Dynamics and time series: theory and applications

Stefano Marmi
Scuola Normale Superiore
Lecture 7, Nov 28 , 2011

Lyapunov exponents for interval maps

- Assume that T is a piecewise smooth map of I=[0,1]
- By the chain rule we have

$$\frac{1}{n}\log|T^n(x)-T^n(y)| \approx \frac{1}{n}\sum_{i=0}^{n-1}\log|T'(T^i x)| .$$

- If μ is an ergodic invariant measure for a.e. x the limit exists and it is given by

$$\int_0^1 \log|T'|\,\mathrm{d}\mu$$

which is also called the Lyapunov exponent of T

Expanding maps and Rokhlin formula

If T is expanding then it has a unique a.c.i.p.m. μ and the entropy h of T w.r.t. μ is equal to the Lyapunov exponent

$$h = \int_0^1 \log |T'(x)| \, \mathrm{d}\mu$$

Law of large numbers

$\{\mathbf{X_i}\}$ independent identically distributed random variables

$$\mathrm{E}(\mathrm{X_i}) = \mu < +\infty$$

Then
$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$$

Weak form:

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathrm{P}(|\overline{X}_n - \mu| < \varepsilon) = 1$$

Strong form:

$$\overline{X}_n \to \mu$$ almost surely

Law of large numbers vs Birkhoff theorem

Random setting

$\{\mathbf{X_i}\}$ i.i.d. random variables

$$\mathrm{E}(\mathrm{X_i}) = \mu < +\infty$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$$

almost surely

Deterministic setting

$$T: X \to X$$

$f \in L^1(X, d\mu)$ observable

$$X_i := \{ f \circ T^i \}$$

are not necessarily independent

If T ergodic

$$\frac{1}{n}\sum_{i=1}^{n} f \circ T^i \to \int f \, d\mu$$

almost surely

Central limit theorem

$\{\mathbf{X_i}\}$ independent identically distributed random variables

$$\mathrm{E}(\mathrm{X_i}) = \mu < +\infty \qquad \mathrm{Var}(\mathrm{X_i}) = \sigma^2 > 0$$

$$Z_n := \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{weak} N(0,1)$$

$$\lim_{n\to\infty} \mathrm{P}(Z_n \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Central limit theorem for deterministic systems

(X, A, μ, T) ergodic measurable dynamical system

$f \in L^2(X, d\mu)$ $\qquad\qquad$ $\int f\, d\mu = 0$

$$\sigma^2 = \int f^2 d\mu + 2\sum_{n=1}^{\infty} \int f(f \circ T^n)\, d\mu$$

Analogously to the independent case, we would like to have

$$\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} f \circ T^i \to N(0,1)$$

Central limit theorem for deterministic systems

(X, A, μ, T) ergodic measurable dynamical system

$f \in L^2(X, d\mu)$ $\quad\quad$ $\int f\, d\mu = 0$

$$\sigma^2 = \int f^2 d\mu + 2\sum_{n=1}^{\infty}\int f(f \circ T^n)\, d\mu$$

Analogously to the independent case, we would like to have

$$P\left(\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} f \circ T^i \geq z\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2} dt$$

Hypotheses for deterministic CLT

1. Mixing-type condition

$$c(n, f) := \sup\{\int f(g \circ T^n) d\mu : \int g^2 d\mu = 1\}$$

$$\sum_{n=0}^{\infty} c(n, f) < +\infty$$

2. Cohomological equation has no solution

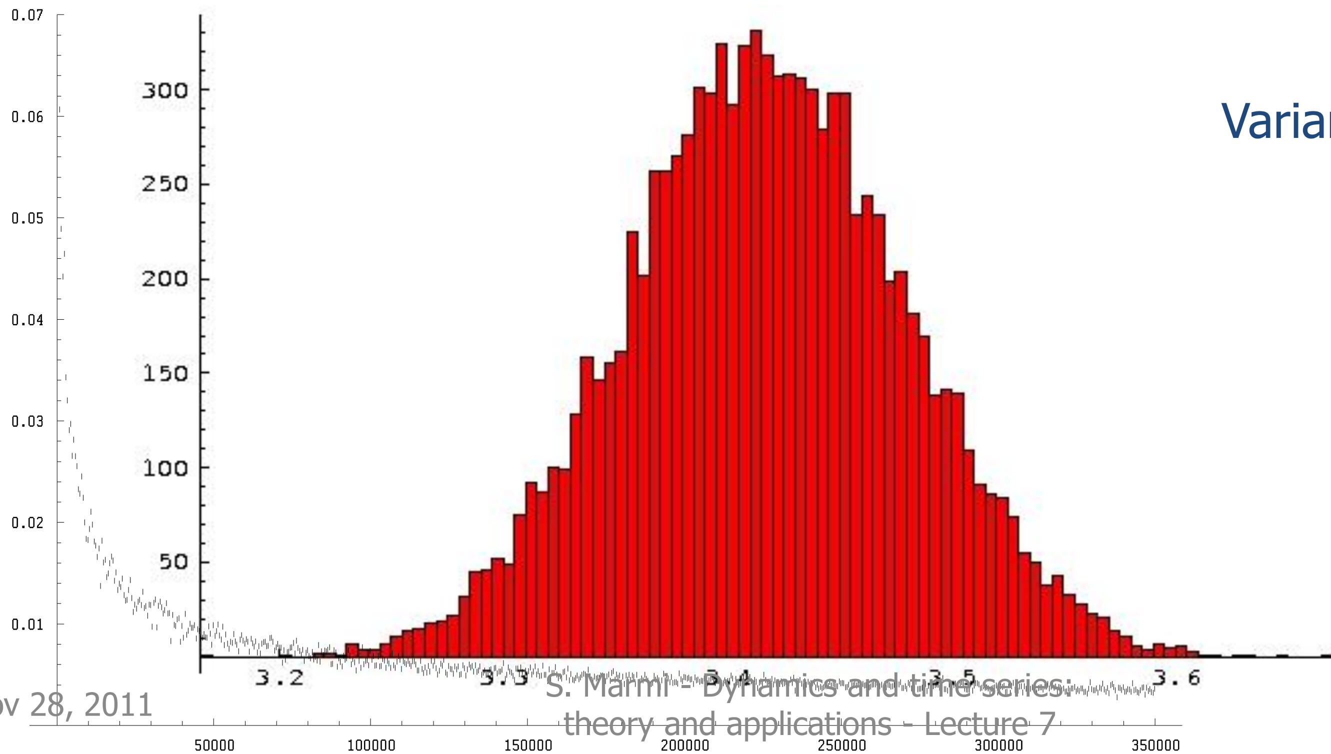
$$\sigma = 0 \Leftrightarrow \exists u \in L^2(X, \mu)\ s.t.\ f = u \circ T - u$$

Then $P\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} f \circ T^i \geq z\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$

An example: distribution of Birkhoff averages for the entropy

$$h_\mu(T) = \int \log|T'|\,d\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \log|T' \circ T^i|$$

KS entropy = Birkhoff average for observable $f(x) = \log|T'(x)|$



Variance decays as

$$\frac{1}{\sqrt{n}}$$

Speed of convergence (Berry-Esseen theorem)

$\{\mathbf{X_i}\}$ independent identically distributed random variables

$$\mathrm{E}(\mathrm{X_i}) = \mu < +\infty \qquad \mathrm{Var}(\mathrm{X_i}) = \sigma^2 > 0$$

$$\mathrm{E}(|\,\mathrm{X_i}\,|^3) = \rho < +\infty$$

By CLT $$\frac{X_1 + \ldots + X_n}{\sigma\sqrt{n}} \to N(0,1)$$

Speed of convergence (Berry-Esseen theorem)

$\{\mathbf{X_i}\}$ independent identically distributed random variables

$$\mathrm{E}(\mathrm{X_i}) = \mu < +\infty \qquad \mathrm{Var}(\mathrm{X_i}) = \sigma^2 > 0$$

$$\mathrm{E}(|\mathrm{X_i}|^3) = \rho < +\infty$$

By CLT $F_n(x) := \mathrm{P}\left(\dfrac{X_1 + \ldots + X_n}{\sigma\sqrt{n}} \geq x \right) \to \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2} dt$

Moreover

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| \leq \frac{0.7056\,\rho}{\sigma^3 \sqrt{n}}$$

Topological entropy

Topological entropy represents the exponential growth rate of the number of orbit segments which are distinguishable with an arbitrarily high but finite precision. It is invariant under topological conjugacy.

