Dynamics and time series: theory and applications

Stefano Marmi
Scuola Normale Superiore
Lecture 4, Jan 29, 2009

- Lecture 1: An introduction to dynamical systems and to time series. Periodic and quasiperiodic motions. (Tue Jan 13, 2 pm - 4 pm Aula Bianchi)
- Lecture 2: Ergodicity. Uniform distribution of orbits. Return times. Kac inequality Mixing (Thu Jan 15, 2 pm - 4 pm Aula Dini)
- Lecture 3: Kolmogorov-Sinai entropy. Randomness and deterministic chaos. (Tue Jan 27, 2 pm - 4 pm Aula Bianchi)
- **Lecture 4: Time series analysis and embedology. (Thu Jan 29, 2 pm - 4 pm Dini)**
- Lecture 5: Fractals and multifractals. (Thu Feb 12, 2 pm - 4 pm Dini)
- Lecture 6: The rhythms of life. (Tue Feb 17, 2 pm - 4 pm Bianchi)
- Lecture 7: Financial time series. (Thu Feb 19, 2 pm - 4 pm Dini)
- Lecture 8: The efficient markets hypothesis. (Tue Mar 3, 2 pm - 4 pm Bianchi)
- Lecture 9: A random walk down Wall Street. (Thu Mar 19, 2 pm - 4 pm Dini)
- Lecture 10: A non-random walk down Wall Street. (Tue Mar 24, 2 pm – 4 pm Bianchi)

- Seminar I: Waiting times, recurrence times ergodicity and quasiperiodic dynamics (D.H. Kim, Suwon, Korea; Thu Jan 22, 2 pm - 4 pm Aula Dini)
- Seminar II: Symbolization of dynamics. Recurrence rates and entropy (S. Galatolo, Università di Pisa; Tue Feb 10, 2 pm - 4 pm Aula Bianchi)
- Seminar III: Heart Rate Variability: a statistical physics point of view (A. Facchini, Università di Siena; Tue Feb 24, 2 pm - 4 pm Aula Bianchi)
- Seminar IV: Study of a population model: the Yoccoz-Birkeland model (D. Papini, Università di Siena; Thu Feb 26, 2 pm - 4 pm Aula Dini)
- Seminar V: Scaling laws in economics (G. Bottazzi, Scuola Superiore Sant'Anna Pisa; Tue Mar 17, 2 pm - 4 pm Aula Bianchi)
- Seminar VI: Complexity, sequence distance and heart rate variability (M. Degli Esposti, Università di Bologna; Thu Mar 26, 2 pm - 4 pm Aula Dini)
- Seminar VII: Forecasting (M. Lippi, Università di Roma; late april, TBA)

Examples of time-series in natural and social sciences

- Weather measurements (temperature, pressure, rain, wind speed, …) . If the series is very long …climate
- Earthquakes
- Lightcurves of variable stars
- Sunspots
- Macroeconomic historical time series (inflation, GDP, employment,…)
- Financial time series (stocks, futures, commodities, bonds, …)
- Populations census (humans or animals)
- Physiological signals (ECG, EEG, …)

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Technological Forecasting & Social Change 74 (2007) 1508–1514

Technological Forecasting and Social Change

Sunspots, GDP and the stock market

Theodore Modis

Growth-Dynamics, Via Selva 8, Massagno, 6900 Lugano, Switzerland

Received 6 May 2007; accepted 13 June 2007

Abstract

A correlation has been observed between the US GDP and the number of sunspots as well as between the Dow Jones Industrial Average and the number of sunspots. The data cover 80 years of history. The observed correlations permit forecasts for the GDP and for the stock market in America with a future horizon of 10 years. Both being above their long-term trend they are forecasted to go over a peak around Jun-2008.

The work reported here presents hard-to-dispute evidence for the existence of a correlation between stock-market movements as measured by the DJIA (Dow Jones Industrial Average) and sunspot activity, as well as between GDP growth and sunspot activity. No causality arguments are made and there is no attempt to understand the mechanisms behind the observed correlation. The author would be satisfied with as little explanation as the possibility that sunspot activity may influence the climate on earth, which in turn may influence the economy.

Still, given the correlation and the rather reliable forecasts for sunspot activity provided by NASA, the author ventures long-range forecasts for GDP growth and the stock market in the United States.

Table 1
All dates are in decimal fractions of a year

DJIA peaks	Sunspot peaks	Delta
1937.17	1938.25	1.08
1946.33	1948.67	2.34
1956.25	1958.42	2.17
1966.00	1969.25	3.25
1976.50	1980.83	4.33
1987.58	1990.50	2.92
1999.92	2001.17	1.25
		Ave. delta=2.48
Forecast:		
2008.44	2010.92	

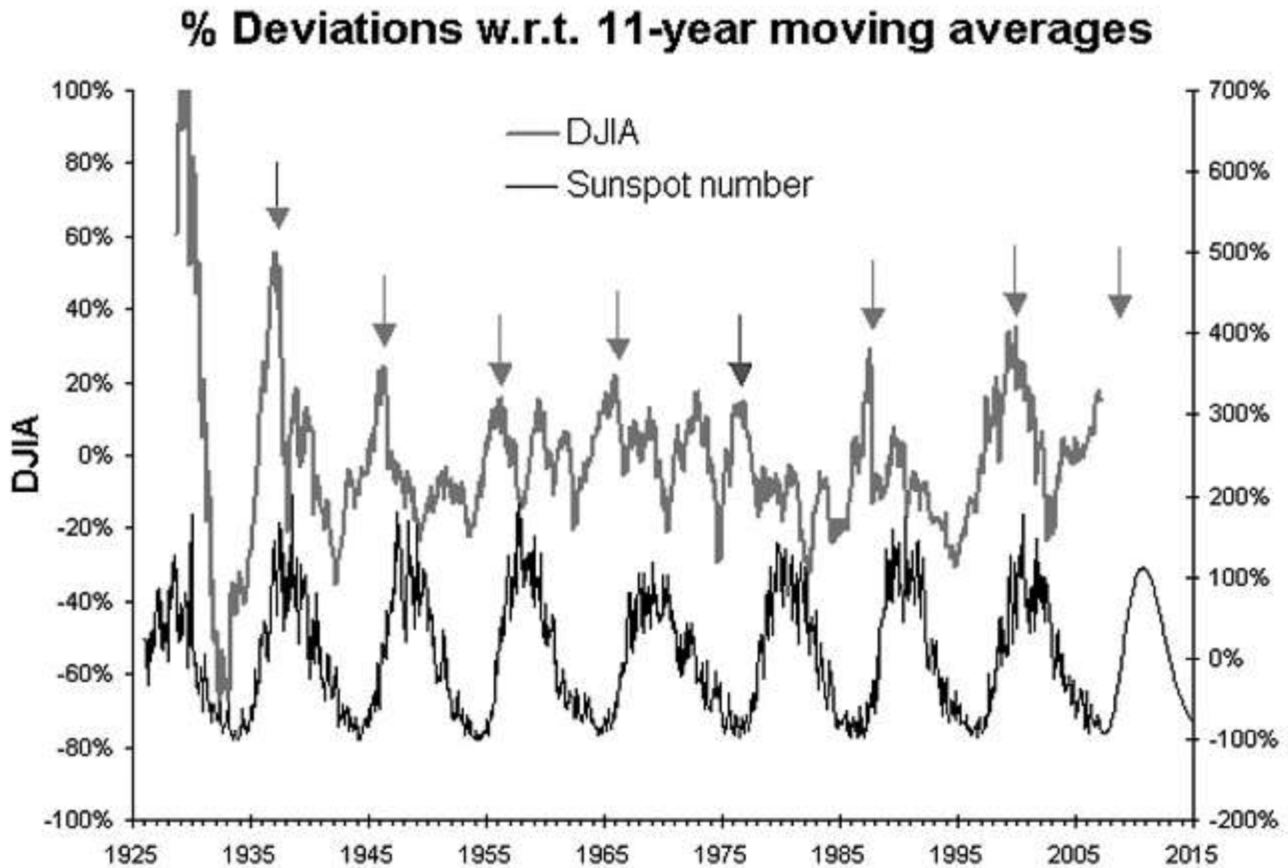


Fig. 2. Percent deviations with respect to the long-term trends as calculated via 11-year moving averages. The arrows point at the "significant" DJIA peaks. The last arrow is a forecast (Jun-2008), see text.

