Dynamics and time series: theory and applications

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Measure-preserving transformations

X phase space, μ probability measure

$\Phi: X \to \mathbf{R}$ observable (a measurable function, say L^2).

Let A be subset of X (event).

$\mu(\Phi) = \int_X \Phi \, d\mu$ is the expectation of Φ

$T: X \to X$ induces a time evolution

on observables: $\Phi \to \Phi \circ T$

on events: $A \to T^{-1}(A)$

T is measure preserving if $\mu(\Phi) = \mu(\Phi \circ T)$ i.e.

$\mu(A) = \mu(T^{-1}(A))$

Birkhoff theorem and ergodicity

Birkhoff theorem: if T preserves the measure μ then with probability one the time averages of the observables exist (statistical expectations). The system is ergodic if these time averages do not depend on the orbit (statistics and a-priori probability agree)

$$\frac{1}{N}\sum_{0}^{N-1}\varphi\circ T^i(x) := \frac{1}{N}S_N\varphi(x) \longrightarrow \int_X \varphi(t)d\mu(t)$$

$$\frac{1}{N}\#\left\{i\in[0,N), T^i(x)\in A\right\} \longrightarrow \mu(A)$$

Law of large numbers: Statistics of orbits = a-priori probability

Strong vs. weak mixing: on events

- Strongly mixing systems are such that for every E, F we have
$$\mu(T^n(E) \cap F) \rightarrow \mu(E)\,\mu(F)$$
as n tends to infinity; the Bernoulli shift is a good example. Informally, this is saying that shifted sets become asymptotically independent of unshifted sets.

- Weakly mixing systems are such that for every E, F we have
$$\mu(T^n(E) \cap F) \rightarrow \mu(E)\,\mu(F)$$
as n tends to infinity *after excluding a set of exceptional values of n of asymptotic density zero*.

- Ergodicity does not imply $\mu(T^n(E) \cap F) \rightarrow \mu(E)\,\mu(F)$ but says that this is true for Cesaro averages:
$$1/n \sum^{n-1}_{0} \mu(T^j(E) \cap F) \rightarrow \mu(E)\,\mu(F)$$

Mixing: on observables

Order n correlation coefficient:

$$c_n(\varphi,\psi) := \int \varphi.\psi \circ T^n d\mu - \int \varphi d\mu \int \psi d\mu$$

Ergodicity implies

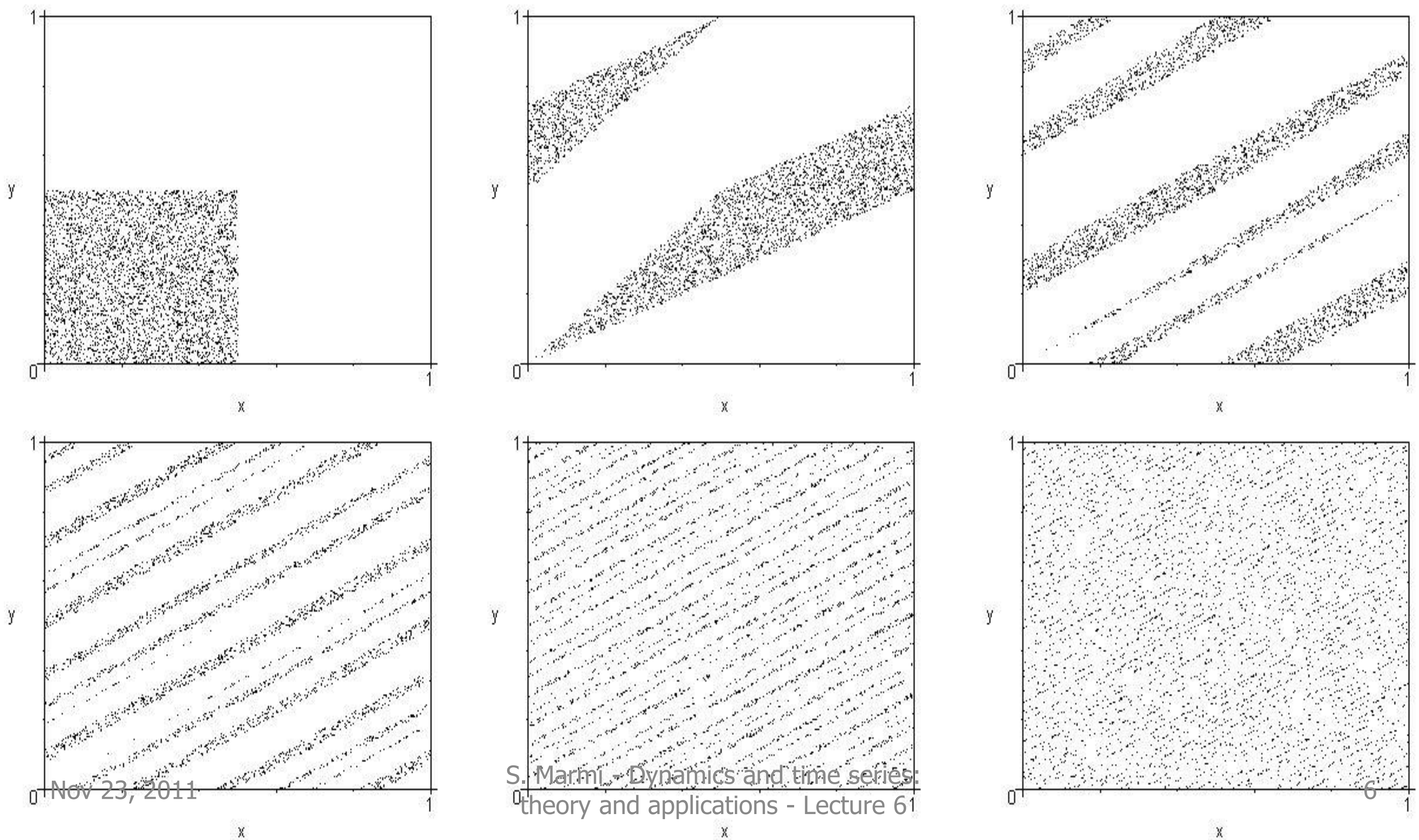
$$\frac{1}{N} \sum_0^{N-1} c_n(\varphi,\psi) \longrightarrow 0$$