Here the phase space is supposed to be a compact metric space (X,d)

Definition 4.1 *Let $S \subset X$, $n \in N$ and $\varepsilon > 0$. S is a (n, ε)–spanning set if for every $x \in X$ there exists $y \in S$ such that $d(f^j(x), f^j(y)) \leq \varepsilon$ for all $0 \leq j \leq n$.*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log r(n, \varepsilon)$$

Here r(n, ε) is the minimal cardinality of a (n, ε)-spanning set

Alternative definition

Let α be an open cover of X and let $N(\alpha)$ be the number of sets in a finite subcover of α with smallest cardinality

$$h_{top}(f) = \sup_{\alpha} \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right)$$

Here the join $\alpha \vee \beta$ of two covers is $\alpha \vee \beta = \{ A \cap B : A \in \alpha, B \in \beta \}$

The variational principle

Let T be a continuous map on X compact Hausdorff: then

$$h_{top}(T) = \sup_{\mu \in M(X,T)} h_\mu(T)$$

(the sup is taken over all invariant Borel probability measures μ)

Example: Bernoulli schemes

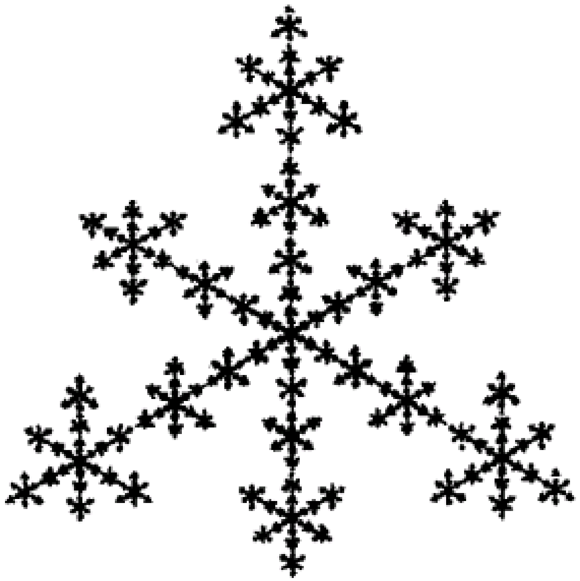
$$h_\mu(T) = -\sum_i p_i \log p_i \leq \log N = h_{top}(T)$$

Remark: the sup need not be achieved (Gurevich, 1969)

Self-similarity and fractals

A subset **A** of Euclidean space will be considered a "fractal" when it has most of the following features:

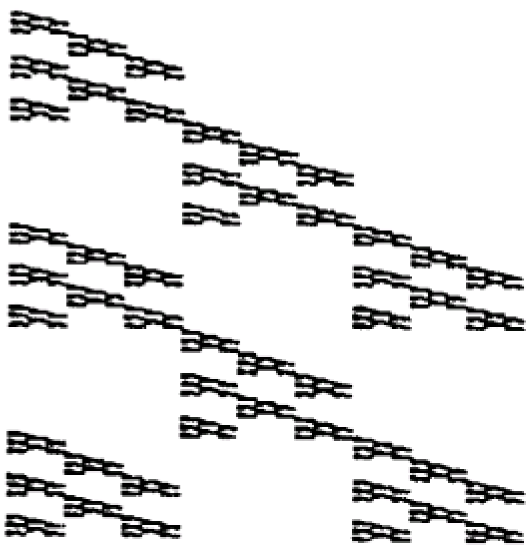
- **A** has fine structure (wiggly detail at arbitrarily small scales)
- **A** is too irregular to be described by calculus (e.g. no tangent space)
- **A** is self-similar or self-affine (maybe approximately or statistically)
- the fractal dimension of **A** is non-integer
- **A** may have a simple (recursive) definition
- **A** has a "natural" appearance: "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line . . ." (B. Mandelbrot)



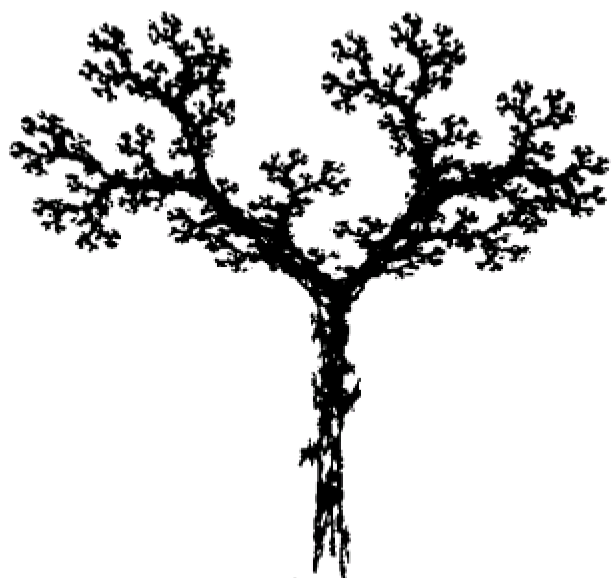
(a)

(b)

self-similar fractals



(c)



(d)

self-affine fractals

From: K. Falconer, Techniques in Fractal Geometry, Wiley 1997

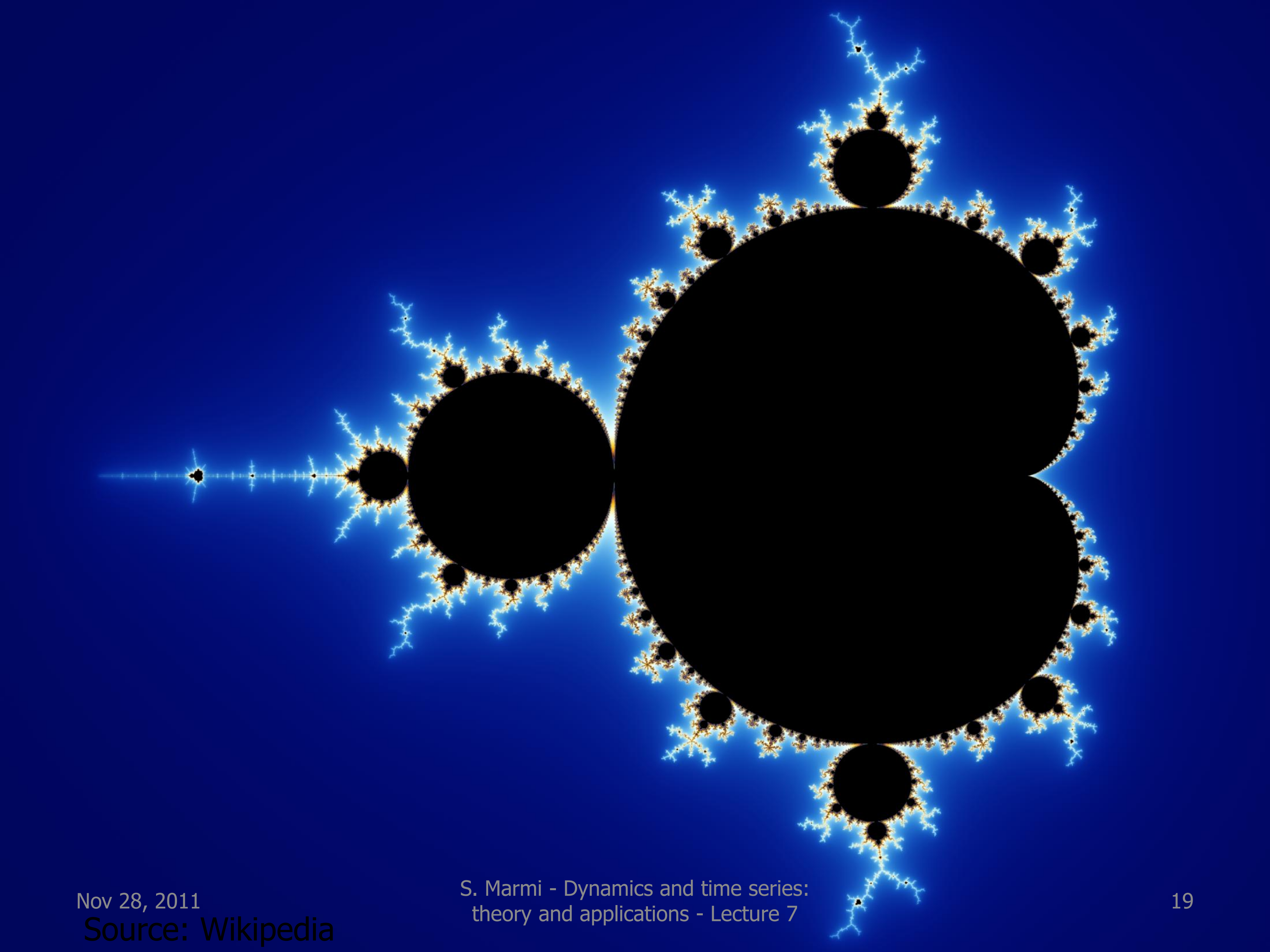
(e)

self-conformal
fractals

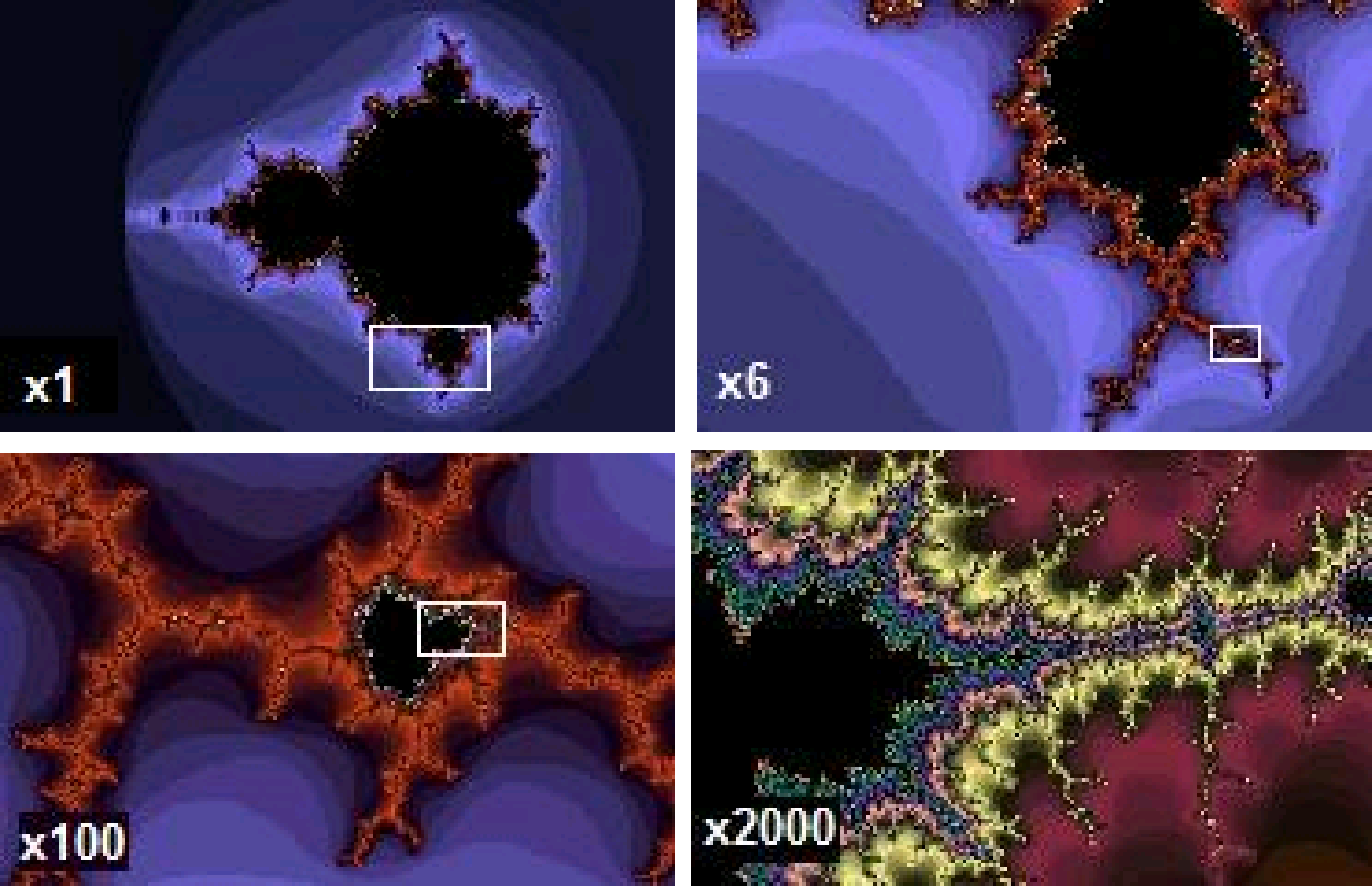
(f)