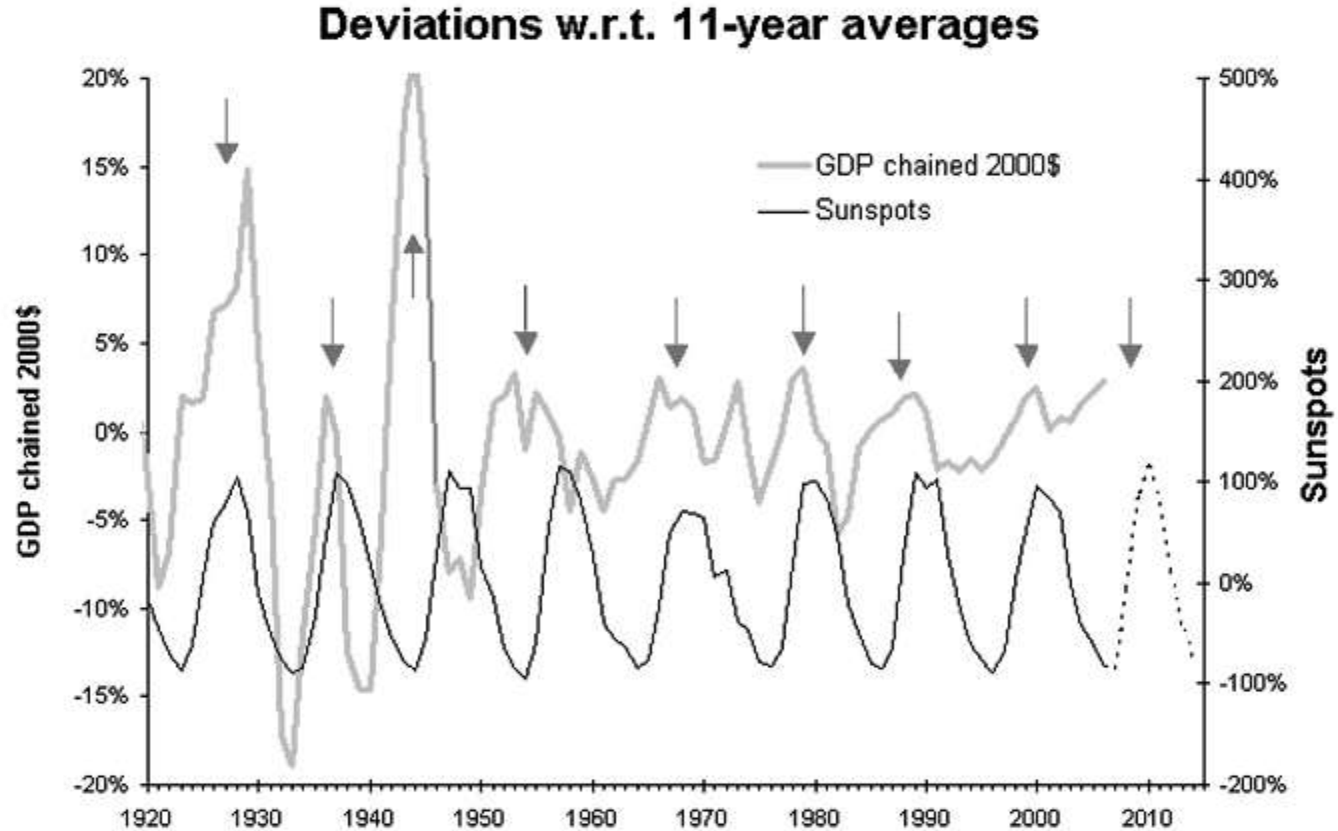


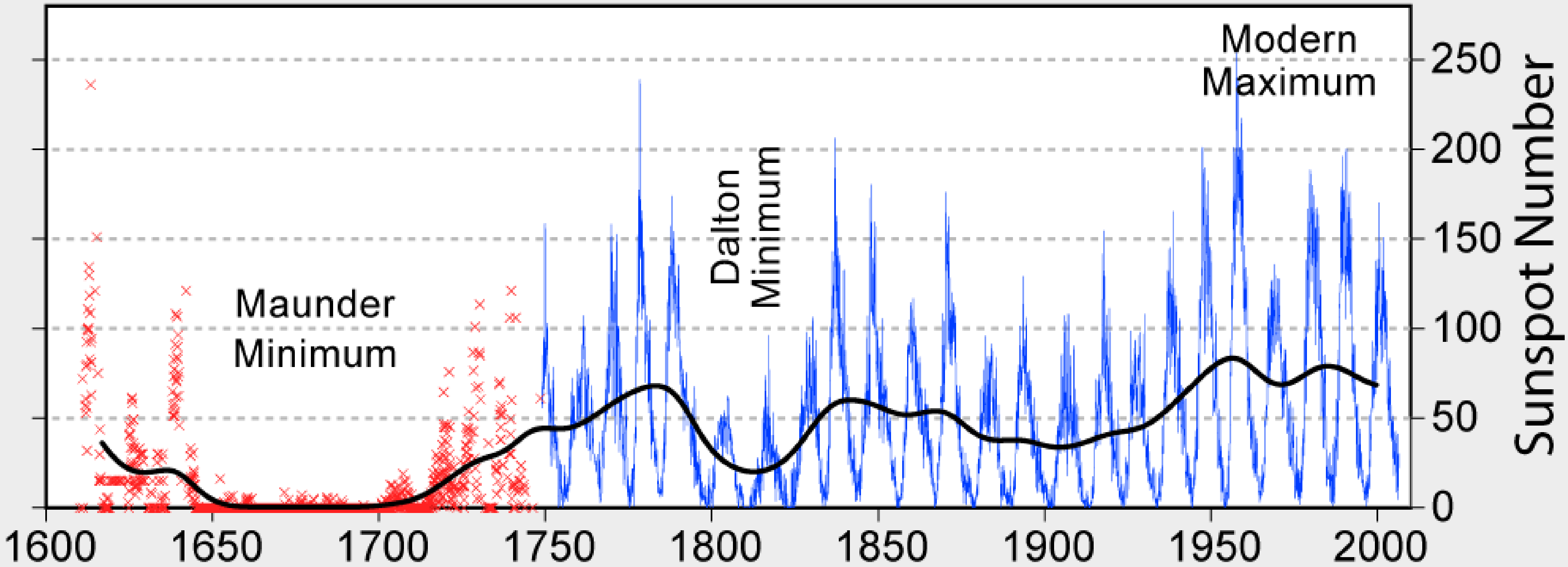
Fig. 4. Percent deviations with respect to the long-term trends as calculated via 11-year moving averages. The arrows point at the "significant" GDP peaks. The last arrow is a forecast (Jun-2008), see text.

Yes!!! These paper really claims that U.S. GDP can be forecasted using sunspots….

….what are Neokeynesians good for??? What is Obanomics good for???

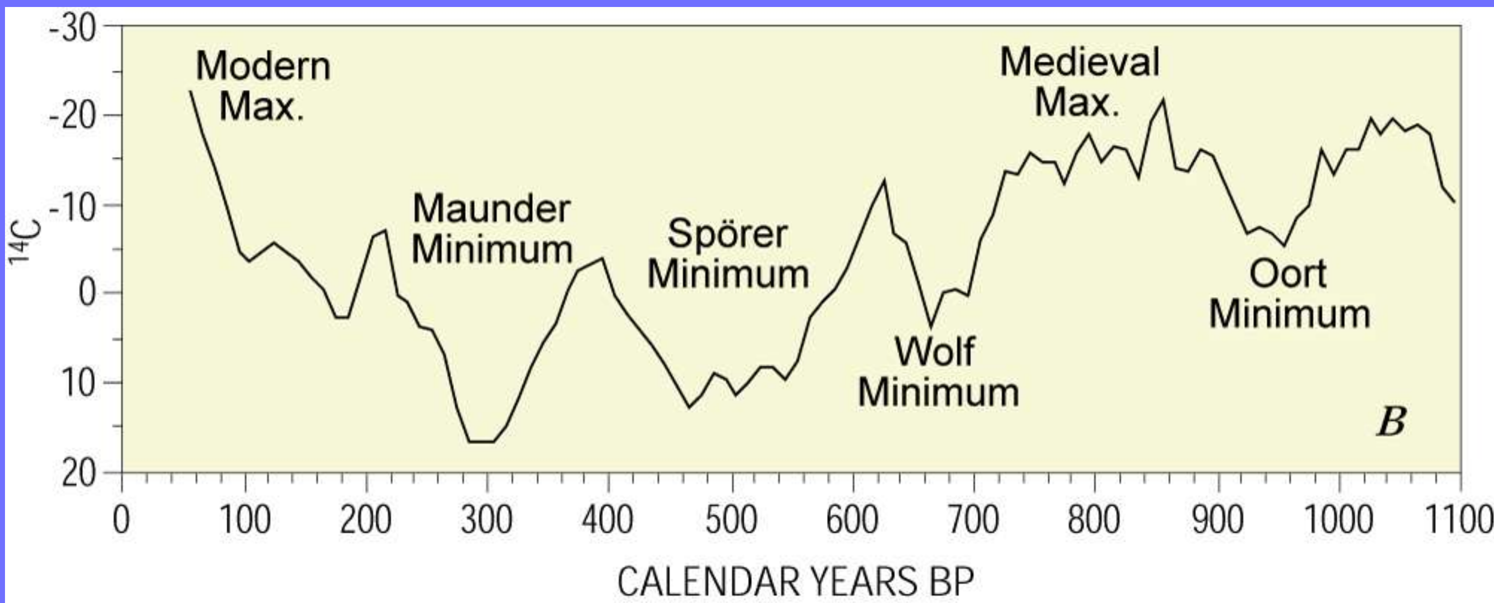
What happened in the U.S. during the Maunder minimum???

400 Years of Sunspot Observations



Source: Wikipedia

Changes in carbon-14 concentration in the[Earth's atmosphere](Earth's atmosphere), which serves as a long term proxy of solar activity.



Stochastic or chaotic?

- An important goal of time-series analysis is to determine, given a times series (e.g. HRV) if the underlying dynamics (the heart) is:
  - Intrinsically random
  - Generated by a deterministic nonlinear chaotic system which generates a random output
  - A mix of the two (stochastic perturbations of deterministic dynamics)

CHAOS 18, 030201 (2008)

Announcement: A new feature—"Controversial Topics in Nonlinear Science: Is the Normal Heart Rate Chaotic?"

