approximately 1988-2008)

Theoretical and observed frequency of outliers in the history of 15 stockmarkets

Exhibit 4: Outliers – Expected and Observed

This exhibit shows, for the indexes and sample periods in Exhibit 2, the expected (Exp) and observed (Obs) number of daily returns three standard deviations (SD) below and above the arithmetic mean return (AM); the ratio between the number of these observed and expected returns; and the total number of expected (TE) and observed (TO) returns more than three SDs away from the mean. 'Exp' figures are rounded to the nearest integer.

	Lower Tail				Upper Tail						
Market	AM–3·SD	Exp	Obs	Ratio	AM+3·SD	Exp	Obs	Ratio	TE	TO	Ratio
Australia	–2.46%	17	73	4.4	2.52%	17	53	3.2	33	126	3.8
Canada	–2.48%	11	73	6.9	2.55%	11	43	4.1	21	116	5.5
France	–3.11%	13	79	6.2	3.19%	13	61	4.8	25	140	5.5
Germany	–3.51%	16	85	5.3	3.57%	16	76	4.8	32	161	5.1
Hong Kong	–5.53%	12	77	6.2	5.67%	12	80	6.5	25	157	6.4
Italy	–3.82%	12	71	6.0	3.91%	12	48	4.0	24	119	5.0
Japan	–3.12%	19	132	6.8	3.19%	19	112	5.8	39	244	6.3
New Zealand	–2.51%	12	61	4.9	2.56%	12	57	4.6	25	118	4.7
Singapore	–3.12%	14	90	6.4	3.18%	14	86	6.1	28	176	6.3
Spain	–3.22%	11	52	4.8	3.31%	11	61	5.6	22	113	5.2
Switzerland	–2.74%	13	101	7.9	2.79%	13	62	4.8	26	163	6.4
Taiwan	–4.55%	15	103	6.8	4.65%	15	81	5.3	30	184	6.0
Thailand	–4.40%	10	62	6.0	4.48%	10	81	7.8	21	143	6.9
UK	–3.00%	13	69	5.3	3.07%	13	60	4.6	26	129	5.0
USA	–3.35%	28	180	6.4	3.40%	28	173	6.1	56	353	6.3
Average	**–3.39%**	**14**	**87**	**6.0**	**3.47%**	**14**	**76**	**5.2**	**29**	**163**	**5.6**

Estrada, Javier: Black Swans and Market Timing: How Not to Generate Alpha. Available at SSRN: http://ssrn.com/abstract=1032962

Distribution of returns of DJIA stocks: from "Foundations of Finance", Fama (1976)