Statistically
self-similar
fractals

From: K. Falconer, Techniques in Fractal Geometry, Wiley 1997

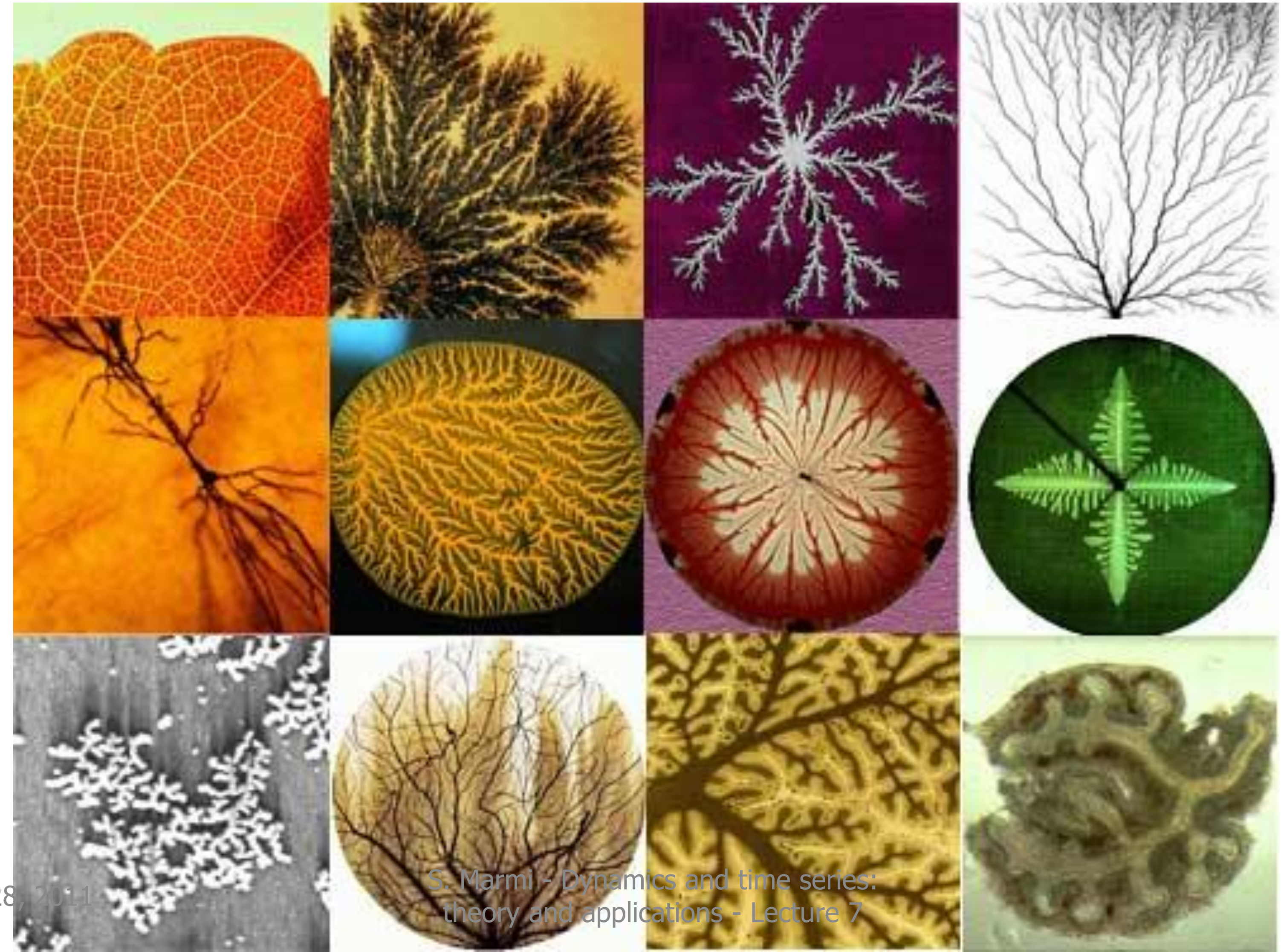


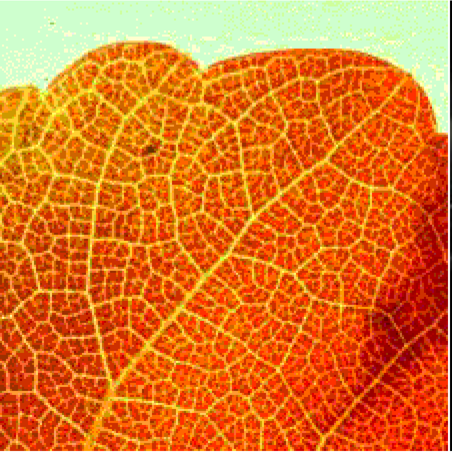
Source: Wikipedia



Source: Wikipedia

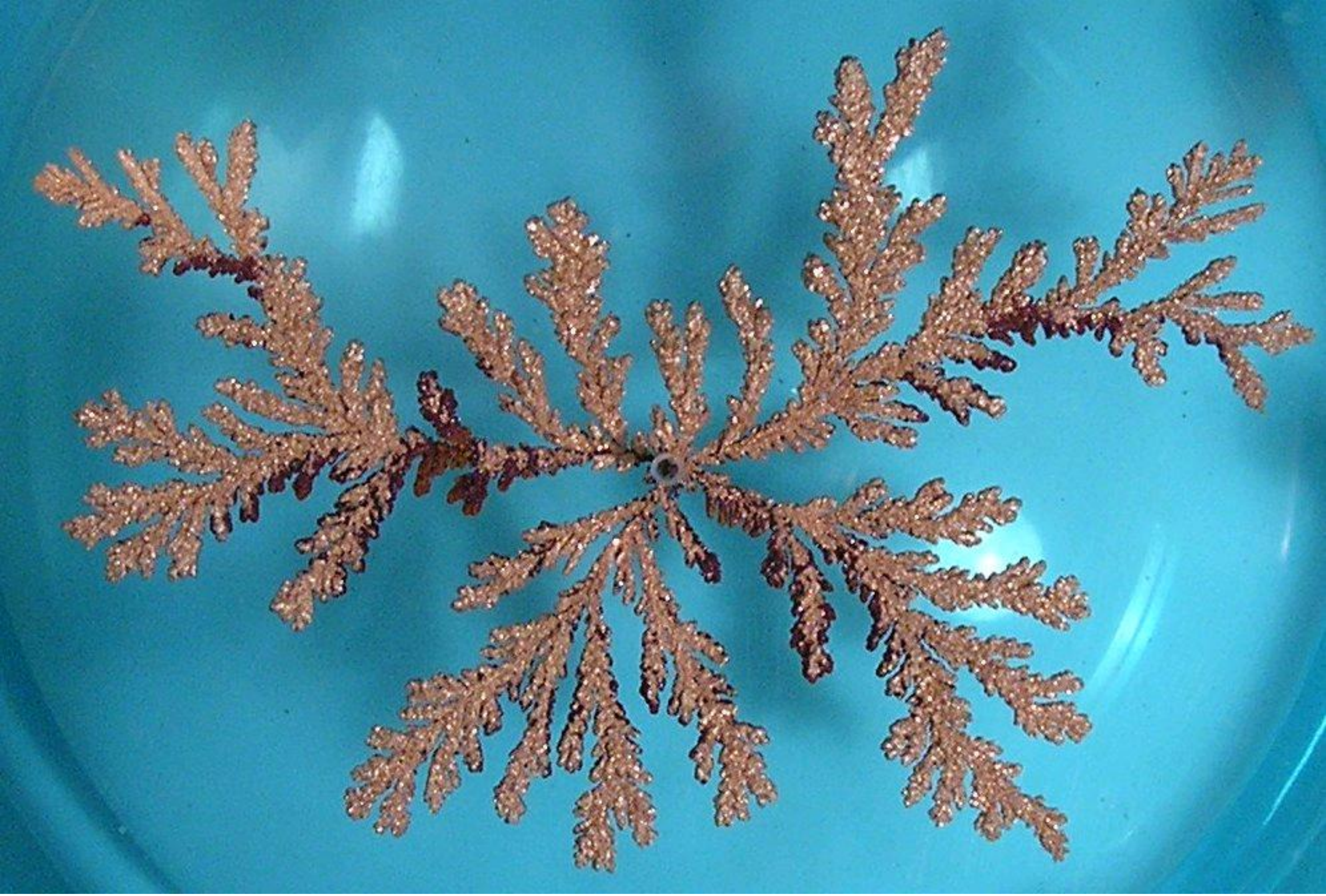
Mathematics, shapes and nature





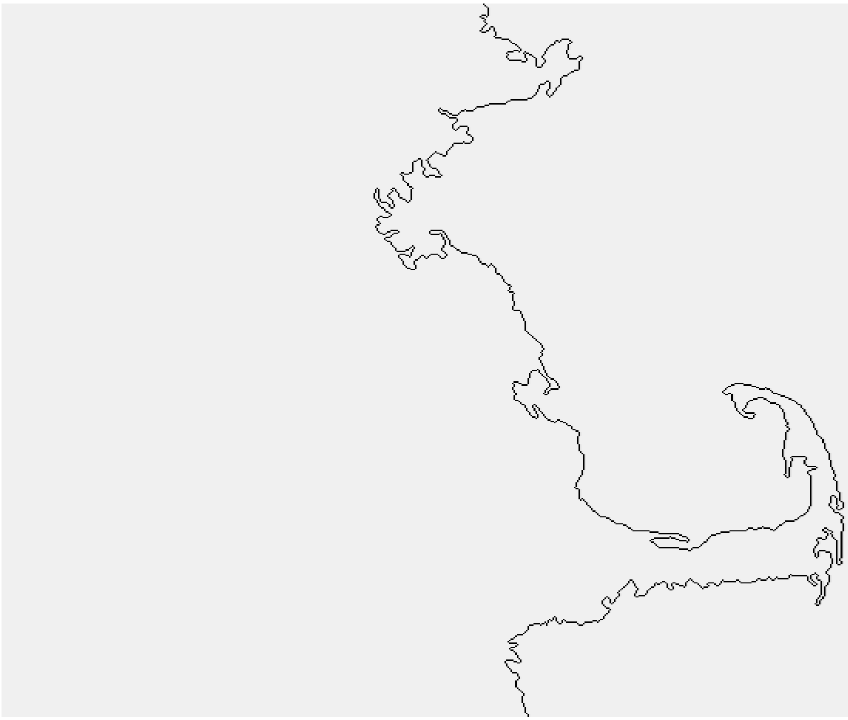
http://classes.yale.edu/fractals/Panorama/Nature/NatFracGallery/Gallery/Gallery.html

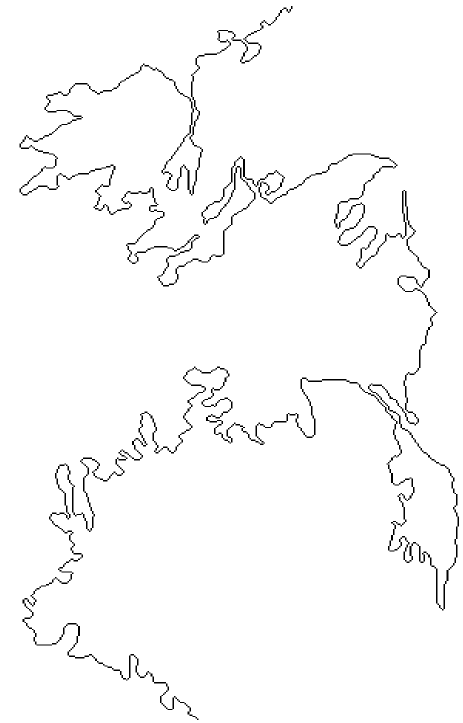
From http://en.wikipedia.org/wiki/Image:Square1.jpg
Lichtenberg Figure
High voltage dielectric breakdown within a block of plexiglas creates a beautiful fractal pattern called a Lichtenberg_figure. The branching discharges ultimately become hairlike, but are thought to extend down to the molecular level.
Bert Hickman, http://www.teslamania.com



A diffusion-limited aggregation (DLA) cluster. Copper aggregate formed from a copper sulfate solution in an electrode position cell. Kevin R. Johnson, Wikipedia

Coastlines





Massachusetts

D=1.15

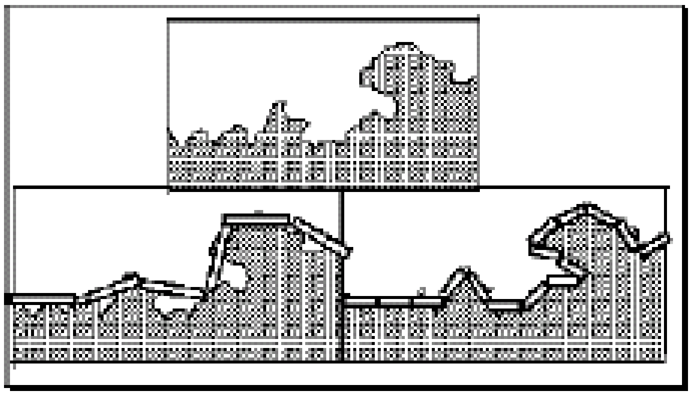
Greece

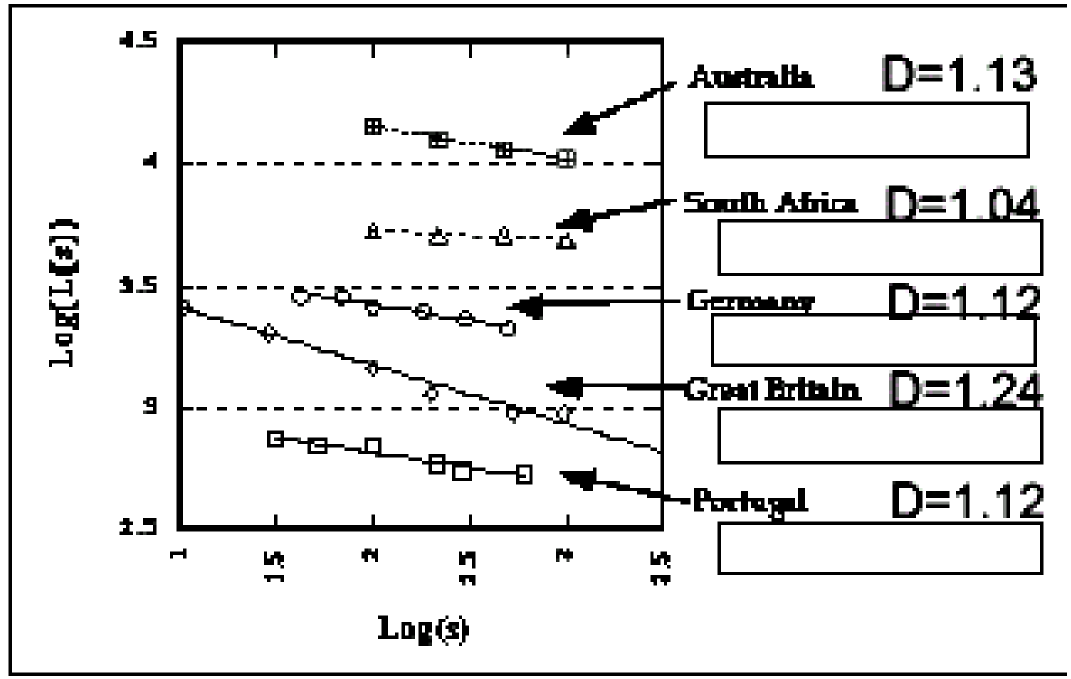
D=1.20

200 km 100 km 50 km