Leon Glass
Department of Physics, McGill University, Montréal, Québec H3G 1Y6, Canada

(Received 16 June 2008; published online 6 August 2008)

The normal heart rhythm in humans is set by a small group of cells called the sinoatrial node. Although over short time intervals, the normal heart rate often appears to be regular, when the heart rate is measured over extended periods of time, it shows significant fluctuations. There are a number of factors that affect these fluctuations: changes of activity or mental state, presence of drugs, presence of artificial pace- makers, occurrence of cardiac arrhythmias that might mask the sinoatrial rhythm or make it difficult to measure. Following the widespread recognition of the possibility of deterministic chaos in the early 1980s, considerable attention has been focused on the possibility that heart rate variability might reflect deterministic chaos in the physiological control system regulating the heart rate. A large number of papers related to the analysis of heart rate variability have been published in Chaos and elsewhere. However, there is still considerable debate about how to characterize fluctuations in the heart rate and the significance of those fluctuations. There has not been a forum in which these disagreements can be aired. Accordingly, Chaos invites submissions that address one or more of the following questions:

- Is the normal heart rate chaotic?
- If the normal heart rate is not chaotic, is there some more appropriate term to characterize the fluctuations e.g., scaling, fractal, multifractal?
- How does the analysis of heart rate variability elucidate the underlying mechanisms controlling the heart rate?
- **Do any analyses of heart rate variability provide clinical information that can be useful in medical assessment e.g., in helping to assess the risk of sudden cardiac death**. If so, please indicate what additional clinical studies would be useful for measures of heart rate variability to be more broadly accepted by the medical community.

Chaotic brains at work!

C.R. Acad. Sci. Paris, Sciences de la vie / Life Sciences 324 (2001) 773–793

S0764446901013774/REV

Point sur / Concise review

Is there chaos in the brain? I. Concepts of nonlinear dynamics and methods of investigation

Philippe Faure, Henri Korn*

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Received 18 June 2001; accepted 2 July 2001

Communicated by Pierre Buser

Abstract – In the light of results obtained during the last two decades in a number of laboratories, it appears that some of the tools of nonlinear dynamics, first developed and improved for the physical sciences and engineering, are well-suited for studies of biological phenomena. In particular it has become clear that the different regimes of activities undergone by nerve cells, neural assemblies and behavioural patterns, the linkage between them, and their modifications over time, cannot be fully understood in the context of even integrative physiology, without using these new techniques. This

Chaotic brains at work!

networks and in the study of higher brain functions, will be critically reviewed. It will be shown that the tools of nonlinear dynamics can be irreplaceable for revealing hidden mechanisms subserving, for example, neuronal synchronization and periodic oscillations. The benefits for the brain of adopting chaotic regimes with their wide range of potential behaviours and their aptitude to quickly react to changing conditions will also be considered.

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C. R. Biologies 326 (2003) 787–840

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BIOLOGIES

Neurosciences

Is there chaos in the brain? II. Experimental evidence and related models

Henri Korn *, Philippe Faure

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Received 16 September 2003; accepted 17 September 2003

Presented by Pierre Buser

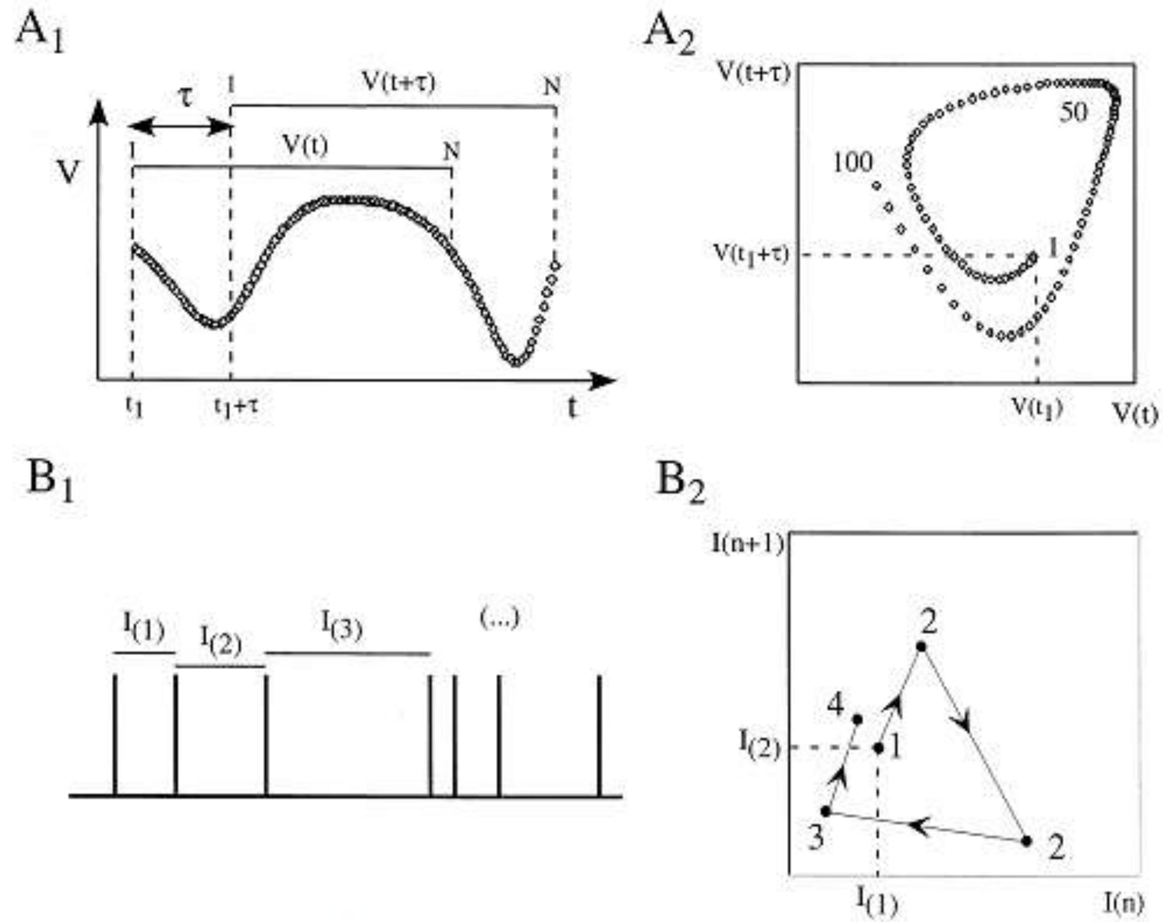


Figure 9. Reconstruction of phase spaces with the delay method. (A1-A2) Case of a continuous signal, as for example the recording of membrane potential, *V*. (A1) The time series is subdivided into two sequences of measurements of the same length N (here equal to 100 points). Their starting point is shifted by the time lag τ. (A2) The trajectory in a two dimensional phase space is obtained by plotting, for each point of the time series, V_t against $V_{t+\tau}$. (B1-B2) In the case of a discrete signal, such as time intervals between action potentials in a spike train (B1), the same procedure is applied to time intervals I_1, I_2...I_N (B2).