Mixing requires that

$$c_N(\varphi,\psi) \longrightarrow 0$$

namely φ and $\varphi \circ T^n$ become independent of each other as $n \to \infty$

Mixing of hyperbolic automorphisms of the 2-torus (Arnold's cat)



ψ,φ observables with expectations (ψ) and $E(\varphi)$

$$\sigma(\psi)^2 = [\, E(\psi^2) - E(\psi)^2 \,]$$ **variance**

The **correlation coefficient** of ψ,φ is

$$\rho(\psi,\varphi) = \mathrm{covariance}(\psi,\varphi) / (\sigma(\psi)\,\sigma(\varphi))$$
$$= \mu\,[(\psi - \mu(\psi))(\varphi - \mu(\varphi))] / (\sigma(\psi)\,\sigma(\varphi))$$
$$= \mu\,[\psi\,\varphi - \mu(\psi)\mu(\varphi)] / (\sigma(\psi)\,\sigma(\varphi))$$

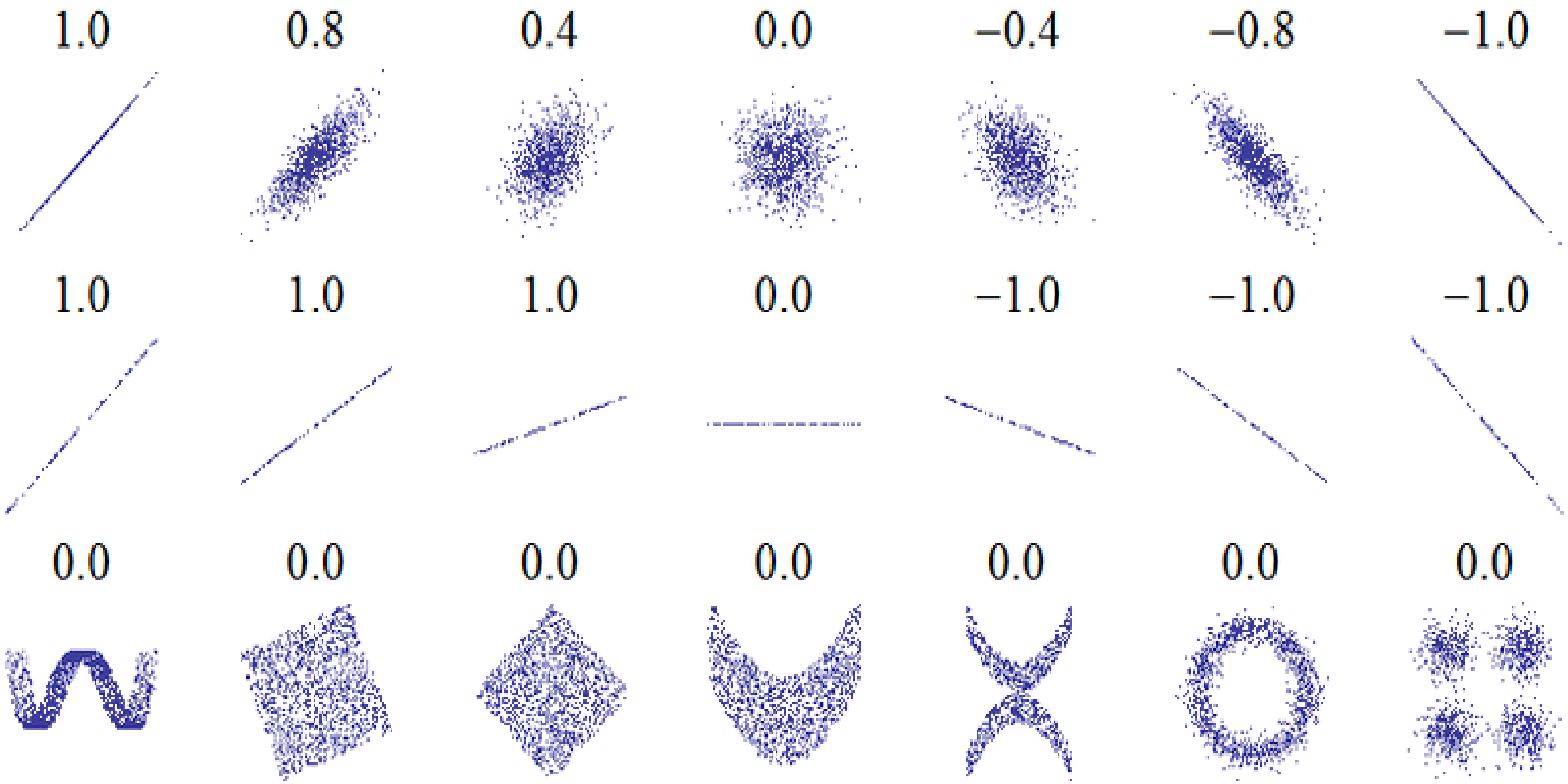
The correlation coefficient varies between -1 and 1 and equals 0 for independent variables but this is only a necessary condition (e.g. φ uniform on [-1,1] has zero correlation with its square)