http://upload.wikimedia.org/wikipedia/commons/2/20/Britain-fractal-coastline-combined.jpg

How long is a coastline?





The answer depends on the scale at which the measurement is made: if s is the reference length the coastline length L(s) will be

$$\text{Log } L(s) = (1-D) \log s + \text{cost}$$

(Richardson 1961, Mandelbrot Science 1967)

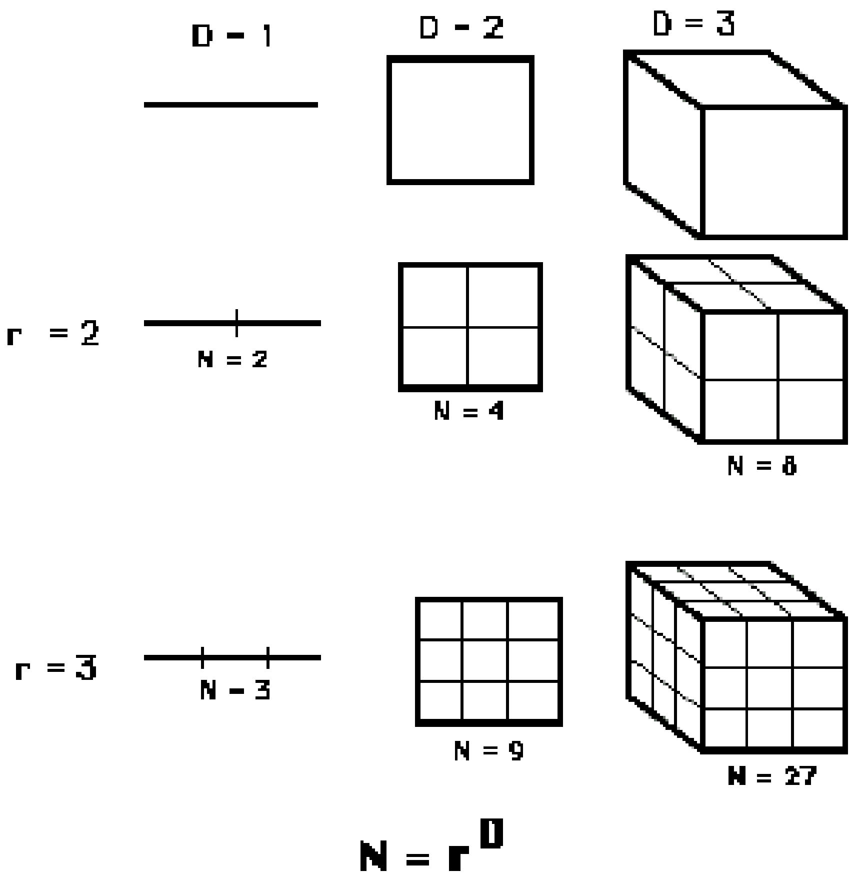
How long is the coast of Britain? Statistical self-similarity and fractional dimension
Science: 156, 1967, 636-638
B. B. Mandelbrot

Seacoast shapes are examples of highly involved curves with the property that - in a statistical sense - each portion can be considered a reduced-scale image of the whole. This property will be referred to as "statistical self-similarity." The concept of "length" is usually meaningless for geographical curves. They can be considered superpositions of features of widely scattered characteristic sizes; as even finer features are taken into account, the total measured length increases, and there is usually no clear-cut gap or crossover, between the realm of geography and details with which geography need not be concerned.