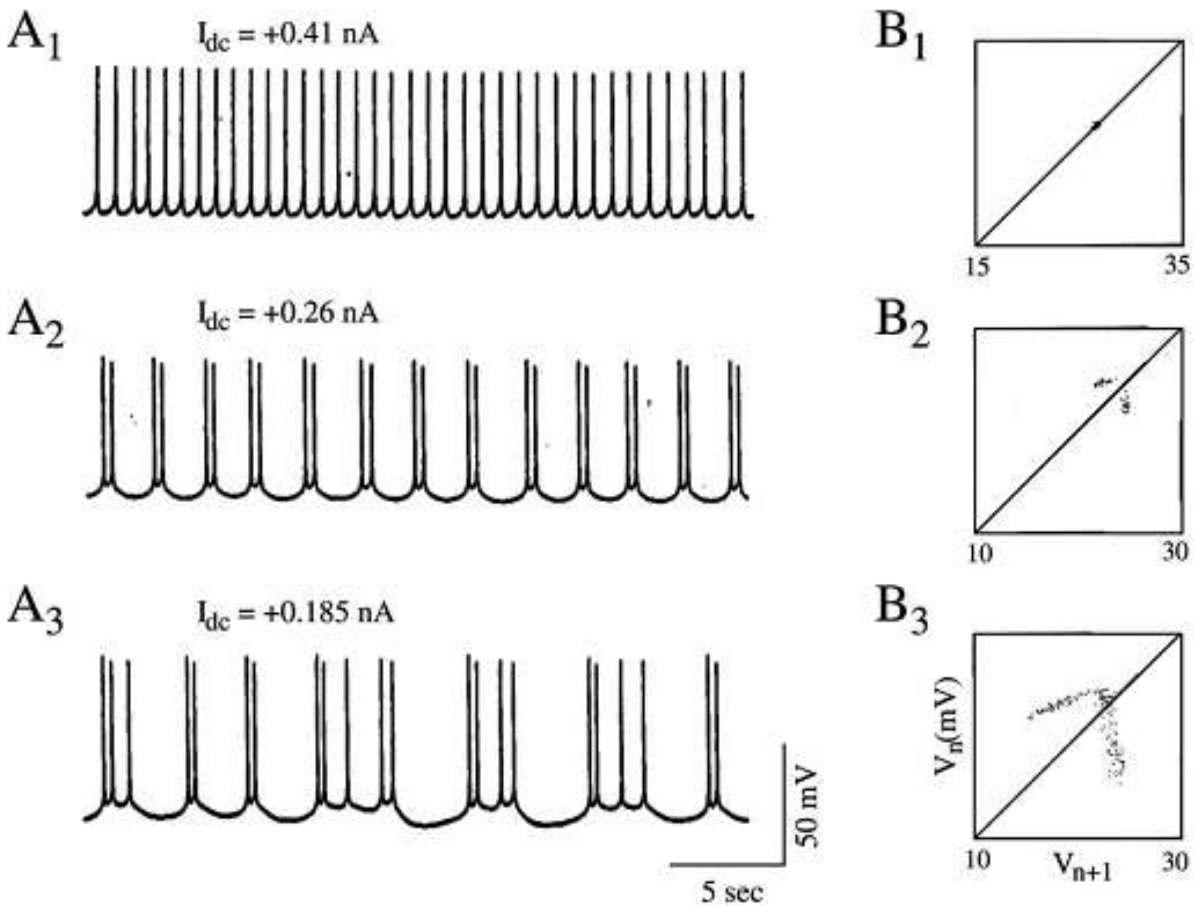
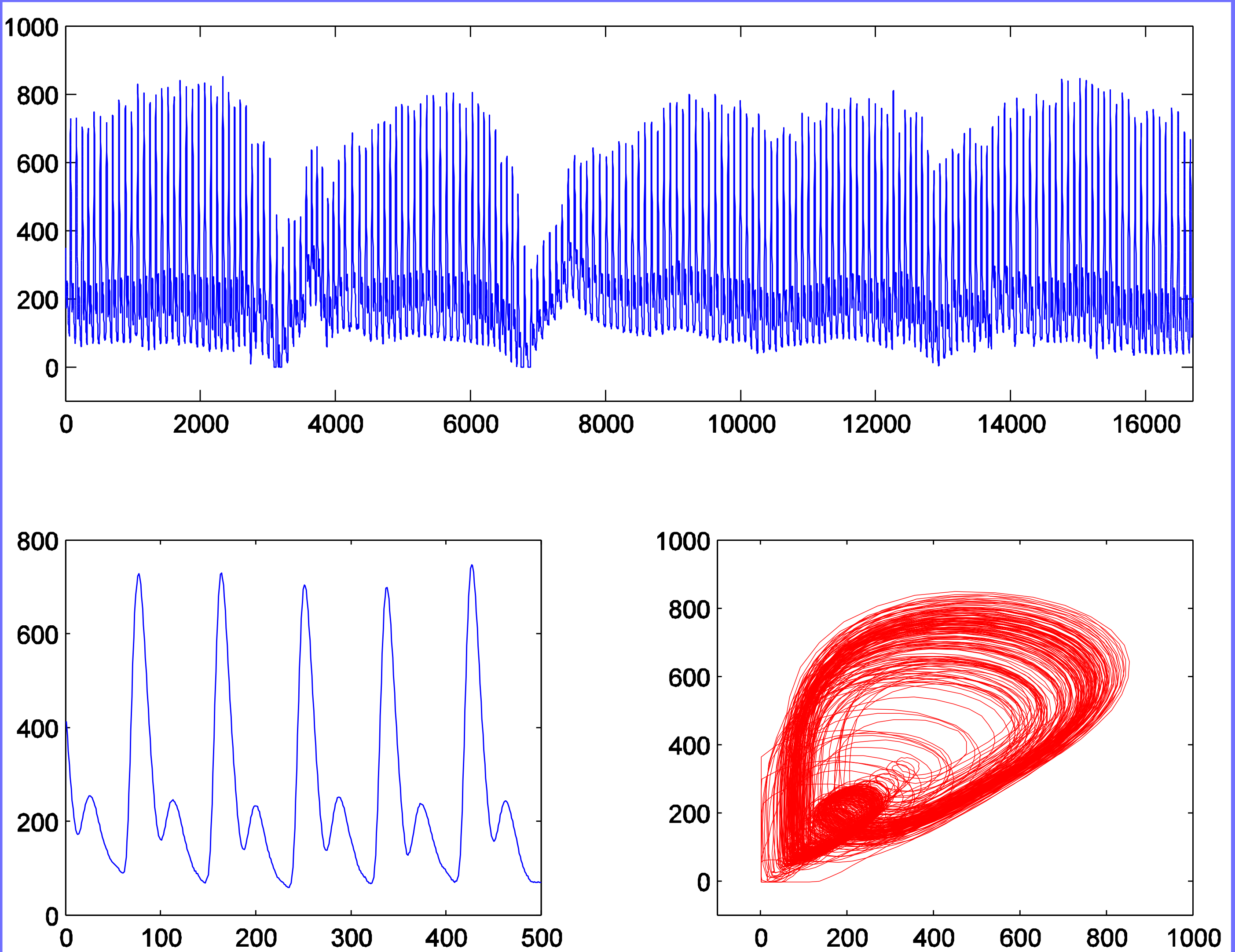


Fig. 5. Discharge patterns of a pacemaker neuron caused by a dc current (**A1–A3**) representative samples of the recorded membrane potential. (**B1–B3**) One-dimensional Poincaré maps of the corresponding sequence of spikes constructed using the delay method (see [1] for explanations). (**A1–B1**) Regular discharges of action potentials. (**A2–B2**) Periodic firing with two spikes per burst. (**A3–B3**) Chaotic bursting discharges. (Adapted from [45], with permission of the Journal of Theoretical Biology.)

Continuous Blood Pressure Waveform: Healthy Subject

http://www.viskom.oeaw.ac.at/~joy/March15,%202004.ppt

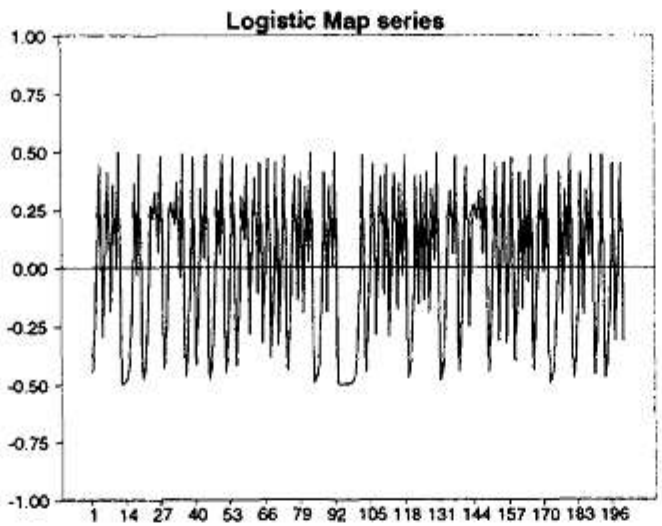


Randomness and the physical law

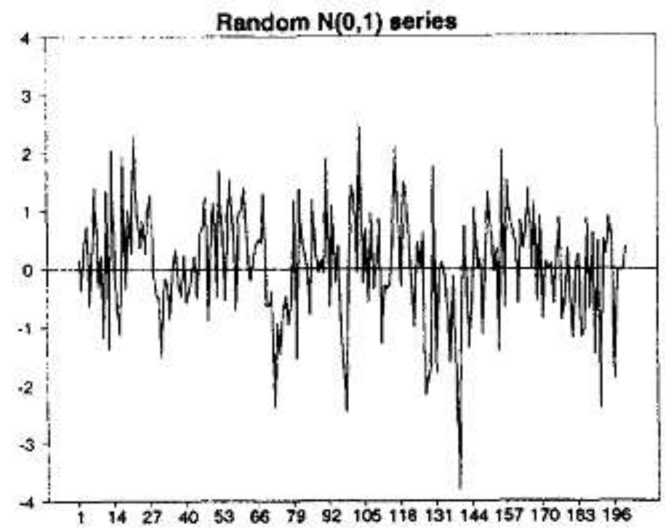
- It may well be that the universe itself is completely deterministic (though this depends on what the "true" laws of physics are, and also to some extent on certain ontological assumptions about reality), in which case randomness is simply a mathematical concept, modeled using such abstract mathematical objects as probability spaces. Nevertheless, the concept of *pseudorandomness*- objects which "behave" randomly in various statistical senses - still makes sense in a purely deterministic setting. A typical example are the digits of π=3.14159265358979323 85…this is a deterministic sequence of digits, but is widely believed to behave pseudorandomly in various precise senses (e.g. each digit should asymptotically appear 10% of the time). If a deterministic system exhibits a sufficient amount of pseudorandomness, then random mathematical models (e.g. statistical mechanics) can yield accurate predictions of reality, even if the underlying physics of that reality has no randomness in it.

http://terrytao.wordpress.com/2007/04/05/simons-lecture-i-structure-and-randomness-in-fourier-analysis-and-number-theory/