If we have a series of n measurements of X and Y written as x(i) and y(i) where i = 1, 2, ..., n, then the Pearson product-moment correlation coefficient can be used to estimate the correlation of X and Y . The Pearson coefficient is also known as the "sample correlation coefficient". The Pearson correlation coefficient is then the best estimate of the correlation of X and Y . The Pearson correlation coefficient is written:

$$r_{xy} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{(n-1)s_x s_y} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - (\sum x_i)^2}\ \sqrt{n\sum y_i^2 - (\sum y_i)^2}}.$$

$$r_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y},$$

Correlation between two observables or series



Entropy

In information theory, **entropy** is a measure of the uncertainty associated with a random variable.

- Experiment with outcomes $A = \{a_1, \ldots, a_k\}$
- probability of obtaining the result a_i is p_i

  $0 <= p_i <= 1, \quad p_1 + \ldots + p_k = 1$
- If one of the a_i, let us say a_1 occurs with probability that is close to 1, then in most trials the outcome would be a_1 . There is not much information gained after the experiment
- We quantitatively measure the magnitude of 'being surprised' as

  $$\text{information} = -\log(\text{probability})$$
- (magnitude of our perception is proportional to the logarithm of the magnitude of the stimulus)

Metric entropy of a partition

Thus the entropy associated to the experiment is

$$H = -\sum_{i=1}^{k} p_i log\, p_i$$

In view of the definition of information = - log (probability), entropy is simply the expectation of information

$$\Delta^{(m)} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \in [0,1],\ \sum_{i=1}^{m} x_i = 1\}$$

Uniqueness of entropy

Definition 4.15 *A continuous function* $H^{(m)} : \Delta^{(m)} \to [0, +\infty]$ *is called an* entropy *if it has the following properties :*

(1) symmetry : $\forall\, i, j \in \{1, \ldots, m\}$ $H^{(m)}(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m) = H(p_1, \ldots, p_j,$ [redacted] $\ldots, p_i, \ldots, p_m)$;

(2) $H^{(m)}(1, 0, \ldots, 0) = 0$;

(3) $H^{(m)}(0, p_2, \ldots, p_m) = H^{(m-1)}(p_2, \ldots, p_m)$ $\forall\ m \geq 2$, $\forall\ (p_2, \ldots, p_m) \in \Delta^{(m-1)}$;

(4) $\forall\ (p_1, \ldots, p_m) \in \Delta^{(m)}$ *one has* $H^{(m)}(p_1, \ldots, p_m) \leq H^{(m)}\left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ *where equality is possible if and only if* $p_i = \frac{1}{m}$ *for all* $i = 1, \ldots, m$;

(5) Let $(\pi_{11}, \ldots, \pi_{1l}, \pi_{21}, \ldots, \pi_{2l}, \ldots, \pi_{m1}, \ldots, \pi_{ml}) \in \Delta^{(ml)}$ *; for all* $(p_1, \ldots, p_m) \in \Delta^{(m)}$ *one must have*

$$H^{(ml)}(\pi_{1l}, \ldots, \pi_{1l}, \pi_{21}, \ldots, \pi_{ml}) = H^{(m)}(p_1, \ldots, p_m) + \\ + \sum_{i=1}^{m} p_i H^{(l)}\left(\frac{\pi_{i1}}{p_i}, \ldots, \frac{\pi_{il}}{p_i}\right) .$$