TABLE 1.2

Frequency Distributions for Daily Returns on Dow-Jones Industrials

INTERVALS

	T (1)	$\bar{R} - .5s(R) \le R \le \bar{R} + .5s(R)$ Expected no. (2)	Actual no. (3)	$\bar{R} - 1.0s(R) \le R < \bar{R} - .5s(R)$ and $\bar{R} + .5s(R) < R \le \bar{R} + 1.0s(R)$ Expected no. (4)	Actual no. (5)	$\bar{R} - 1.5s(R) \le R < \bar{R} - 1.0s(R)$ and $\bar{R} + 1.0s(R) < R \le \bar{R} + 1.5s(R)$ Expected no. (6)	Actual no. (7)
Allied Chemical	1,223	468.5	562	366.5	349	224.8	163
Alcoa	1,190	455.8	521	356.6	343	218.7	172
American Can	1,219	466.9	602	365.1	336	224.1	157
AT & T	1,219	466.9	710	365.1	285	224.1	131
American Tobacco	1,283	491.4	692	384.4	311	235.8	138
Anaconda	1,193	456.9	513	357.4	331	219.3	204
Bethlehem Steel	1,200	459.6	575	359.5	307	220.6	180
Chrysler	1,692	648.0	736	506.9	493	311.0	259
Du Pont	1,243	476.1	539	372.4	363	228.5	195
Eastman Kodak	1,238	474.2	546	370.9	379	227.5	162
General Electric	1,693	648.4	784	507.2	479	311.2	222
General Foods	1,408	539.3	632	421.8	423	258.8	194
General Motors	1,446	553.8	682	433.2	396	265.8	203
Goodyear	1,162	445.0	539	348.1	331	213.6	164
International Harvester	1,200	459.6	529	359.5	365	220.6	182
International Nickel	1,243	476.1	587	372.4	362	228.5	149
International Paper	1,447	554.2	643	433.5	442	266.0	180
Johns Manville	1,205	461.5	526	361.0	363	221.5	163
Owens Illinois	1,237	473.7	591	370.6	323	227.4	188
Procter & Gamble	1,447	554.2	726	433.5	389	266.0	171
Sears	1,236	473.4	666	370.3	305	227.2	144
Standard Oil (California)	1,693	648.4	776	507.2	468	311.2	233
Standard Oil (New Jersey)	1,156	442.8	582	346.3	314	212.6	139
Swift & Co.	1,446	553.8	672	433.2	409	265.8	194
Texaco	1,159	443.9	533	347.3	311	213.0	164
Union Carbide	1,118	428.1	466	335.0	338	205.5	178
United Aircraft	1,200	459.6	550	359.5	348	220.6	165
U.S. Steel	1,200	459.6	495	359.5	337	220.6	219
Westinghouse	1,448	554.6	636	433.8	424	266.1	221
Woolworth	1,445	553.5	718	432.9	390	265.6	170

INTERVALS

	$\bar{R} - 2.0s(R) \le R < \bar{R} - 1.5s(R)$ and $\bar{R} + 1.5s(R) < R \le \bar{R} + 2.0s(R)$ Expected no. (8)	Actual no. (9)	$R < \bar{R} - 2s(R)$ and $R > \bar{R} + 2s(R)$ Expected no. (10)	Actual no. (11)	$R < \bar{R} - 3s(R)$ and $R > \bar{R} + 3s(R)$ Expected no. (12)	Actual no. (13)	$R < \bar{R} - 4s(R)$ and $R > \bar{R} + 4s(R)$ Expected no. (14)	Actual no. (15)	$R < \bar{R} - 5s(R)$ and $R > \bar{R} + 5s(R)$ Expected no. (16)	Actual no. (17)
Allied Chemical	107.7	94	55.5	55	3.3	16	.08	4	.0007	2
Alcoa	104.8	85	54.1	69	3.2	7	.07	0	.0007	0
American Can	107.4	62	55.5	62	3.3	19	.08	6	.0007	3
AT & T	107.4	42	55.5	51	3.3	17	.08	9	.0007	6
American Tobacco	113.0	73	58.4	69	3.5	20	.08	7	.0008	4
Anaconda	105.1	88	54.3	57	3.2	8	.08	1	.0007	0
Bethlehem Steel	105.7	76	54.6	62	3.2	15	.08	4	.0007	1
Chrysler	149.1	117	77.0	87	4.6	16	.11	4	.0010	1
Du Pont	109.5	80	56.5	66	3.4	8	.08	3	.0007	1
Eastman Kodak	109.1	85	56.3	66	3.3	13	.08	2	.0007	2
General Electric	149.2	111	77.0	97	4.6	22	.11	5	.0010	1
General Foods	124.0	84	64.1	75	3.8	22	.09	3	.0008	1
General Motors	127.4	103	65.8	62	3.9	13	.09	6	.0009	3
Goodyear	102.4	71	52.9	57	3.1	10	.07	4	.0007	2
International Harvester	105.7	61	54.6	63	3.2	15	.08	4	.0007	1
International Nickel	109.5	72	56.5	73	3.4	16	.08	6	.0007	0
International Paper	127.5	100	65.8	82	3.9	19	.09	5	.0009	0
Johns Manville	106.2	91	54.8	62	3.2	11	.08	3	.0007	1
Owens Illinois	109.0	69	56.3	66	3.3	20	.08	3	.0007	1
Procter & Gamble	127.5	71	65.8	90	3.9	20	.09	6	.0009	2
Sears	108.9	58	56.2	63	3.3	21	.08	8	.0007	5
Standard Oil (California)	149.2	121	77.0	95	4.6	14	.11	5	.0010	1
Standard Oil (New Jersey)	101.8	70	52.5	51	3.1	12	.07	3	.0007	2
Swift & Co.	127.4	85	65.8	86	3.9	18	.09	4	.0009	0
Texaco	102.1	95	52.7	56	3.1	14	.07	2	.0007	0
Union Carbide	98.5	69	50.9	67	3.0	6	.07	1	.0007	0
United Aircraft	105.7	77	54.6	60	3.2	11	.08	3	.0007	1
U.S. Steel	105.7	90	54.6	59	3.2	8	.08	1	.0007	0
Westinghouse	127.6	95	65.9	72	3.9	14	.09	3	.0009	2
Woolworth	127.3	91	65.7	76	3.9	23	.09	5	.0009	2