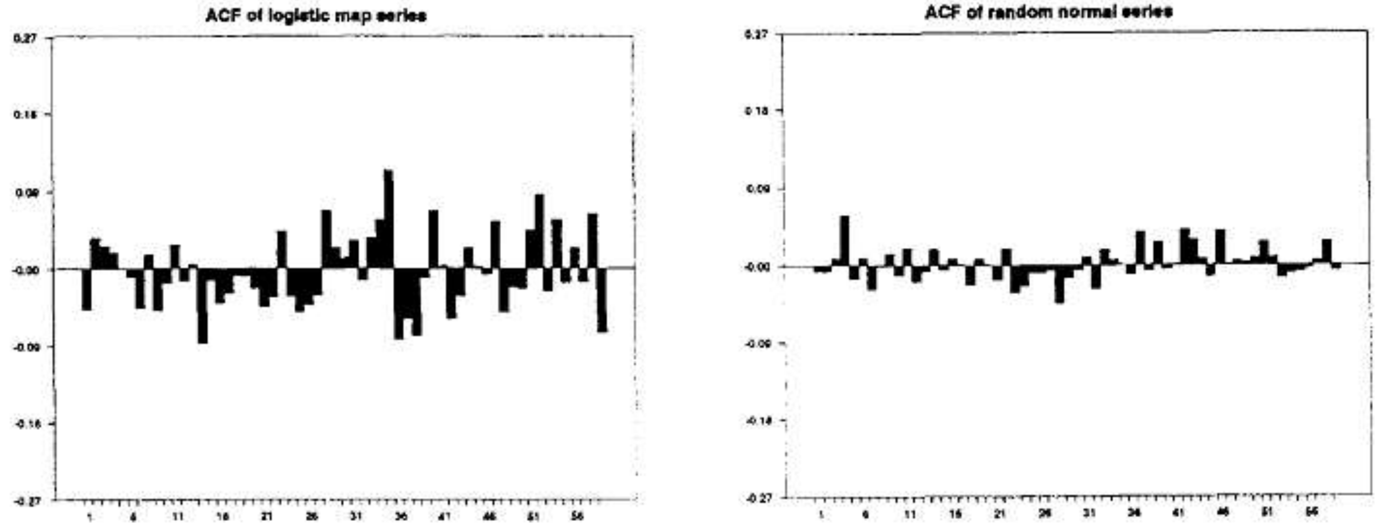
Logistic map series (adjusted with mean)



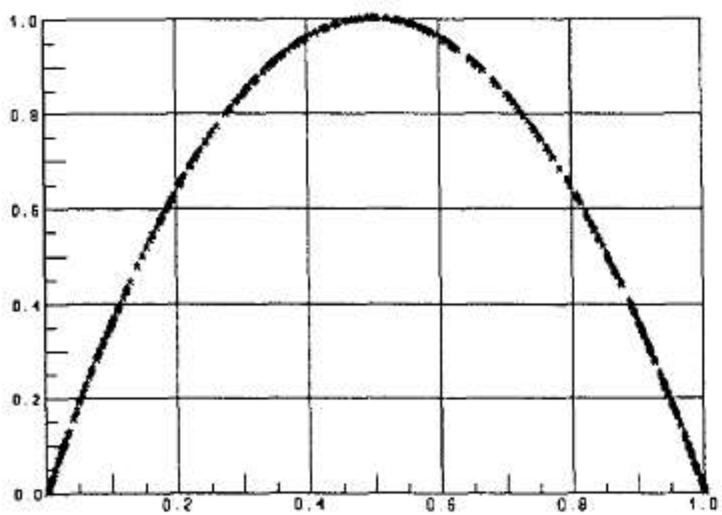
Random N(0,1) series



Autocorrelations



Logistic map series



Random N(0,1) series

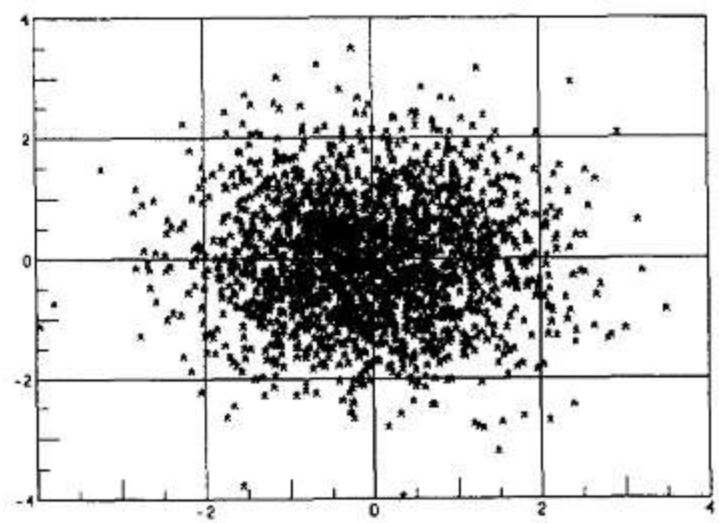


Fig. 2. Comparison of logistic map and random series.

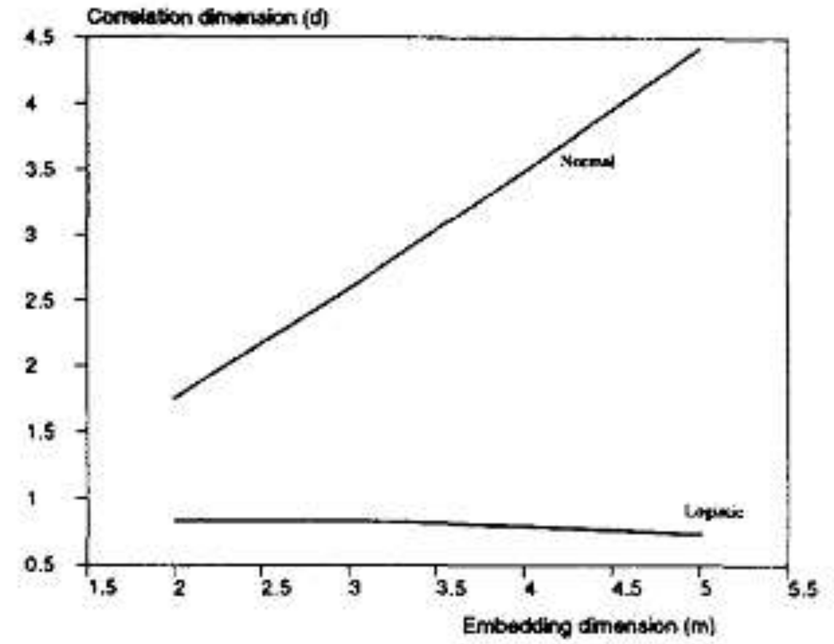
K. Takala, M. Virén / European Journal of Operational Research 93 (1996) 155–172

Embedding dimension = m

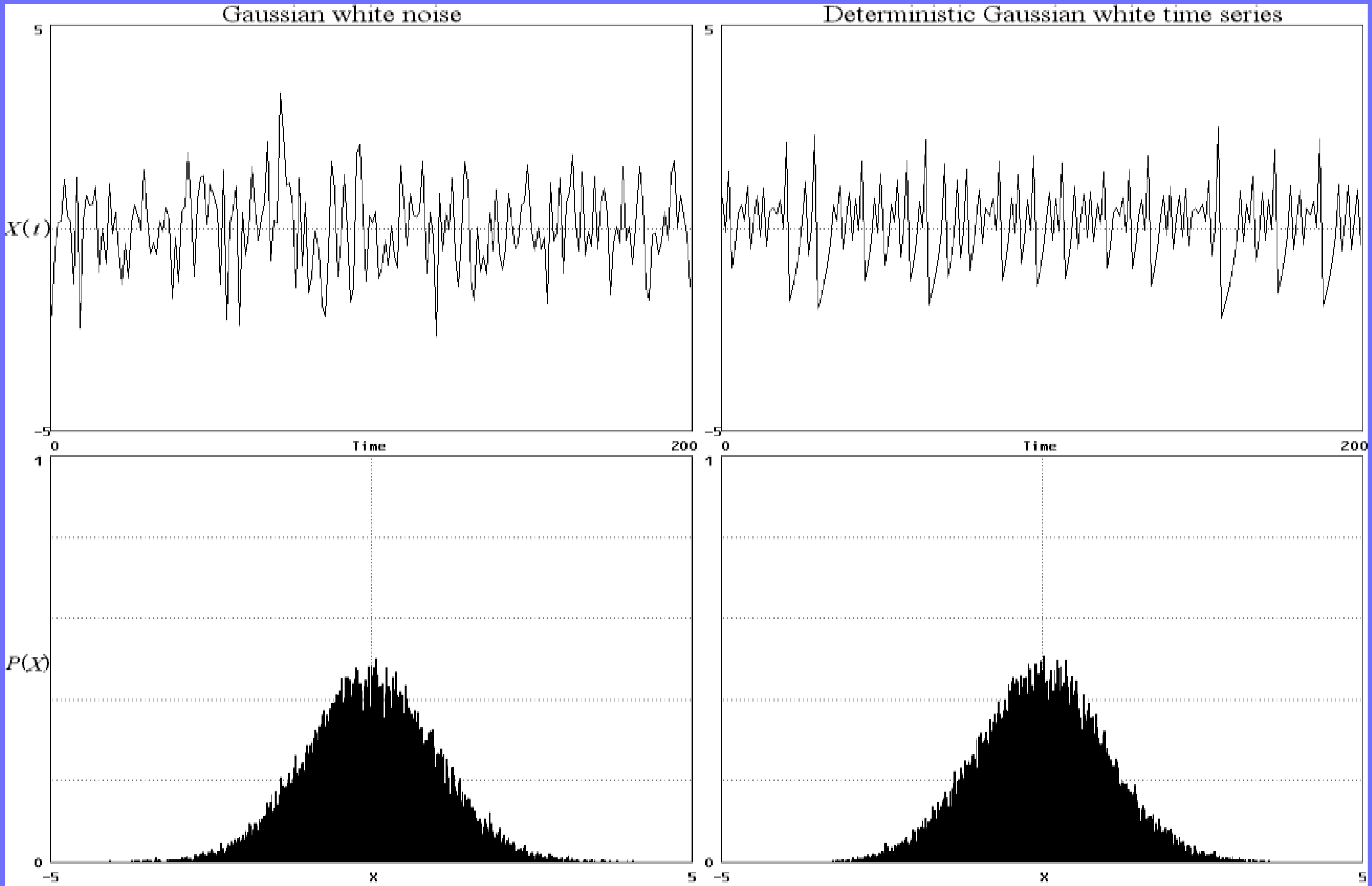
$$C_m(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \#\,(x_{m,i}, x_{m,j}), \left\| x_{m,i} - x_{m,j} \right\| < \varepsilon$$

$$d(m) = \lim_{\varepsilon\to 0} \frac{\log C_m(\varepsilon)}{\log(\varepsilon)}$$