Theorem 4.16 *An entropy is necessarily a positive multiple of*

$$H(p_1, \ldots, p_m) = -\sum_{i=1}^{m} p_i \log p_i \ .$$

Entropy of a dynamical system (Kolmogorov-Sinai entropy)

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$

$\mathcal{P} \vee \mathcal{Q}$ the **join** of $\mathcal{P}$ and $\mathcal{Q}$

$B \cap C$ where $B \in \mathcal{Q}$ and $C \in \mathcal{Q}$

$T : X \to X$ measure preserving

$$\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-(n-1)}\mathcal{P}$$

$$h(T,\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P}_n) \qquad h(T) = \sup_{\mathcal{P}} h(T,\mathcal{P})$$

Properties of the entropy

Let $T:X\to X$, $S:Y\to Y$ be measure preserving (T preserves μ, S preserves ν)

$$\mathit{If}\ n \geq 1,\ \mathit{then}\ h(T^n) = n\, h(T)$$

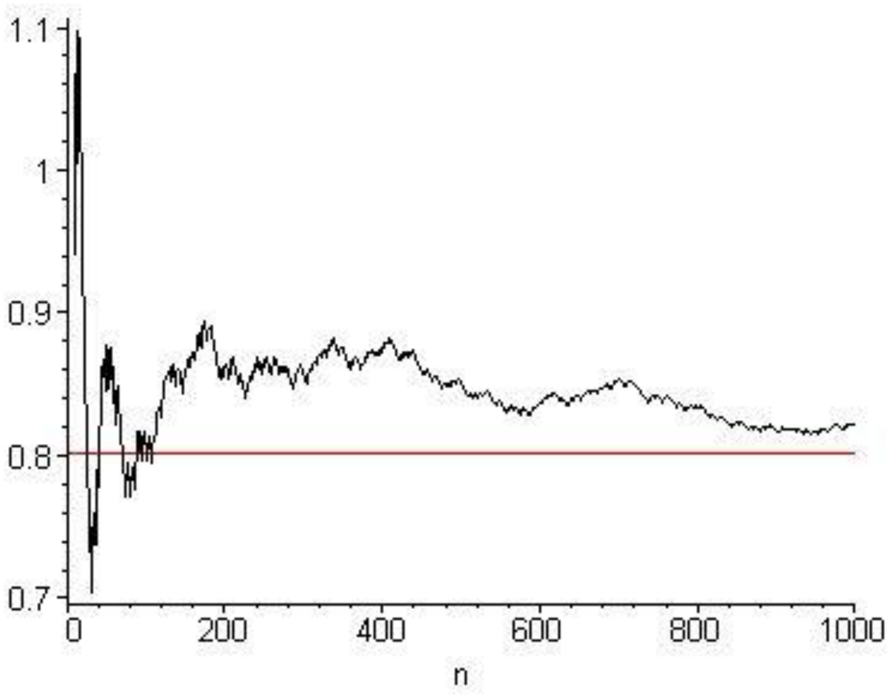
$$\mathit{If}\ T\ \mathit{is\ invertible},\ \mathit{then}\ h(T^{-1}) = h(T)$$

If S is a factor of T then $h(S,\nu)\leq h(T,\mu)$

If S and T are isomorphic then $h(S,\nu)=h(T,\mu)$

On $X\text{x}Y$ one has $h(T\text{x}S,\mu\text{x}\nu)= h(T,\mu)\ \text{x}\ h(S,\nu)$

Shannon-Breiman-McMillan theorem



Let $\mathscr{P}$ be a generating partition

Let P(n,x) be the element of

$$\bigvee_{i=0}^{n-1} T^{-i}\mathscr{P}$$

which contains x

The SHANNON-BREIMAN-MCMILLAN theorem says that for ergodic T, for a.e. x one has

$$h(T,\mu) = -\lim_{n\to\infty} \frac{\operatorname{Log} \mu(P(n,x))}{n}$$

Asymptotic equipartition property

Suppose that $\mathcal{P}$ is a finite generating partition of X. For every $\varepsilon > 0$ and $n \geq 1$ there exist subsets in $\mathcal{P}_n$, which are called (n, ε)-typical subsets, satisfying the following:
(i) for every typical subset $\mathcal{P}_n(x)$,

$$2^{-n(h+\varepsilon)} < \mu(\mathcal{P}_n(x)) < 2^{-n(h-\varepsilon)} ,$$

(ii) the union of all (n, ε)-typical subsets has measure greater than $1-\varepsilon$, and
(iii) the number of (n, ε)-typical subsets is between $(1-\varepsilon)2^{n(h-\varepsilon)}$ and $2^{n(h+\varepsilon)}$.

