A Berry-type paradox

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We want to present and make known a statement of Berry paradox which has been ignored – buried in the graveyard of dead languages – and which is due to Beppo Levi, in 1908, independently from Russell. Berry paradox should actually be called Russell's paradox, as Chaitin has observed, in [Chaitin 1999, pp. 8-9], on the basis of Alejandro Garciadiego's findings.

Beppo Levi's version is much more modern than Russell's informal one, since it is cast in arithmetical terms, with the appropriate numerical computations, and so it is ready for use when supplemented with the ideas which will transform the epistemological paradoxes in positive arguments in the theory of undecidable problems, as foreseen by Gödel and as realized in [Chaitin 1995] or [Boolos 1989].

Beppo Levi doesn't mention Berry paradox and he doesn't quote the paper [Russell 1906], which he quite certainly did not know; his only references are to [Russell 1903]. However he gives a mathematically rigorous version of the paradox in the course of a rather lengthy analysis of Richard's antinomy, where he follows and improves Peano's discussion of the latter.

1. Levi did not belong to Peano's school, although he graduated in Torino in 1896. At the beginning of his career he worked in set theory, starting out with the perusal of Baire's thesis, then reverting to measure and category theory of the line after a failed attack at the continuum hypothesis. His name in mentioned in connection with the history of the axiom of choice, as he seems to have been the first to recognize (and formulate) the principle of partition in Berstein's proof that there are continuum many closed sets, and to give a new proof avoiding choice. Azriel Levy for example has as reference for the axiom of choice "(Beppo Levi 1902; Erhard Schmidt 1904 - see Zermelo 1904)", in [Lévy 1979, p. 159], but Zermelo rightly only attributed

to him the principle of partition. Levi's precise contributions are spelled out in [Moore 1982, pp. 78-80] and [Lolli 1999].

Further work by Beppo Levi in logic concerned mainly improvements and criticism of Peano's logic, and has only historical interest. As a mathematician he is best remembered for results in analysis related to Lebesgue theory, such as the theorem of the passage to limit under integral sign – equivalent to the later Lebesgue's dominated convergence - and in number theory, as explained in [Schappacher, Schoof 1996].

Beppo Levi had a crystal clear conception of the new axiomatic method as it had been recognized at the end of the nineteenth century; to him, on a par with Pasch, Hilbert, Enriques, Pieri, are due some of the most neat explications of the nature of mathematical theories, of the impossibility for primitive notions to be completely determined and of the unavoidability of multiple interpretations.

He relied on these principles also in the analysis of logical antinomies he gave in [Levi 1908].

2. In discussing Richard's antinomy Levi denotes by E the set of real numbers definable with a finite number of words, and by R an enumeration of E.

He concludes that E does not exist, but the reason he gives is different from that of Poincaré and Russell imputing the antinomy to the impredicative character of the definition. He also notices the circularity of the definition of E, which to him, from his axiomatic point of view, means that E ed R are not defined and independent; they are primitive ideas subjected as a whole to some postulates, which he tries to identify:

1. E is a set of numbers comprised between 0 and 1;

2. R is an order of the numbers of E;

3. All numbers of E and only they can be expressed with a finite number of symbols, including E and R.

Richard's diagonal number N proves only that the postulates imposed on E and R are non mutually consistent.

Levi does not pinpoint the source of the contradiction, observing that "when a set of postulates is inconsistent, the contradiction lies in their union, not in any single one of them".

He dwells however on the ordering R, which is naturally thought of as obtained by a lexicographic ordering of the set F of all statements by pruning it of those which do not define a number or define a number previously listed. Let us call it briefly the lexicographic order.

Here comes the paradox (we had to change the counting with respect to Levi's because of the different length of sentences in Italian and in English).

> Let us call B, as we did, the number of symbols needed to compose our statements: among these symbols we'll assume to be included, for simplicity, the symbol for exponentiation $\upharpoonleft$ ("to the"). It is easy to see that $B > 40$: suppose we write it in the decimal system, and let β be the number of its digits (we can reasonably assume $\beta = 2$). Now consider the proposition
>
> $$\ll \text{The number whose place in } R \text{ is } B \upharpoonleft B \gg ;$$
>
> this proposition contains $32 + 2\beta$ symbols[1]; its place in the enumeration of F is therefore $< B^{32+2\beta} < B \upharpoonleft B$; if numbers of E are listed in the order described in n. 8 [by enumerating and pruning the members of F], the above defined number must have a place $< B \upharpoonleft B$. *Hence that order cannot be R.*
>
> It is obvious that similar contradictions can be concocted for many other definitions of orderings one can imagine substituted to that of n. 8 [the lexicographic one]. Richard's contradiction presents itself for any way R can be thought to be defined.

We give also the italian original text for the historian's sake[2]:

> Si chiami B, come pocanzi, il numero dei segni che servono a comporre le nostre frasi: fra questi segni supporremo compreso, per comodità, il segno $\upharpoonleft$ (elevato a). Si vede facilmente che $B > 40$: lo supporremo scritto in cifre nel sistema decimale, e chiameremo β il numero di queste cifre (presumibilmente $\beta = 2$). Si consideri allora la proposizione

[1] [Blank spaces count as occurrences of a symbol for Levi.]

[2] [Levi 1908, p. 638].

$\ll$ Il numero di posto $B \upharpoonright B$ in $R \gg$;

> essa consta di $25 + 2\beta$ segni; il suo posto nell'ordinamento di F è quindi $< B^{25+2\beta} < B \upharpoonright B$; se dunque i numeri di E si numerano come si disse nel n. 8, il numero considerato dovrà avere posto $< B \upharpoonright B$. *Quella numerazione non può dunque essere R.*
>
> È chiaro che simili contraddizioni si possono costruire per molte altre definizioni di numerazioni che si vogliano immaginare sostituite a quella del n. 8. La contraddizione del Richard si presenta comunque R si voglia immaginar determinata.

Levi is unaware that his argument can be made to stand independently of Richard's setting in the real numbers and of diagonalization, to which it offers an alternative; he could have noticed, with his axiomatic sensibility, that no mention is made of the nature of the definable entities, and that he could as well refer to the definability of natural numbers.

The general phenomenon Levi ran into is that of descriptions which are shorter than what they describe. What is missing of course, to arrive at an instance of the incompleteness phenomenon, is the idea of proving in a formal system which is the place of the number defined as "the number whose place is $B \upharpoonright B$". Since this description is shorter than $B \upharpoonright B$, the number defined should occupy a place $< B \upharpoonright B$, so no place can be proved to exist for it.

What Beppo Levi instead argues is the following, as can be inferred by his last vague comment: think of the order R as the lexicographic order, then the number "whose place is $B \upharpoonright B$" shows that the order cannot be that one. But think of any other ordering, and a similar argument will show that the order cannot be the one you thought. So there is none, which is consistent with E.

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