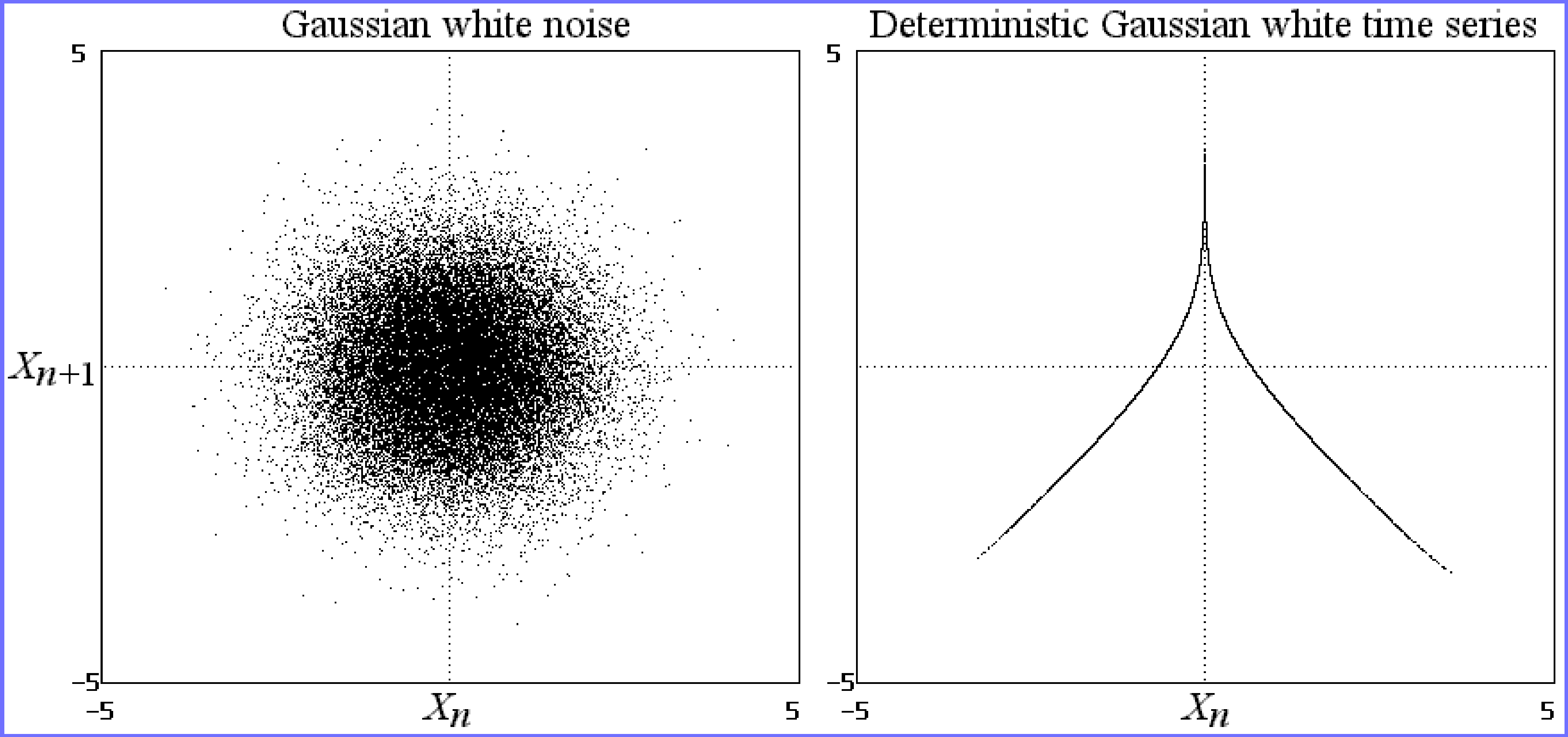
Correlation dimensions of logistic map and random normal processes



Deterministic or random? Appearance can be misleading…



Time delay map



Source: **sprott**.physics.wisc.edu/lectures/**tsa**.ppt

Logit and logistic

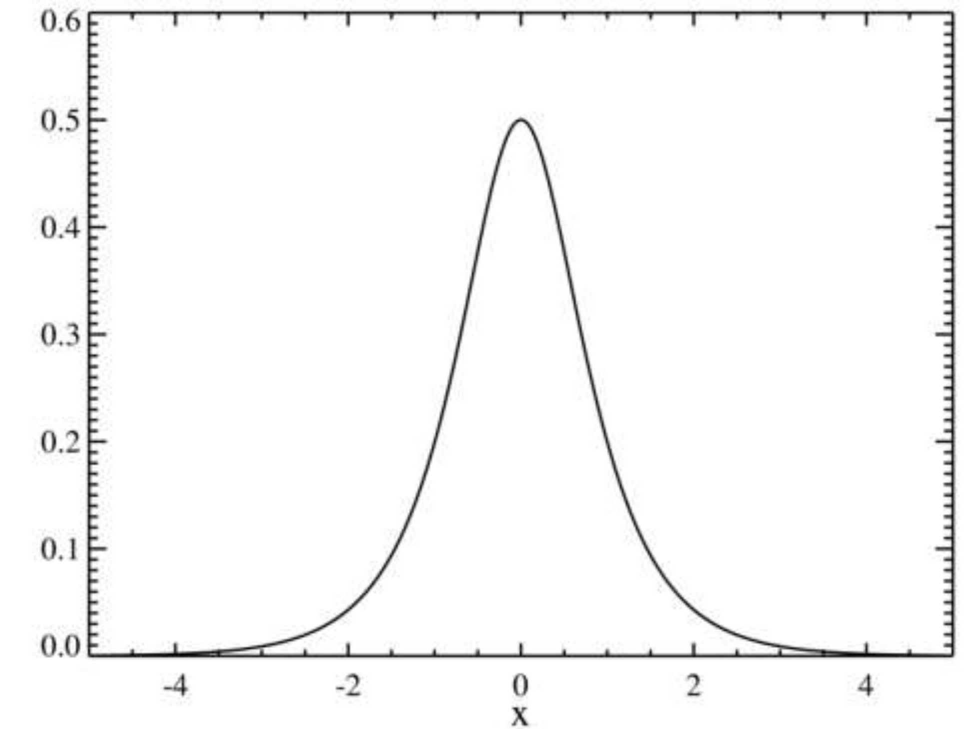
The logistic map $x \to L(x)=4x(1-x)$ preserves the probability measure $d\mu(x)=dx/(\pi\sqrt{x(1-x)})$

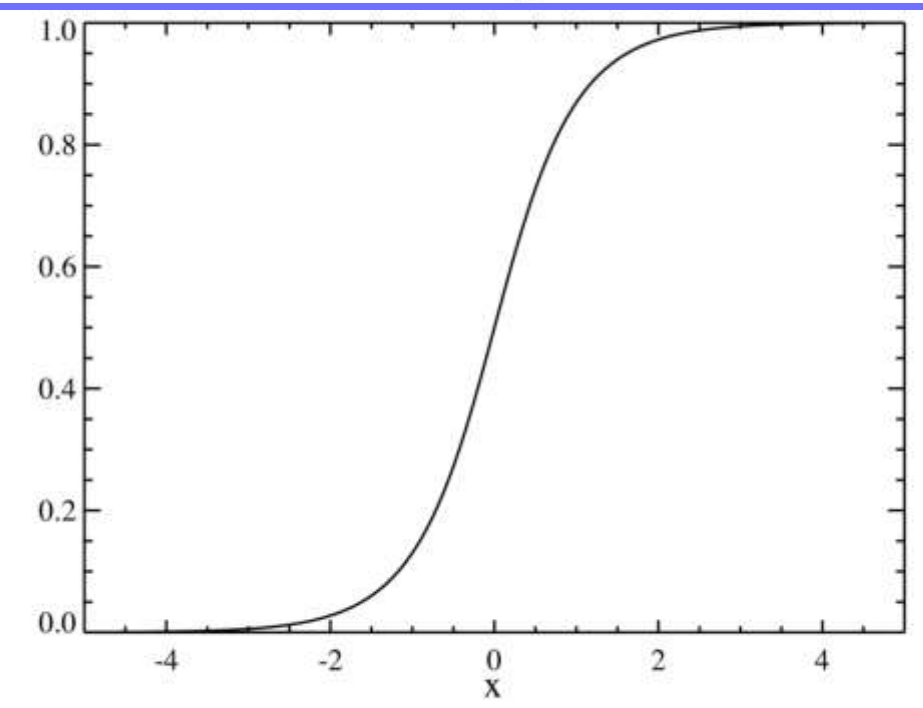
The transformation $h:[0,1] \to \mathbf{R}$, $h(x)=\ln x-\ln(1-x)$ conjugates L with a new map G

$$h \circ L = G \circ h$$

definined on **R.** The new invariant probability measure is $d\mu(x)=dx/[\pi(e^{x/2}+e^{-x/2})]$

