Models for Many-Valued Probabilistic Reasoning

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Abstract
In this article, we compare models for many-valued probabilistic reasoning from the point of view of the sets of satisfiable formulas, positive satisfiable formulas, and tautologies. The results arising from this comparison will be used in the final part of the present article to provide results about the computational complexity for the problem of deciding if a formula belongs to one of the previously discussed sets.

Keywords: SMV-algebras, states on MV-algebras, probabilistic Kripke models, PSPACE containment.

1 Introduction

In his book [9], Hájek introduces a discussion about the connections between fuzzy logic and probability. He points out several differences: for instance, fuzzy logic is a logic for the treatment of vagueness, whereas probabilistic logic is a logic for the treatment of uncertainty. Moreover, the most prominent fuzzy logics are truth functional, whereas probabilistic logic is not. Nevertheless, Hájek shows that probabilistic logic can be treated inside fuzzy logic, by means of a modality Pr to be interpreted as probably. Then, Flaminio and Godo [6], introduce a logic, called $FP(Ł,Ł)$, in which it is possible to express the probability of fuzzy events. This logic is not algebraizable in the sense of Blok and Pigozzi (cf. [1]), and its semantics is not given by means of universal algebras but rather by means of Kripke models.

On the algebraic side, Mundici in [13] introduces states on MV-algebras. These operators can be interpreted as probabilistic measures over many-valued formulas. Moreover, as shown by Mundici in [15], states are also related to subjective probability. Indeed, he shows that a probabilistic assessment over many-valued events can be extended to a state iff there is no Dutch-book for it (cf. Section 2.3 for details).

Thus, both states and probabilistic fuzzy logics have to do with probability over fuzzy events. However, so far these two approaches to probability have been investigated separately. In [7, 8], the present authors try to connect them. To this purpose, they introduce an algebraizable logic, called $SFP(Ł,Ł)$, which extends $FP(Ł,Ł)$ (see Section 2.1), and whose equivalent algebraic semantics is constituted by a variety of universal algebras, called SMV-algebras. These algebras are MV-algebras equipped with a unary operation whose properties somehow simulate the properties of states.

The purpose of the present article is to connect the semantics of MV-algebras with a state to the semantics given by Kripke models, showing that both semantics are equivalent to a semantics given by a special kind of SMV-algebras, called $σ$-simple SMV-algebras. As a consequence, we establish some complexity theoretic results for the classes of $SFP(Ł,Ł)$-formulas which are 1-tautologies or 1-satisfiable (with respect to various kinds of semantics).
The article is organized as follows. In the next section, we recall the syntax of \( SFP(Ł,Ł) \) and we point out the improvements this logic brings, with respect to the weaker logic \( FP(Ł,Ł) \).

In Section 2, we recall some preliminary definitions about MV-algebras with a state and SMV-algebras, and we prove some properties of tensor product of MV-algebras.

In Section 3, we introduce various kinds of semantics for \( SFP(Ł,Ł) \), namely, the SMV semantics given by general SMV-algebras, the standard semantics given by the so called \( σ \)-simple SMV-algebras, the state semantics, given by the states on the Lindenbaum algebra of Łukasiewicz logic, and the Kripke semantics, given by Kripke models. For each kind of semantics, we investigate the set of 1-tautologies and the set of 1-satisfiable formulas. For the 1-tautologicity, we prove that the standard semantics, the state semantics and the Kripke semantics are mutually equivalent, and we do not know whether these semantics are equivalent to the SMV semantics or not. For the 1-satisfiability, the situation is more complex: we have two (non-equivalent) kinds of 1-satisfiability, a global one and a local one. The local one is called local 1-satisfiability, and the global one is called just 1-satisfiability. For 1-satisfiability, we prove that both the Kripke semantics and the standard semantics are equivalent to the SMV semantics, but not to the state semantics; for the local 1-satisfiability, we prove that the state semantics, the Kripke semantics and the standard semantics are mutually equivalent, and we do not know whether they are also equivalent to the SMV semantics or not.