These formulas assume that the entropy is measured
in bits, i.e. using the base 2 logarithm

Entropy of Bernoulli schemes

Let $N \geq 2$, $\Sigma_N = \{1, \ldots N\}^{\mathbb{Z}}$.

$$d(x,y) = 2^{-a(x,y)} \quad \text{where } a(x,y) = \inf\{|n|\,,\, n \in \mathbb{Z}\,,\, x_n \neq y_n\}$$

shift $\sigma \ : \Sigma_N \to \Sigma_N \qquad \sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$

The topological entropy of (Σ_N, σ) is $\log N$

$(p_1, \ldots, p_N) \in \Delta^{(N)} \qquad \nu(\{i\}) = p_i$

Definition 4.26 *The* Bernoulli scheme $BS(p_1, \ldots, p_N)$ *is the measurable dynamical system given by the shift map* $\sigma \ : \Sigma_N \to \Sigma_N$ *with the (product) probability measure* $\mu = \nu^{\mathbb{Z}}$ *on* Σ_N.

Proposition 4.27 *The Kolmogorov–Sinai entropy of the Bernoulli scheme* $BS(p_1, \ldots, p_N)$ *is* $-\sum_{i=1}^N p_i \log p_i$.

Topological Markov chains or subshifts of finite type

$$\Sigma_A = \{x \in \Sigma_N\,,\,(x_i, x_{i+1}) \in \Gamma\,\forall i \in \mathbb{Z}\} \qquad \Gamma \subset \{1,\dots N\}^2$$

Σ_A is a compact shift invariant subset of Σ_N

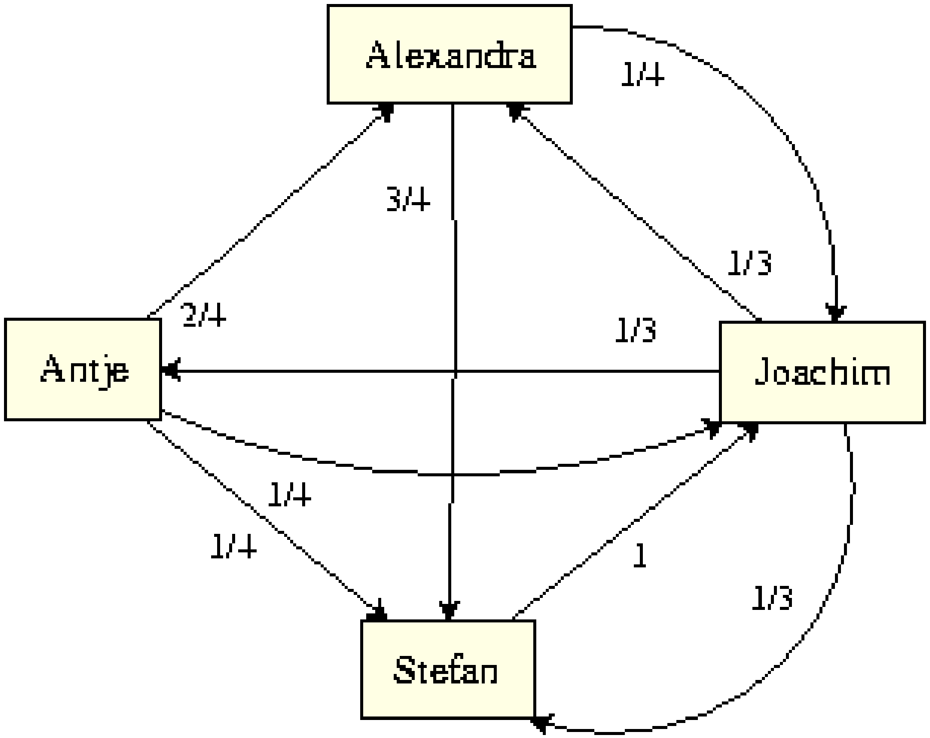
$A = A_\Gamma$ the $N \times N$ matrix with entries $a_{ij} \in \{0,1\}$

$$a_{ij} = \begin{cases} 1 & \iff (i,j) \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

The restriction of the shift σ to Σ_A is denoted σ_A

$A^m = (a_{ij}^m)$ and $a_{ij}^m > 0$ for all i, j (primitive matrix)

Markov chains



Topological: some moves are allowed and some are not

Metric: any allowed move happens with some fixed probability

Entropy of Markov chains

Theorem 4.35 (Perron–Frobenius, see [Gan]) If A is primitive then there exists an eigenvalue $\lambda_A > 0$ such that :

(i) $|\lambda_A| > \lambda$ for all eigenvalues $\lambda \neq \lambda_A$;

(ii) the left and right eigenvectors associated to λ_A are strictly positive and are unique up to constant multiples ;

(iii) λ_A is a simple root of the characteristic polynomial of A.

the topological entropy of σ_A is $\log \lambda_A$ (clearly $\lambda_A > 1$ since all the integers $a_{ij}^m > 0$)

Let $P = (P_{ij})$ be an $N \times N$ matrix such that

(i) $P_{ij} \geq 0$ for all i, j, and $P_{ij} > 0 \iff a_{ij} = 1$;

(ii) $\sum_{j=1}^{N} P_{ij} = 1$ for all $i = 1, \ldots, N$;

(iii) P^m has all its entries strictly positive.