G and L have the same dynamics (the only difference is a coordinates change)





Hyperbolic secant distribution

Source: wikipedia

Parameters	*none*
Support	$x \in (-\infty, +\infty)$
Probability density function (pdf)	$\tfrac{1}{2}\operatorname{sech}(\tfrac{1}{2}\pi x)$
Cumulative distribution function (cdf)	$\frac{2\arctan(\exp(\frac{1}{2}\pi x))}{\pi}$
Mean	0
Median	0
Mode	0
Variance	1
Skewness	0
Excess kurtosis	2
Entropy	$4/\pi\, G \approx 1.16624$

G = 0.915 965 594 177 219 015 054 603 514 932 384 110 774... Catalan's constant

Takens theorem

- $\phi : X \to X$ map, $f : X \to R$ smooth observable
- Time-delay map (reconstruction of the dynamics from periodic sampling):
- $F(f,\phi) : X \to R^n$ n is the number of delays
- $F(f,\phi)(x) = (f(x), f(\phi(x)), f(\phi\circ\phi(x)), \ldots, f(\phi^{n-1}(x)))$
- Under mild assumptions if the dynamics has an attractor with dimension k and n>2k then for almost any choice of the observable the reconstruction map is injective

Immersions and embeddings

- A smooth map F on a compact smooth manifold A is an immersion if the derivative map DF(x) (represented by the Jacobian matrix of F at x) is one-to-one at every point x∈A. Since DF(x) is a linear map, this is equivalent to DF(x) having full rank on the tangent space. This can happen whether or not F is one-to-one. Under an immersion, no differential structure is lost in going from A to F(A).
- An embedding of A is a smooth diffeomorphism from A onto its image F(A), that is, a smooth one-to-one map which has a smooth inverse. For a compact manifold A, the map F is an embedding if and only if ,F is a one- to-one immersion.
- The set of embeddings is open in the set of smooth maps: arbitrarily small perturbations of an embedding will still be embeddings!

Embedology (Sauer, Yorke, Casdagli, J. Stat. Phys. 65 (1991)

Whitney showed that a generic smooth map ,F from a d-dimensional smooth compact manifold M to R^n , n>2d is actually a diffeomorphism on M. That is, M and F(M) are diffeomorphic. We generalize this in two ways:

- first, by replacing "generic" with "probability-one" (in a prescribed sense),
- second, by replacing the manifold M by a compact invariant set A

contained in some Rk that may have noninteger box-counting dimension (boxdim). In that case, we show that almost every smooth map from a neighborhood of A to R^n is one-to-one as long as n>2 * boxdim(A)

We also show that almost every smooth map is an embedding on compact subsets of smooth manifolds within I. This suggests that embedding techniques can be used to compute positive Lyapunov exponents (but not necessarily negative Lyapunov exponents). The positive Lyapunov exponents are usually carried by smooth unstable manifolds on attractors.

Embedology (Sauer, Yorke, Casdagli, J. Stat. Phys. 65 (1991)

Takens dealt with a restricted class of maps called delay-coordinate maps: these are time series of a single observed quantity from an experiment. He showed (F. Takens, Detecting strange attractors in turbulence, in Lecture Notes in Mathematics, No. 898 (Springer-Verlag, 1981) that if the dynamical system and the observed quantity are generic, then the delay-coordinate map from a d-dimensional smooth compact manifold M to R^n , n>2d is a diffeomorphism on M.

- we replace generic with probability-one
- and the manifold M by a possibly fractal set.

Thus, for a compact invariant subset A under mild conditions on the dynamical system, almost every delay-coordinate map to R^n is one-to-one on A provided that n>2.boxdim(A). Also, any manifold structure within I will be preserved in F(A).

- Only C^1 smoothness is needed.;
- For flows, the delay must be chosen so that there are no periodic orbits with period exactly equal to the time delay used or twice the delay

Embedding method

- Plot $x(t)$ vs. $x(t-\tau)$, $x(t-2\tau)$, $x(t-3\tau)$, ...
- $x(t)$ can be any observable
- The embedding dimension is the # of delays
- The choice of τ and of the dimension are critical
- For a typical deterministic system, the orbit will be diffeomorphic to the attractor of the system (Takens theorem)

Choice of Embedding Parameters

Theoretically, a time delay coordinate map yields an valid embedding for any sufficiently large embedding dimension and for any time delay when the data are noise free and measured with infinite precision.

But, there are several problems:

(i) Data are not clean
(ii) Large embedding dimension are computationally expensive and unstable
(iii) Finite precision induces noise

Effectively, the solution is to search for:

(i) Optimal time delay τ
(ii) Minimum embedding dimension d
or
(i) Optimal time window τ_w

There is no one unique method solving all problems and neither there is an unique set of embedding parameters appropriate for all purposes.

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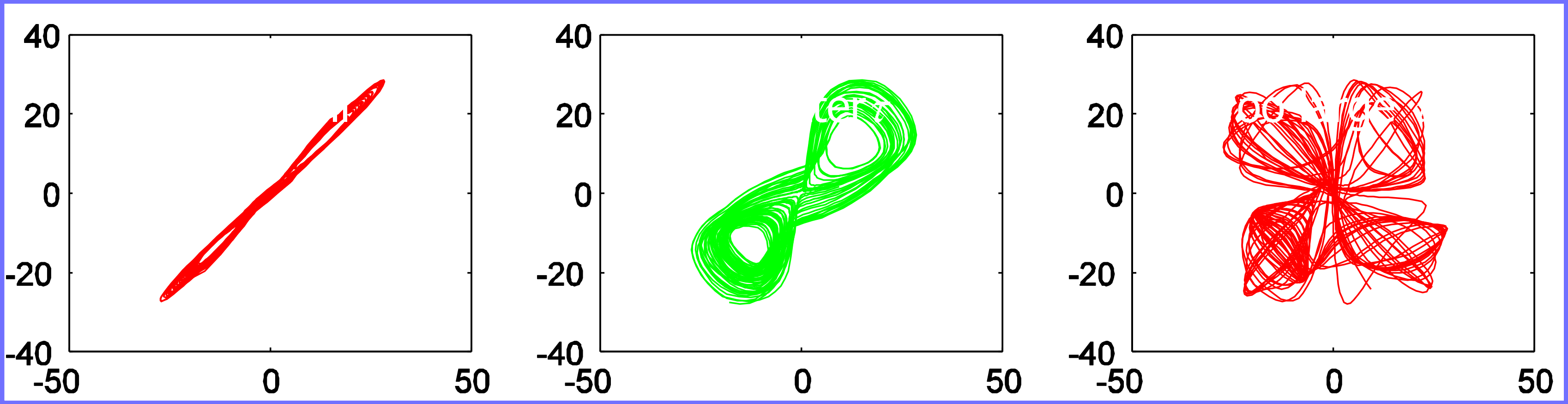
The Role of Time Delay τ

If τ is too small, $x(t)$ and $x(t-\tau)$ will be very close, then each reconstructed vector will consist of almost equal components → *Redundancy (τ_R)*

The reconstructed state space will collapse into the main diagonal

If τ is too large, $x(t)$ and $x(t-\tau)$ will be completely unrelated, then each reconstructed vector will consist of irrelevant components → *Irrelevance (τ_I)*

The reconstructed state space will fill the entire state space.