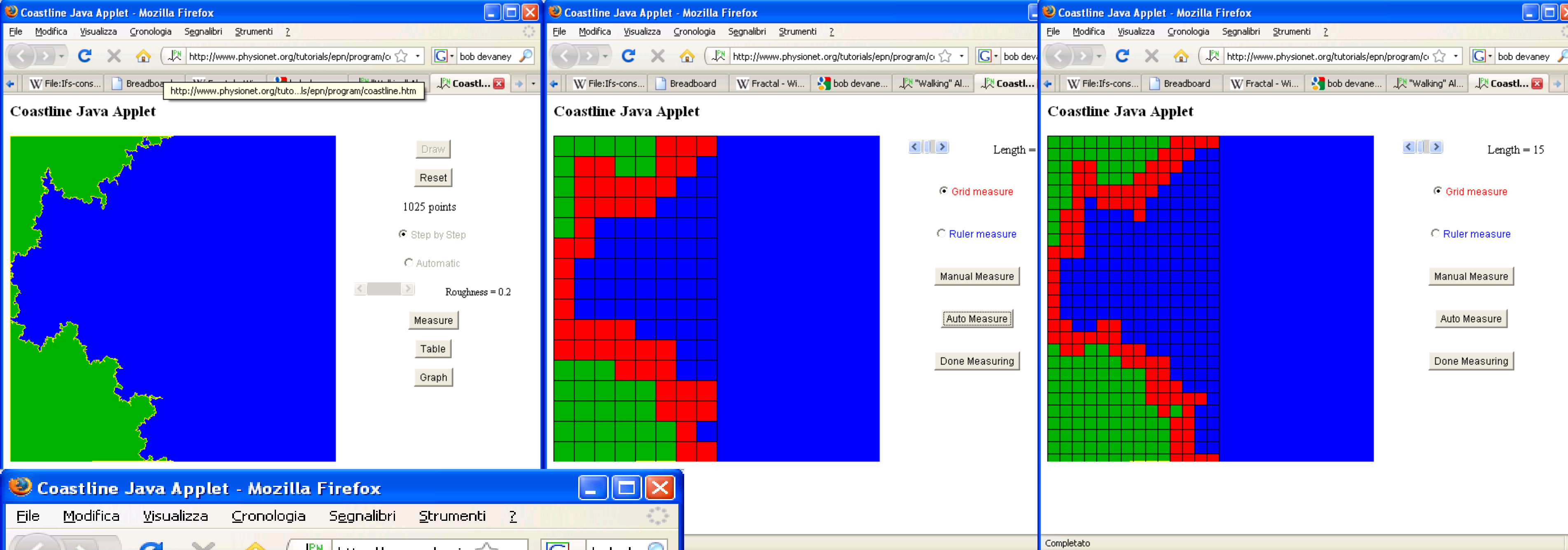
How long is the coast of Britain? Statistical self-similarity and fractional dimension Science: 156, 1967, 636-638 B. B. Mandelbrot

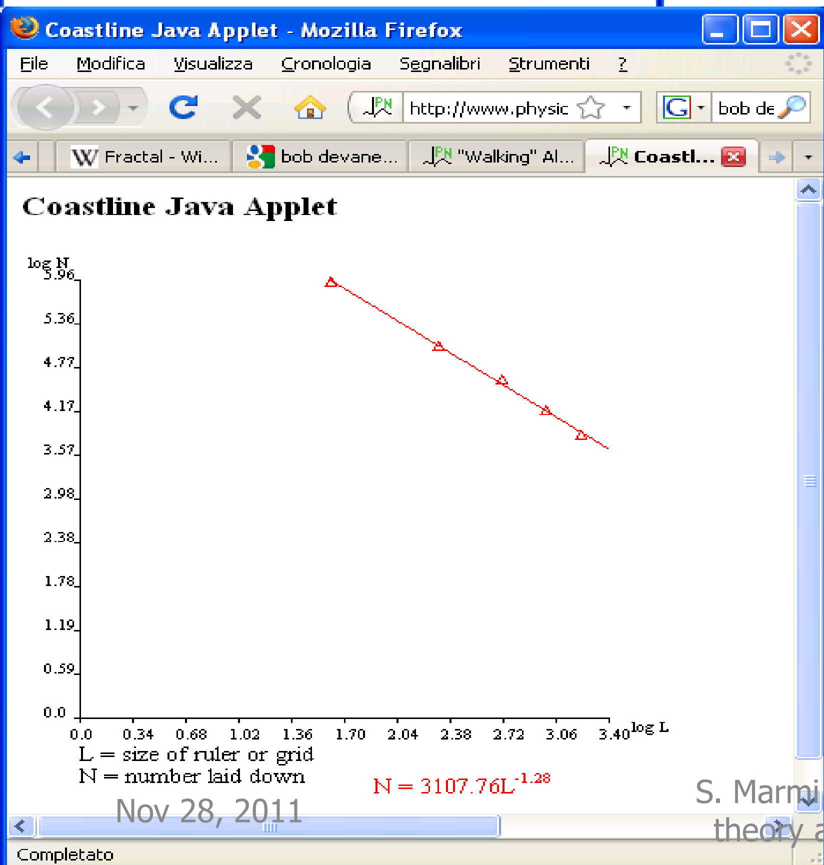
Quantities other than length are therefore needed to discriminate between various degrees of complication for a geographical curve. When a curve is self-similar, it is characterized by an exponent of similarity, D, which possesses many properties of a dimension, though it is usually a fraction greater that the dimension 1 commonly attributed to curves. I propose to reexamine in this light, some empirical observations in Richardson 1961 and interpret them as implying, for example, that the dimension of the west coast of Great Britain is D = 1.25. Thus, the so far esoteric concept of a "random figure of fractional dimension" is shown to have simple and concrete applications of great usefulness.

“Box counting” dimension



$$D = \lim_{s \to 0} \frac{\log N(s)}{\log(1/s)}$$





s	N(s)
25	47
20	67
15	100
10	159
5	386

$$\text{Log } N(s) = -D \log s + \text{cost}$$

http://www.physionet.org/tutorials/epn/program/coastline.htm

Box counting (Minkowski) dimension

Let E be a non-empty bounded subset of $\mathbf{R}^n$ and let N_r (E) be the smallest number of sets of diameter r needed to cover E

- Lower dimension $\dim_B E = \liminf_{r\to 0} \log N_r(E) / -\log r$
- Upper dimension $\dim^B E = \limsup_{r\to 0} \log N_r(E) / -\log r$

- Box-counting dimension: if the lower and upper dimension agree then we define

$$\dim E = \lim_{r\to 0} \log N_r(E) / -\log r$$

The value of these limits remains unaltered if N_r (E) is taken to be the smallest number of balls of radius r (cubes of side r) needed to cover E, or the number of r-mesh cubes that intersect E

Hausdorff dimension

A finite or countable collection of subsets $\{U_i\}$ of $\mathbf{R^n}$ is a δ-cover of a set E if $|U_i| < \delta$ for all i and E is contained in $\bigcup_i U_i$

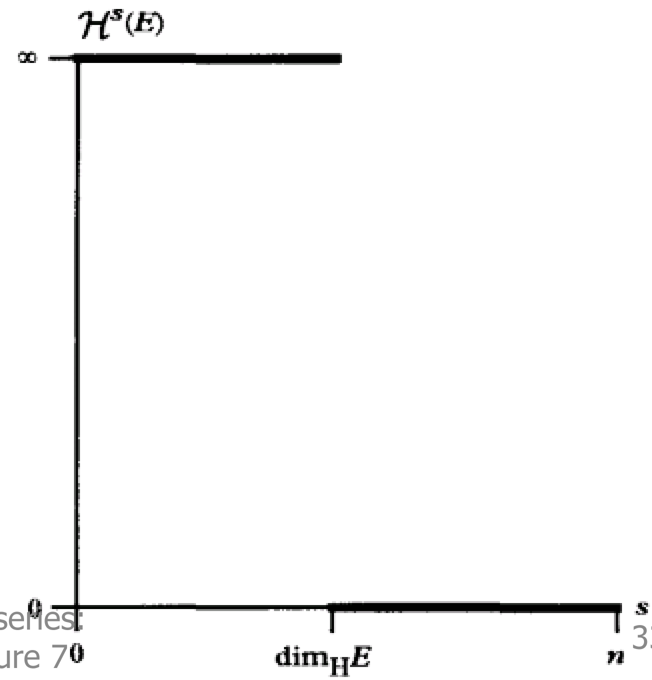
$H^s_\delta(E) = \inf \{\Sigma_i |U_i|^s, \{U_i\}$ is a a δ-cover of E$\}$

s-dimensional Hausdorff measure of E: $H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$