Such a matrix is called a *stochastic matrix*. Applying Perron–Frobenius theorem to P we see that 1 is a simple eigenvalue of P and there exists a normalized eigenvector $p = (p_1, \dots, p_N) \in \Delta^{(N)}$ such that $p_i > 0$ for all i and

$$\sum_{i=1}^{N} p_i P_{ij} = p_j \ , \quad \forall\, 1 \le i \le N \ .$$

We define a probability measure μ on Σ_A corresponding to P prescribing its value on the cylinders :

$$\mu\left(C\begin{pmatrix} j_0, \dots, j_k \\ i, \dots, i+k \end{pmatrix}\right) = p_{j_0} P_{j_0 j_1} \cdots P_{j_{k-1} j_k} \ ,$$

for all $i \in \mathbb{Z}$, $k \ge 0$ and $j_0, \dots, j_k \in \{1, \dots, N\}$. It is called the *Markov measure* associated to the stochastic matrix P.

the subshift σ_A preserves the Markov measure μ.

$$h_\mu(\sigma_A) = -\sum_{i,j=1}^{N} p_i P_{ij} \log P_{ij} \qquad\qquad h_\mu(\sigma_A) \le h_{top}(\sigma_A)$$

Entropy, coding and data compression

- Computer file= infinitely long binary sequence
- Entropy = best possible compression ratio
- Lempel-Ziv (Compression of individual sequences via variable rate coding, IEEE Trans. Inf. Th. 24 (1978) 530-536): it does not assume knowledge of probability distribution of the source and achieves asymptotic compression ratio=entropy of source

Let $X = \{0,1\}^{\mathbb{N}}$ and σ be a left-shift map.
Define R_n to be the first return time of the initial n-block, i.e.,

$$R_n(x) = \min\{j \geq 1 : x_1 \ldots x_n = x_{j+1} \ldots x_{j+n}\}.$$

$$x = \overbrace{\boxed{1\,0\,1\,0}\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0}^{15}\,\boxed{1\,0\,1\,0} \cdots \Rightarrow R_4(x) = 15.$$

The convergence of $\dfrac{1}{n} \log R_n(x)$ to the entropy h was studied in a relation with data compression algorithm such as the Lempel-Ziv compression algorithm.

In the 1978 paper, Ziv and Lempel described an algorithm that parses a string into phrases, where each phrase is the shortest phrase not seen earlier. This algorithm can be viewed as building a dictionary in the form of a tree, where the nodes correspond to phrases seen so far. The algorithm is particularly simple to implement and has become popular as one of the early standard algorithms for file compression on computers because of its speed and efficiency. The source sequence is sequentially parsed into strings that have not appeared so far. For example, if the string is ABBABBABBBAABABAA . . . , we parse it as A,B,BA,BB,AB,BBA,ABA,BAA. . . . After every comma, we look along the input sequence until we come to the shortest string that has not been marked off before. Since this is the shortest such string, all its prefixes must have occurred earlier. (Thus, we can build up a tree of these phrases.) In particular, the string consisting of all but the last bit of this string must have occurred earlier. We code this phrase by giving the location of the prefix and the value of the last symbol. Thus, the string above would be represented as (0,A),(0,B),(2,A),(2,B),(1,B),(4,A),(5,A), (3,A), ...

The Lempel-Ziv data compression algorithm provide a universal way to coding a sequence without knowledge of source.
Parse a source sequence into shortest words that has not appeared so far:

$$1011010100010 \cdots \Rightarrow 1, 0, 11, 01, 010, 00, 10, \ldots$$

For each new word, find a phrase consisting of all but the last bit, and recode the location of the phrase and the last bit as the compressed data.

$$(000, 1)\,(000, 0)\,(001, 1)\,(010, 1)\,(100, 0)\,(010, 0)\,(001, 0)\ldots$$

Theorem (Wyner-Ziv(1989), Ornstein and Weiss(1993))

For ergodic processes with entropy h,

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x) = h \quad \textit{almost surely.}$$

The meaning of entropy

- Entropy measures the information content or the amount of randomness.
- Entropy measures the maximum compression rate.
- Totally random binary sequence has entropy $\log 2 = 1$. It cannot be compressed further.