http://www.viskom.oeaw.ac.at/~joy/March22,%202004.ppt

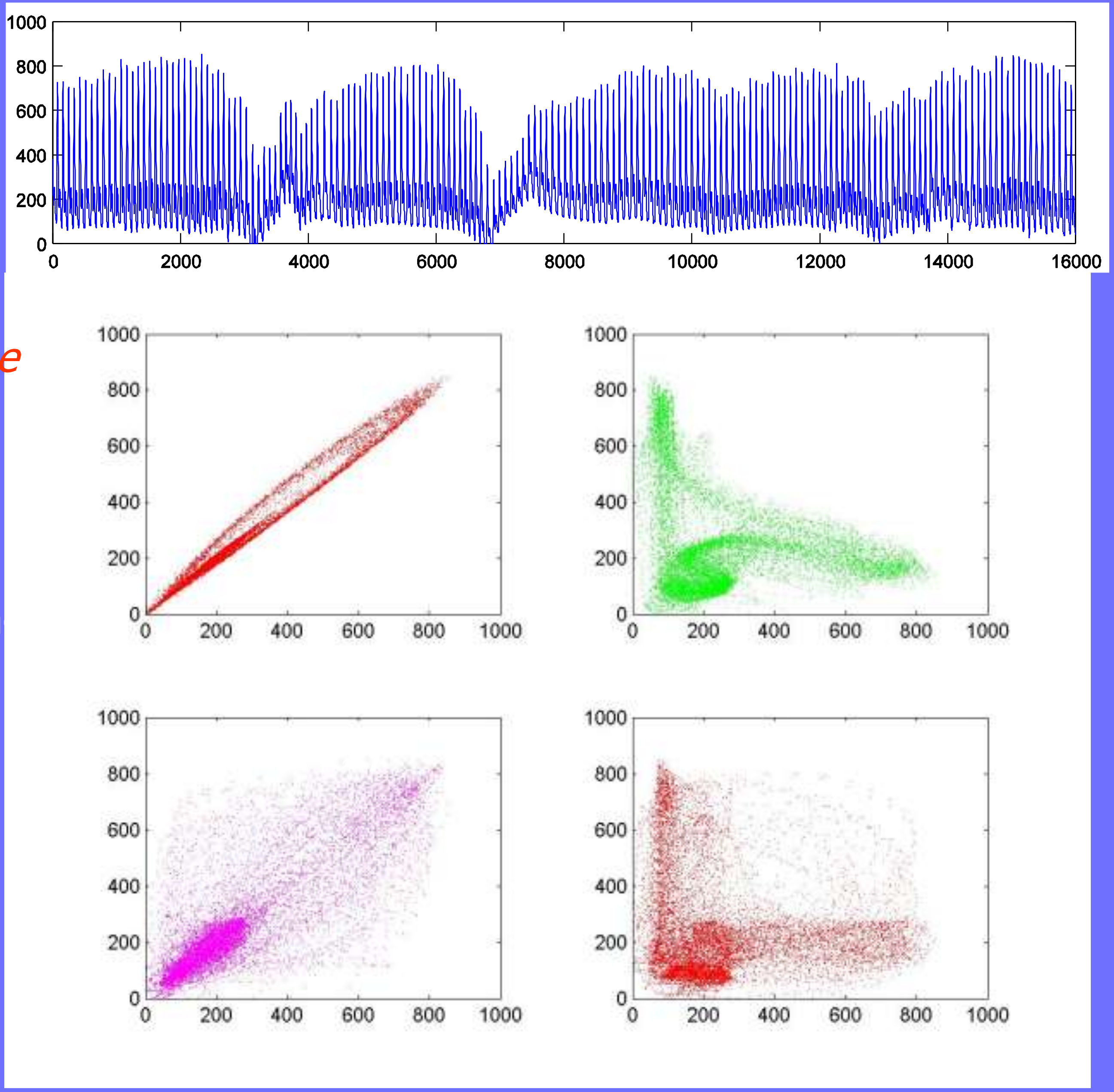
A better choice is:

$\tau_R < \tau_W < \tau_I$

Caution: τ should not be close to main period

Collapsing of state spa

Blood Pressure Signal



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Some Recipes to Choose τ

Based on Autocorrelation

Estimate autocorrelation function: $$C(\tau) = \frac{1}{N-\tau-1}\sum_{t=0}^{N-\tau-1} x(t)x(t+\tau) = \langle x(t)x(t+\tau)\rangle$$

Then, $\tau_{opt} \approx C(0)/e$

or

first zero crossing of $C(\tau)$

Modifications:

1. Consider minima of higher order autocorrelation functions, $<x(\tau)x(t+\tau)x(t+2\tau)>$ and then look for time when these minima for various orders coincide.

Albano et al. (1991) *Physica D*

2. Apply nonlinear autocorrelation functions: $<x^2(\tau)x^2(t+2\tau)>$

Billings, Tao (1991) *Int. J. Control.*

http://www.viskom.oeaw.ac.at/~joy/March22,%202004.ppt

Based on Time delayed Mutual Information

The information we have about the value of $x(t+\tau)$ if we know $x(t)$.

1. Generate the histogram for the probability distribution of the signal $x(t)$.
2. Let p_i is the probability that the signal will be inside the i-th bin and $p_{ij}(t)$ is the probability that $x(t)$ is in i-th bin and $x(t+\tau)$ is in j-th bin.
3. Then the mutual information for delay τ will be

$$I(\tau) = \sum_{i,j} p_{ij}(\tau) \log p_{ij}(\tau) - 2\sum_{i} p_i \log p_i$$

For $\tau \rightarrow 0$, $I(\tau) \rightarrow$ Shannon's Entropy

$\tau_{opt} \approx$ First minimum of $I(\tau)$

http://www.viskom.oeaw.ac.at/~joy/March22,%202004.ppt

Statistical analysis of a time series: moments of the probability distribution

mean	$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$
variance	$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$
standard deviation	σ
skewness	$\zeta = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{x_i - \bar{x}}{\sigma}\right)^3$
kurtosis	$\kappa = -3 + \frac{1}{N} \sum_{i=1}^{N} \left(\frac{x_i - \bar{x}}{\sigma}\right)^4$

Higher moments: simmetry of the distribution and fat tails

- Skewness: measures simmetry of the data about the mean (third moment)
- Kurtosis: peakedness of the distribution relative to the normal distribution (hence the -3 term)
- Leptokurtic distribution (fat tailed): has positive kurtosis

ψ,φ observables with expectations $\mu(\psi)$ and $\mu(\varphi)$

$\sigma(\psi)^2 = [\,(\mu(\psi^2) - \mu(\psi)^2]$ **variance**

The **correlation coefficient** of ψ,φ is

$$\rho(\psi,\varphi) = \text{covariance}(\psi,\varphi) \,/\, (\sigma(\psi)\,\sigma(\varphi))$$
$$= \mu\,[(\psi - \mu(\psi))(\varphi - \mu\,(\varphi))] \,/\, (\sigma(\psi)\,\sigma(\varphi))$$
$$= \mu\,[\psi\,\varphi - \mu(\psi)\mu\,(\varphi)] \,/\, (\sigma(\psi)\,\sigma(\varphi))$$

The correlation coefficient varies between -1 and 1 and equals 0 for independent variables but this is only a necessary condition (e.g. φ uniform on [-1,1] has zero correlation with its square)

Sample correlation coefficient between two finite series of data

$$\{x_i\} \ \text{for} \ \ i = 1, \ldots, N \qquad \{y_i\} \ \text{for} \ \ i = 1, \ldots, N$$

$$\text{for} \ \ \bar{x} \neq 0, \ \ \sigma_x^2 \neq 1, \ \ \bar{y} \neq 0, \ \ \sigma_y^2 \neq 1$$

$$r_{xy} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N} (x_i - \bar{x})^2 \cdot \sum_{i=1}^{N} (y_i - \bar{y})^2}}$$

$$\text{for} \ \ \bar{x} = 0, \ \ \sigma_x^2 = 1, \ \ \bar{y} = 0, \ \ \sigma_y^2 = 1$$

$$r_{xy} = \frac{1}{N-1} \sum_{i=1}^{N} x_i y_i$$

Autocorrelation function

$$\{x_i\} \text{ for } i = 1, \ldots, N \text{ with } \bar{x} = 0 \text{ and } \sigma^2 = 1$$

$$C_{xx}(\tau) = \begin{cases} \frac{1}{N-\tau} \sum_{n=1}^{N-\tau} x_{n+\tau} x_n & \tau \geq 0 \\ C_{xx}(-\tau) & \tau < 0 \end{cases}$$

$$\tau = -(N-1), \ldots, N-1$$

Decay time of autocorrelation

$$\tau_d = \min\{\tau : C_{xx}(\tau) < \frac{1}{e}\}$$

This is an important indicator of the strength of the autocorrelation of time series

It can be used to determine the time delay in embedology

Stationarity

- Stationarity: all parameters of the data series statistical distribution must be time-independent
- Weak-stationarity: we only require that the first two moments (mean and variance) are constant
- Parameters can for example be moments of the probability distribution, but also coefficients in differential equations or autoregressive processes.

Tests of stationarity

- Moving window analysis: Divide a long time series in shorter windows and analyze these short windows separately.
- For example split the series into two parts, compute mean and variance and compare (remember that the standard error will be $\sigma/\sqrt{N}$)

Financial time series: standard deviation and volatility

If the daily logarithmic returns of a stock have a standard deviation of 0.01 and there are 252 trading days in a year, then the time period of returns is 1/252 and annualized volatility is

$$\sigma = \frac{0.01}{\sqrt{1/252}} = 0.1587$$

The formula used to annualize returns is not deterministic, but is an extrapolation valid for a random walk process whose steps have finite variance. Generally, the relation between volatility in different time scales is more complicated, involving the Lévy stability exponent α:

$$\sigma_T = T^{\frac{1}{\alpha}}\sigma$$

α = 2 you get the Wiener process scaling relation, but some people believe α< 2 for financial activities such as stocks, indexes and so on. This was discovered byBenoît Mandelbrot, who looked at cotton prices and found that they followed a Lévy alpha-stable distribution with α = 1.7. Mandelbrot's conclusion is, however, not accepted by mainstream financial econometricians.

In econometrics, an **autoregressive conditional heteroscedasticity** (**ARCH**, Engle (1982)) model considers the variance of the current error term to be a function of the variances of the previous time period's error terms. ARCH relates the error variance to the square of a previous period's error. It is employed commonly in modeling financial time series that exhibit time-varying volatility clustering, i.e. periods of swings followed by periods of relative calm.