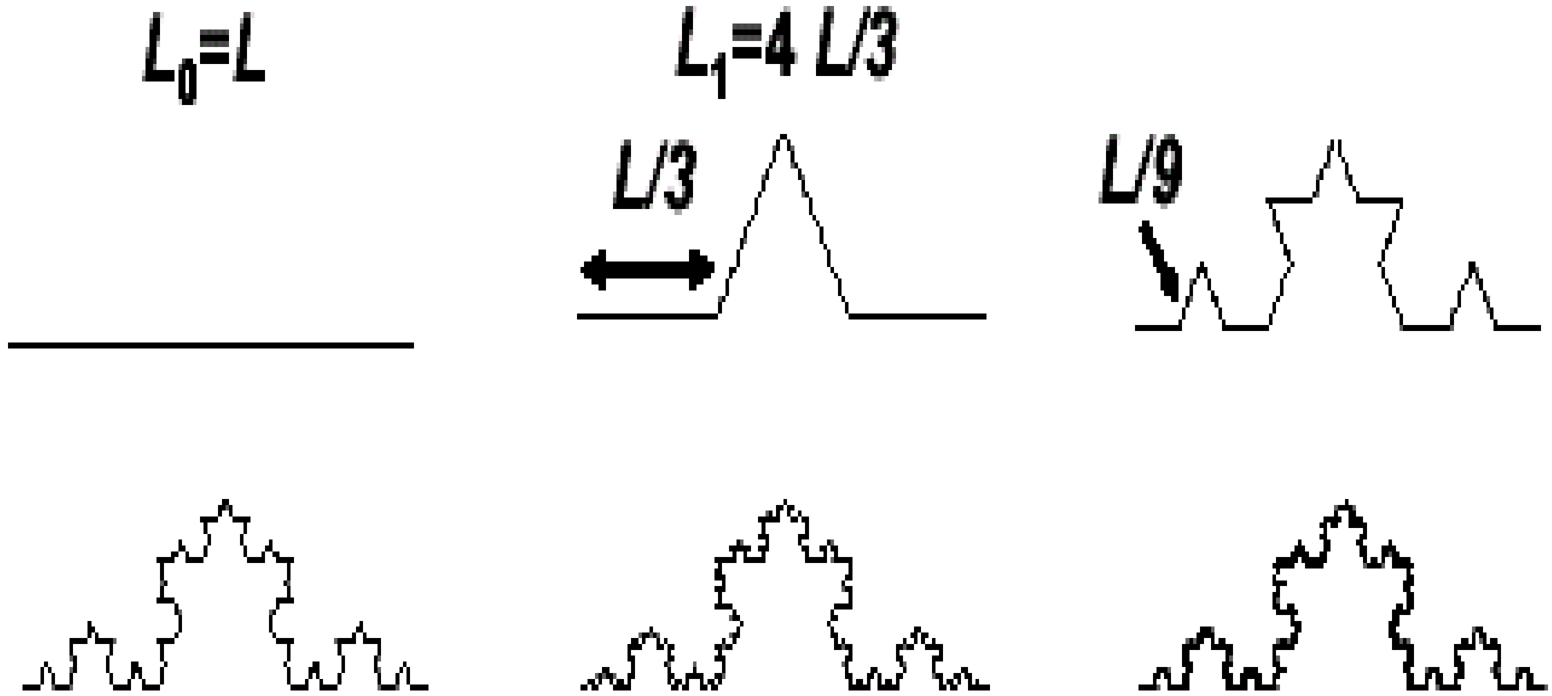
It is a Borel regular measure on $\mathbf{R^n}$, it behaves well under similarities and Lipschitz maps

The Hausdorff dimension $\dim_H E$ is the number at which the Hausdorff measure $H^s(E)$ jumps from ∞ to 0

$\dim_H E \le \dim_B E \le \dim^B E$



Von Koch curve (1904)



D=log4/log3=1.261859…

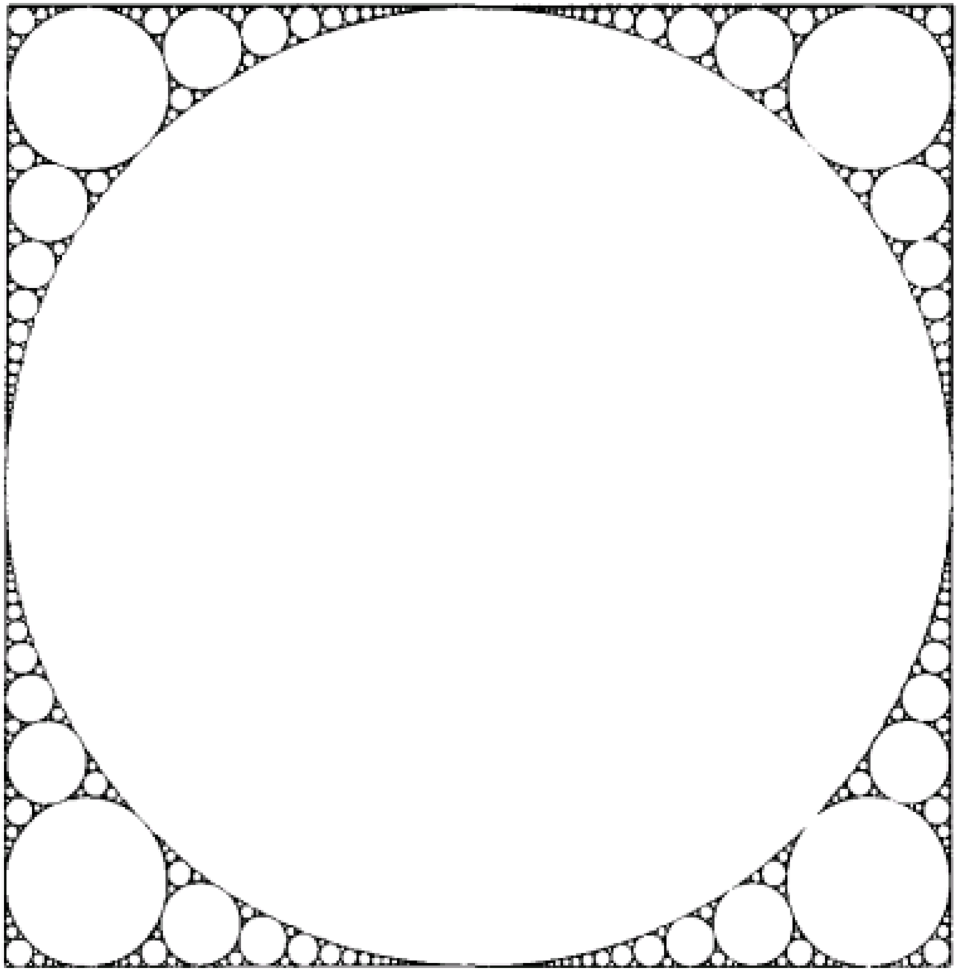
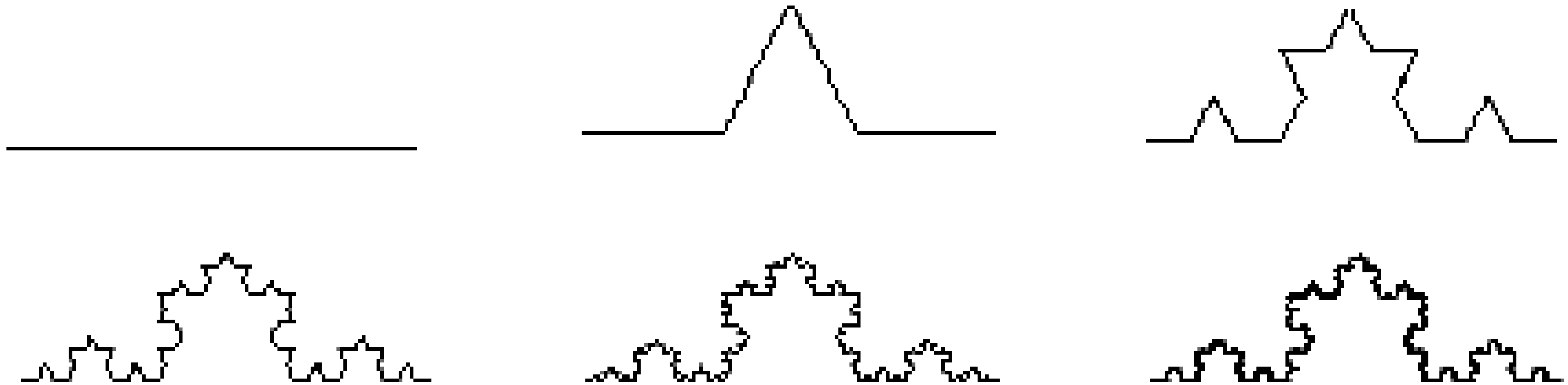


Figure 3.6 A cut-out set in the plane. Here, the largest possible disc is removed at each step. The family of discs removed is called the Apollonian packing of the square, and the cut-out set remaining is called the residual set, which has Hausdorff and box dimension about 1.31