The entropy of English

Is English a stationary ergodic process? Probably not!

Stochastic approximations to English: as we increase the complexity of the model, we can generate text that looks like English. The stochastic models can be used to compress English text. The better the stochastic approximation, the better the compression.

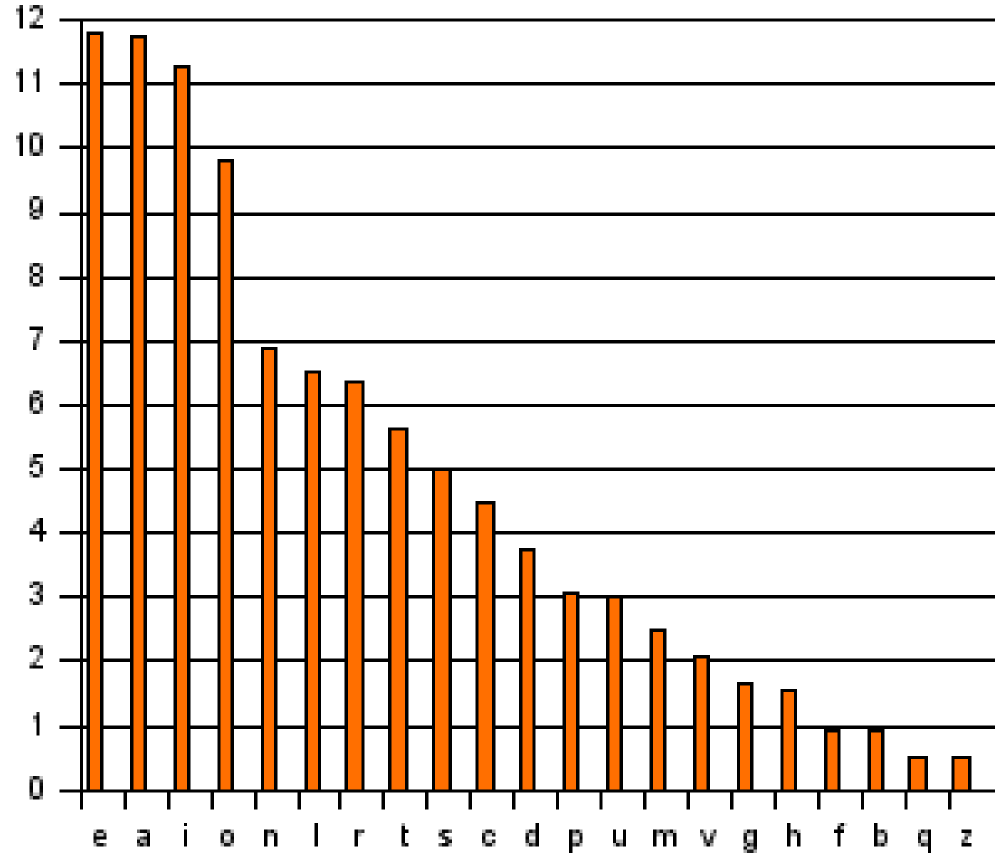
alphabet of English = 26 letters and the space symbol