Source: Adapted from Eugene F. Fama, "The Behavior of Stock Market Prices," *Journal of Business* 38 (January 1965): 47–48.

Table 1.2, constructed from Tables 1 and 3 of Fama (1965), shows frequency distributions for continuously compounded daily returns for each of the 30 stocks of the Dow-Jones Industrial Average, for time periods that vary slightly from stock to stock but which usually run from about the end of 1957 to September 26, 1962. Column (1) of the table shows the number

The obvious finding in Table 1.2 is that the frequency distributions of the daily returns have more observations both in their central portions and in their extreme tails than are expected from normal distributions. For every stock the actual number of daily returns within .5 sample standard deviations from the sample mean return is greater than the expected number. Every stock also has more observations beyond three standard deviations from its mean return than would be expected with normal distributions; all but one have more beyond four standard deviations; and all but three have more beyond two standard deviations.

In more vivid terms, if daily returns are drawn from normal distributions, for any stock a daily return greater than four standard deviations from the mean is expected about once every 50 years. Daily returns this extreme are observed about four times every five years. Similarly, under the hypothesis of normality, for any given stock a daily return more than five standard deviations from the mean daily return should be observed about once every 7,000 years. Such observations seem to occur about every three to four years.

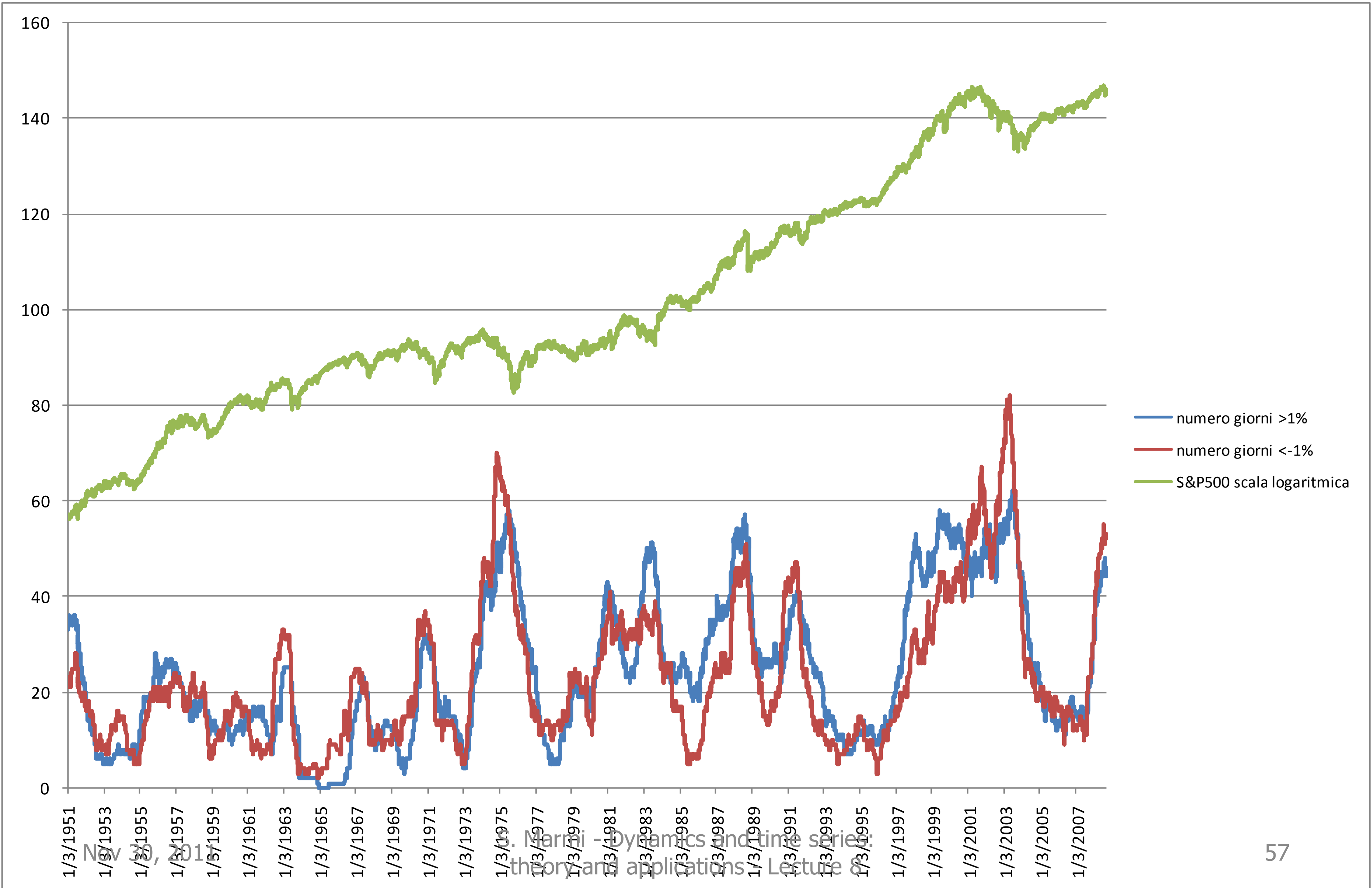
Stylized facts (R. Cont, Quantitative Finance (2001))