Then, in Section 4, we establish some complexity results. In particular, we prove that the set of standard 1-tautologies with respect to either Kripke semantics or the standard semantics or the state semantics, as well as the set of 1-satisfiable, and locally 1-satisfiable formulas with respect to any of the above semantics, are in PSPACE.

## 2 Preliminaries

### 2.1 The logic \( SFP(Ł,Ł) \)

The language of \( SFP(Ł,Ł) \) consists of a set \( \text{Var} = \{p_1, p_2, \ldots \} \) of propositional variables, the constant \( 0 \) (for the falsity), the binary connective \( \oplus \), the unary connective \( ¬ \) and the unary modality \( \text{Pr} \).

Formulas are defined by induction as usual: every propositional variable is a formula, 0 is a formula and, whenever \( \varphi \) and \( \psi \) are formulas, then \( \varphi \oplus \psi \), \( ¬\varphi \) and \( \text{Pr}(\varphi) \) are formulas. The following connectives are definable:

\[
\begin{align*}
\varphi \odot \psi & \quad \text{is} \quad ¬(¬\varphi \oplus ¬\psi) \\
\varphi \rightarrow \psi & \quad \text{is} \quad ¬\varphi \oplus \psi \\
\varphi \leftrightarrow \psi & \quad \text{is} \quad (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi) \\
\varphi \oplus \psi & \quad \text{is} \quad ¬(\varphi \rightarrow \psi) \\
\varphi \lor \psi & \quad \text{is} \quad (\varphi \rightarrow \psi) \rightarrow \psi \\
\varphi \land \psi & \quad \text{is} \quad ¬(¬\varphi \lor ¬\psi)
\end{align*}
\]

We shall henceforth denote by \( SFP \) the set of formulas of \( SFP(Ł,Ł) \).

Axioms are those of Łukasiewicz logic (cf. [9]) and the following schemata for the modality \( \text{Pr} \):

\[
\begin{align*}
(1) \quad & \text{Pr}(0) \leftrightarrow 0 \\
(2) \quad & \text{Pr}(\text{Pr}(\varphi) \oplus \text{Pr}(\psi)) \leftrightarrow \text{Pr}(\varphi) \oplus \text{Pr}(\psi) \\
(3) \quad & \text{Pr}(\varphi \rightarrow \psi) \rightarrow (\text{Pr}(\varphi) \rightarrow \text{Pr}(\psi)) \\
(4) \quad & \text{Pr}(\varphi \odot \psi) \leftrightarrow [\text{Pr}(\varphi) \oplus \text{Pr}(\psi \odot (\varphi \odot \psi))]^{1}
\end{align*}
\]

\[1\text{In [8] the present authors gave a different axiomatization for } SFP(Ł,Ł) \text{: the axiom 3 was substituted for } \text{Pr}(¬\varphi) \leftrightarrow ¬\text{Pr}(\varphi). \text{ The two axiomatizations are equivalent, but this one is preferable because it allows to derive the rule of substitution of equivalents for } \text{Pr} \text{ [from } \varphi \leftrightarrow \psi, \text{ derive } \text{Pr}(\varphi) \leftrightarrow \text{Pr}(\psi) \text{] in an easier way.}\]
Rules are modus ponens: from \( \varphi \) and \( \varphi \rightarrow \psi \), derive \( \psi \), and necessitation: from \( \varphi \), derive \( \Pr(\varphi) \).