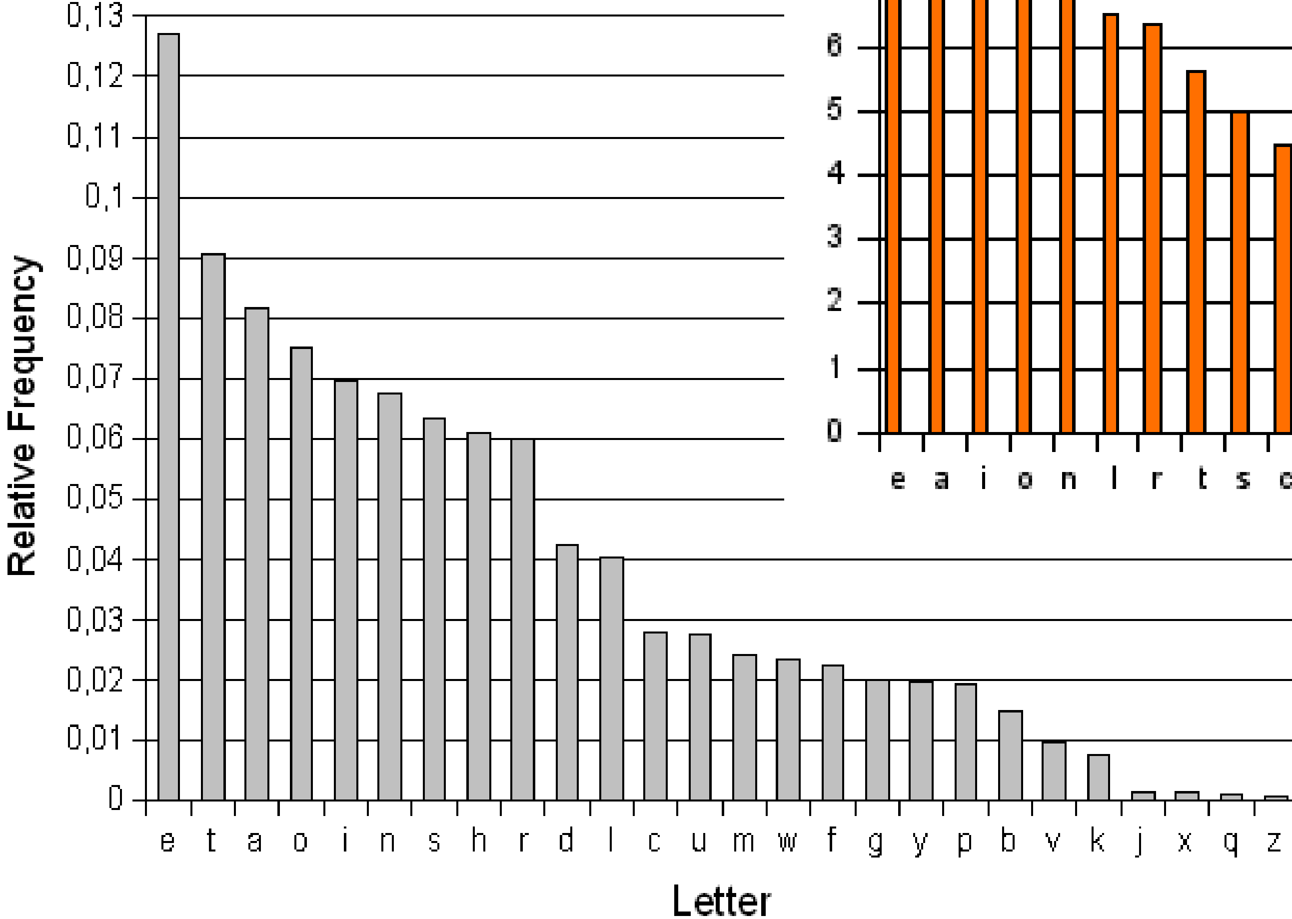
models for English are constructed using empirical distributions collected from samples of text.

E is most common, with a frequency of about 13%,

least common letters, Q and Z, have a frequency of about 0.1%.

Frequency of letters
In Italian





Frequency of letters
In English

Source: Wikipedia

Construction of a Markov model for English

The frequency of pairs of letters is also far from uniform:
Q is always followed by a U, the most frequent pair is TH, (frequency of about 3.7%), etc.

Proceeding this way, we can also estimate higher-order conditional probabilities and build more complex models for the language.

However, we soon run out of data. For example, to build a third-order Markov approximation, we must compute
$p(x_i \mid x_{i-1}, x_{i-2}, x_{i-3})$ in correspondence of $27 \times 27^3 = 531\,441$ entries for this table: need to process millions of letters to make accurate estimates of these probabilities.

Examples

(Cover and Thomas, Elements of Information Theory, 2nd edition , Wiley 2006)

- Zero order approximation (equiprobable h=4.76 bits):

  XFOML RXKHRJFFJUJ ZLPWCFWKCYJ FFJEYVKCQSGXYD QPAAMKBZAACIBZLHJQD

- First order approximation (frequencies match):

  OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI

  ALHENHTTPA OOBTTVA NAH BRL

- Second order (frequencies of pairs match): ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN ANDY TOBE SEACE CTISBE

- Third order (frequencies of triplets match): IN NO IST LAT WHEY CRATICT FROURE BERS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE

- Fourth order approximation (frequencies of quadruplets match, each letter depends on previous three letters; h=2.8 bits):

  THE GENERATED JOB PROVIDUAL BETTER TRANDTHE DISPLAYED CODE, ABOVERY UPONDULTS WELL THE CODERST IN THESTICAL IT DO HOCK BOTHE MERG. (INSTATES CONS ERATION. NEVER ANY OF PUBLE AND TO THEORY. EVENTIAL CALLEGAND TO ELAST BENERATED IN WITH PIES AS IS WITH THE)

- First order WORD approximation (random words, frequencies match): REPRESENTING AND SPEEDILY IS AN GOOD APT OR COME CAN DIFFERENT NATURAL HERE HE THE A IN CAME THE TO OF TO EXPERT GRAY COME TO FURNISHES THE LINE MESSAGE HAD BE THESE.
- Second order (WORD transition probabilities match): THE HEAD AND IN FRONTAL ATTACK ON AN ENGLISH WRITER THAT THE CHARACTER OF THIS POINT IS THEREFORE ANOTHER METHOD FOR THE LETTERS THAT THE TIME OF WHO EVER TOLD THE PROBLEM FOR AN UNEXPECTED