4. Aggregational Gaussianity: as one increases the time scale Δt over which returns are calculated, their distribution looks more and more like a normal distribution. In particular, the shape of the distribution is not the same at different time scales.
5. Intermittency: returns display, at any time scale, a high degree of variability. This is quantified by the presence of irregular bursts in time series of awide variety of volatility estimators.
6. Volatility clustering: different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.
7. Conditional heavy tails: even after correcting returns for volatility clustering (e.g. via GARCH-type models), the residual time series still exhibit heavy tails. However, the tails are less heavy than in the unconditional distribution of returns.

Stylized facts (R. Cont, Quantitative Finance (2001))

8. Slow decay of autocorrelation in absolute returns: the autocorrelation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent $\beta \in [0.2, 0.4]$. This is sometimes interpreted as a sign of long-range dependence.
9. Leverage effect: most measures of volatility of an asset are negatively correlated with the returns of that asset.
10. Volume/volatility correlation: trading volume is correlated with all measures of volatility.
11. Asymmetry in time scales: coarse-grained measures of volatility predict fine-scale volatility better than the other way round.

Volatility clustering and leverage effect



An **autoregressive conditional heteroscedasticity** (**ARCH**, Engle (1982)) model considers the variance of the current error term to be a function of the variances of the previous time period's error terms. ARCH relates the error variance to the square of a previous period's error. It is employed commonly in modeling financial time series that exhibit time-varying volatility clustering, i.e. periods of swings followed by periods of relative calm.