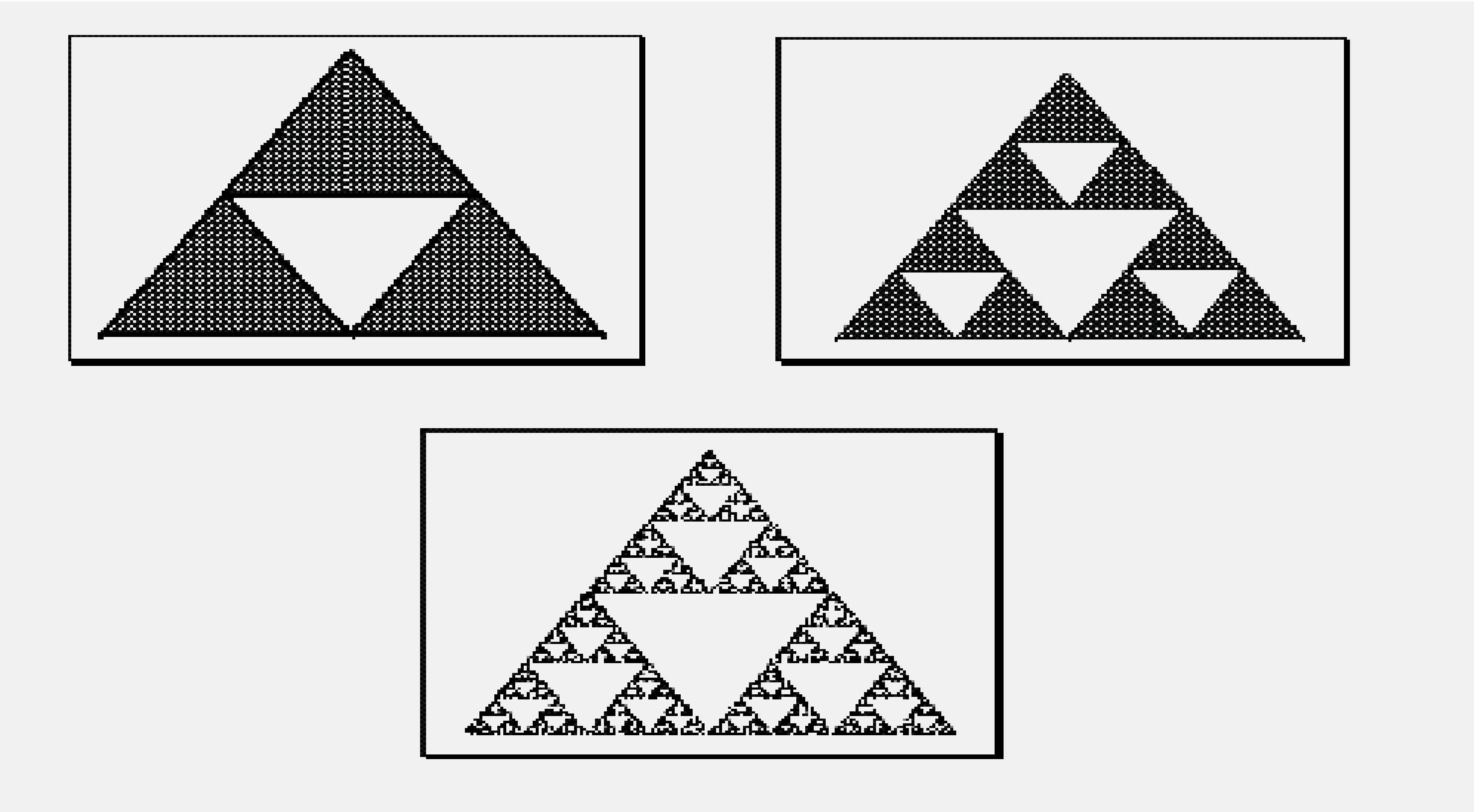
From: K. Falconer, Techniques in Fractal Geometry, Wiley 1997



$$L_0 = 1, \quad L_1 = 4/3, \quad L_2 = 4^2/3^2, \text{ etc... } L_k \to \infty$$

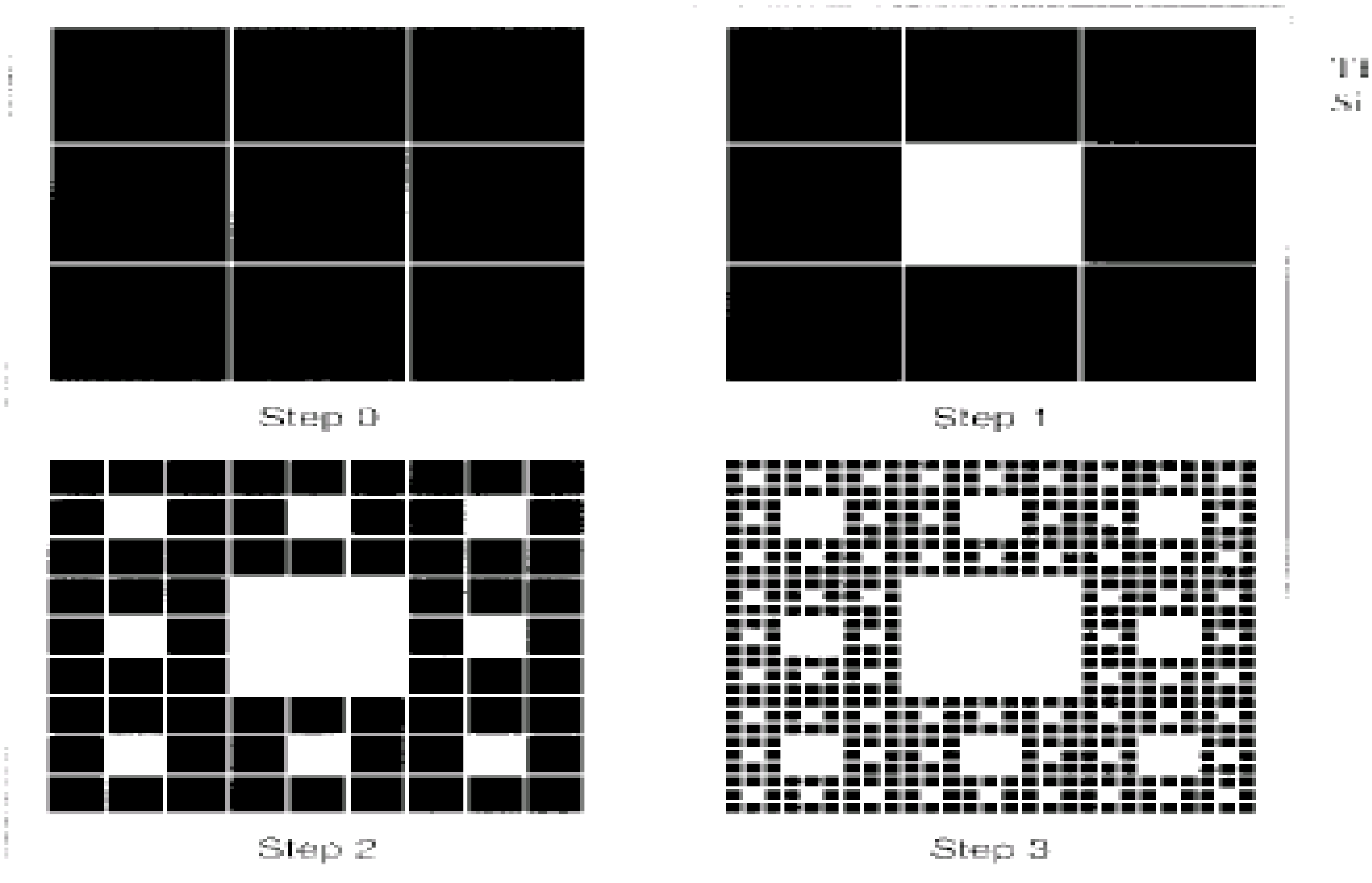
$$s = 1/3^k, \; N(s) = 4^k \to D = \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}$$

Sierpinski triangle (1916)



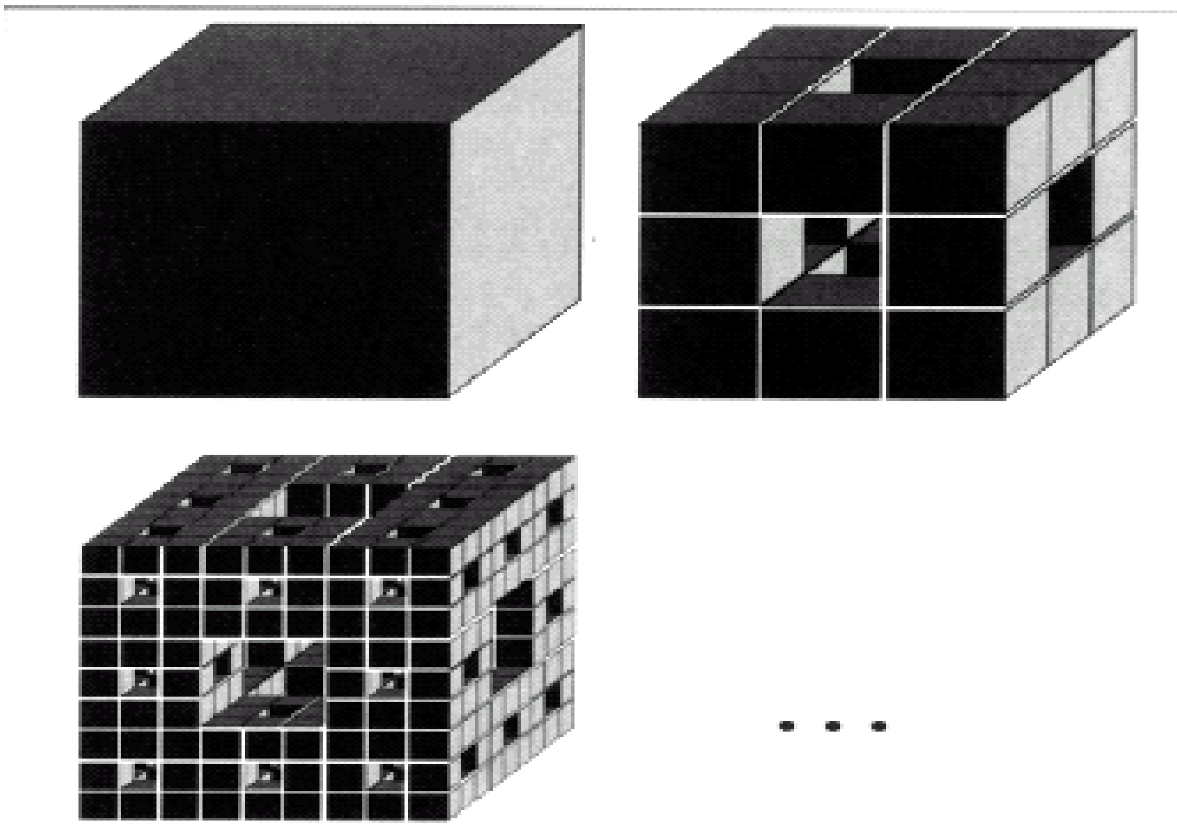
D=log3/log2=1.5849625...

A fractal carpet (zero area)

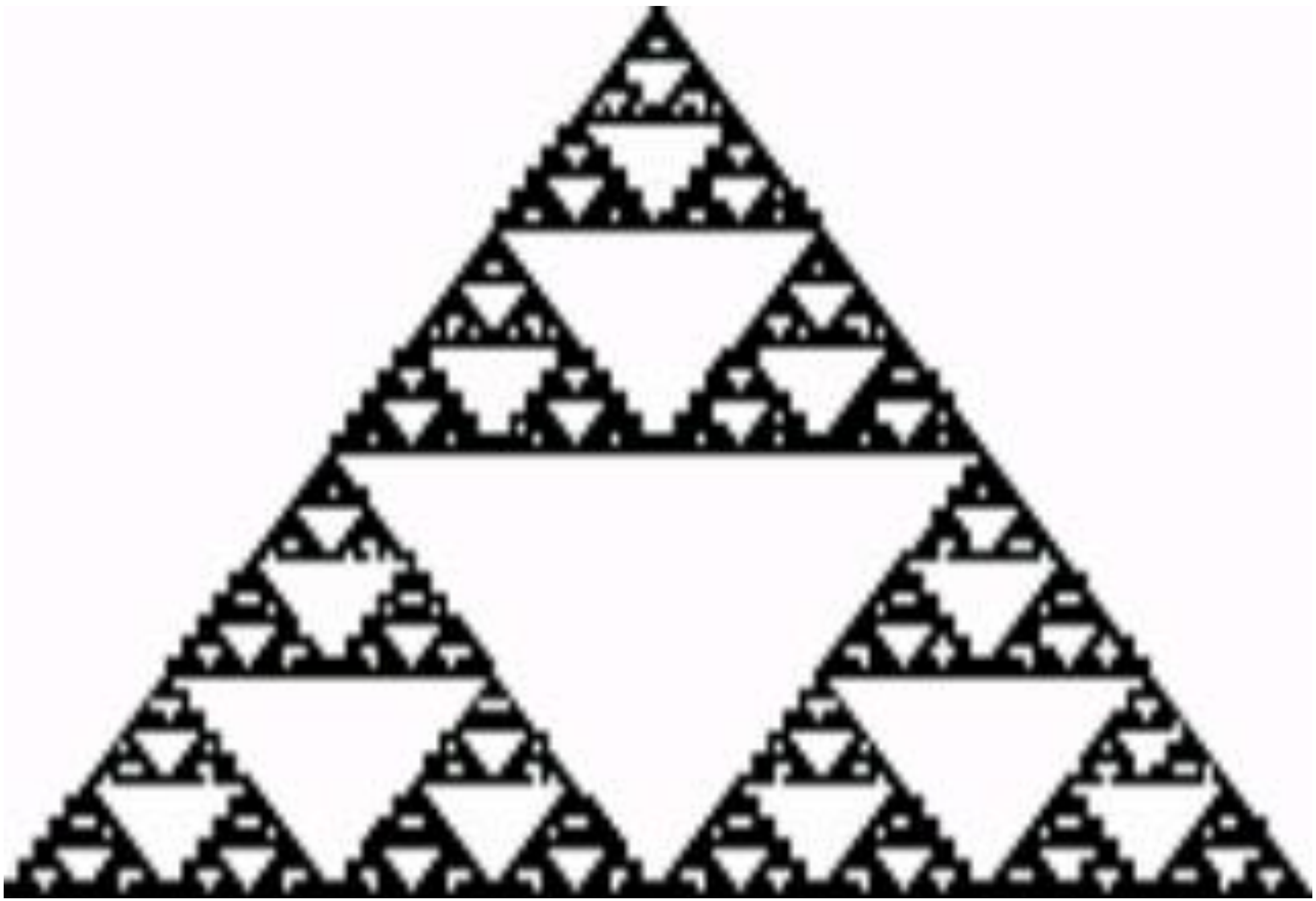


D=3log2/log3=1.892789...

A fractal sponge



Zooming in

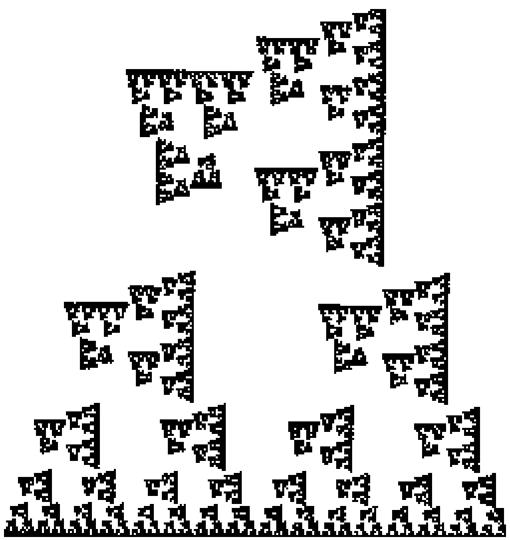


Changing parameters

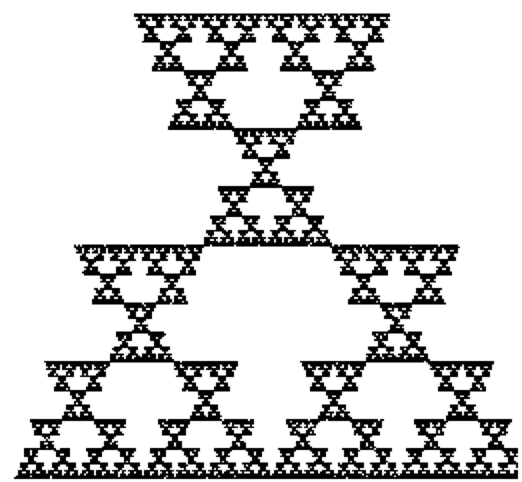
•The triangle of Sierpinski is the attractor of an iterated function system (i.f.s).

•The i.f.s. is made of three affine maps (each contracting by a factor ½ and leaving one of the initial vertices fixed)

•Combining the affine maps with rotations one can change the shape considerably



90° anticlockwise rotation about the top vertex



180° rotation about the same vertex

Hausdorff metric and compact sets

$X=[0,1]^2$

$d((x,y),(x',y'))= |x-x'|+|y-y'|$ Manhattan metric

$\mathcal{H}(X)=\{E$ compact nonempty subsets of $X\}$

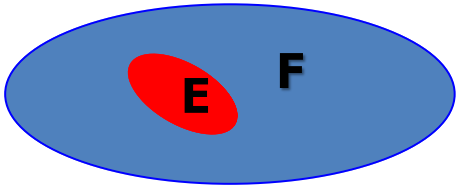
$h(E,F)=\max(d(E,F),d(F,E))$

$d(E,F)=\max_{x\in E}\min_{y\in F} d(x,y)$

$d(E,F)\neq d(F,E)$

$d(E,F)>0$

$d(F,E)=0$



Theorem: $(\mathcal{H}(X),h)$ is a complete metric space

$\rightarrow$ Cauchy sequences have a limit!

Contractions and Hausdorff metric

Proposition: if $w:X\to X$ is a contraction with Lipschitz constant s then w is also a contraction on $(\mathcal{H}(X),h)$ with Lipschitz constant s

To each family $\mathcal{F}$ of contractions on X one can associate a family of contractions on $(\mathcal{H}(X),h)$. By Banach-Caccioppoli to each such $\mathcal{F}$ will correspond a compact nonempty subset $\mathcal{A}$ of X: the attractor associated to $\mathcal{F}$

$$d(w(E),w(F)) = \max_{y\in E} \min_{z\in F} d(y,z) = \max_{e\in E} \min_{f\in F} d(w(e),w(f))$$

$$\leq s \max_{e\in E} \min_{f\in F} d(e,f) = s\, d(E,F)$$

Iterated function systems

$\mathcal{F} = \{w_1, \ldots, w_N\}$ each $w_i : X \to X$ is a contraction of constant s_i, $0 \le s_i < 1$

Let $\mathcal{W} : \mathcal{H}(X) \to X$

$$\mathcal{W}(E) = \bigcup_{1 \le i \le N} w_i(E)$$

Then $\mathcal{W}$ contracts the Hausdorff metric h with Lipschitz constant $s = \max_{1 \le i \le N} s_i$. We denote by $\mathcal{A}$ the corresponding attractor

Given any subset E of X, the iterates $\mathcal{W}^n(E) \to \mathcal{A}$ exponentially fast, in fact $h(\mathcal{W}^n(E), \mathcal{A}) \approx s^n$ as $n \to \infty$

Self similarity and fractal dimension

If the contractions of the i.f.s. $\mathcal{F} = \{w_1, \ldots, w_N\}$ are

- Similarities ⇒ the attractor $\mathcal{A}$ will be said self-similar
- Affine maps ⇒ the attractor $\mathcal{A}$ will be said self-affine
- Conformal maps (i.e. their derivative is a similarity) then the attractor $\mathcal{A}$ will be said self-conformal

If the open set condition is verified, i.e. there exists an open set U such that $w_i(U) \cap w_j(U) = \emptyset$ if $i \neq j$ and $\bigcup_i w_i(U)$ is an open subset of U then the dimension d of the attractor $\mathcal{A}$ is the unique positive solution of $s_1^d + s_2^d + \ldots + s_N^d = 1$