The advantages of dealing with \( SFP(\mathbb{L}, \mathbb{L}) \) instead of \( FP(\mathbb{L}, \mathbb{L}) \), are the following:

(a) \( SFP(\mathbb{L}, \mathbb{L}) \) is more expressive than \( FP(\mathbb{L}, \mathbb{L}) \). Indeed the set of well-founded formulas of \( FP(\mathbb{L}, \mathbb{L}) \) is a proper subset of \( SFP \), because in \( FP(\mathbb{L}, \mathbb{L}) \) \( \Pr(\varphi) \) is a formula iff \( \varphi \) is a formula without occurrences of \( \Pr \) (i.e. it is not allowed to use the modality nested), and a Łukasiewicz connective \( * \) is not admitted in contexts like \( \Pr(\varphi) \rightarrow \psi \), where \( \psi \) is not in the form \( \Pr(\gamma) \). Moreover the algebraic models of \( SFP(\mathbb{L}, \mathbb{L}) \), the SMV-algebras, allow us to interpret formulas which do not have a natural interpretation in MV-algebras equipped with a state.

(b) Whilst we do not know yet if \( FP(\mathbb{L}, \mathbb{L}) \) is complete with respect to the class of Kripke models, \( SFP(\mathbb{L}, \mathbb{L}) \) is strongly complete with respect to the class of SMV-algebras (cf. [8, Theorem 4.5]).

(c) Whilst the connections between Kripke models and states have not been investigated in details, SMV-algebras are closely related to MV-algebras with a state.

2.2 MV-algebras and tensor product

An algebra \( A = (A, \oplus, \neg, 0, 1) \) of type \( (2, 1, 0, 0) \) is said to be an MV-algebra if \( A \) satisfies the following equations:

\[
(x \oplus y) \oplus z = x \oplus (y \oplus z) \\
x \oplus y = y \oplus x \\
x \oplus 0 = x \\
x \oplus 1 = 1 \\
\neg 1 = 0 \text{ and } \neg 0 = 1 \\
\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x
\]

The class of MV-algebras forms a variety, which will be denoted by \( \mathbb{M}V \). In any MV-algebra \( A \), one can define the following operations: \( x \rightarrow y = \neg x \oplus y \), \( x \odot y = \neg(\neg x \odot \neg y) \), \( x \leftrightarrow y = (x \rightarrow y) \odot (y \rightarrow x) \), \( x \lor y = (x \rightarrow y) \rightarrow y \) and \( x \land y = \neg(\neg x \lor \neg y) \).

We shall henceforth use the following abbreviations: for every \( x \in A \), and every \( n \in \mathbb{N} \),

\[
\begin{align*}
mx \text{ stands for } x \underbrace{\oplus \ldots \oplus}_n x, \text{ and } \xn \text{ stands for } x \underbrace{\odot \ldots \odot}_n x.
\end{align*}
\]

Any MV-algebra \( A \) can be equipped with the order relation \( \leq \), defined, for all \( x, y \in A \), by \( x \leq y \) iff \( x \rightarrow y = 1 \).

An MV-algebra \( A \) is said to be linearly ordered (or an MV-chain), provided that the order \( \leq \) is linear.

Example 2.1

(1) Consider the real unit interval \([0, 1]\) equipped with (Łukasiewicz) operations \( \oplus \) and \( \neg \), defined, for all \( x, y \in [0, 1] \), by

\[
x \oplus y = \min\{1, x + y\}, \text{ and } \neg x = 1 - x.
\]

The resulting algebra \([0, 1]_{MV} = ([0, 1], \oplus, \neg, 0, 1)\) is an MV-chain. Actually, Chang (cf. [3]) proved that the variety \( \mathbb{M}V \) is generated by \([0, 1]_{MV} \). \([0, 1]_{MV} \) is called the standard MV-algebra.
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(2) Fix a $k \in \mathbb{N}$, and let $F(k)$ be the set of all the McNaughton functions (cf. [4]) from the hypercube $[0, 1]^k$ into $[0, 1]$. In other words, let $F(k)$ be the set of all those functions $f : [0, 1]^k \to [0, 1]$ which are continuous and piecewise linear with integer coefficients.

The operations $\oplus$ and $\neg$ defined on $F(k)$ by

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\}, \text{ and } \neg f(x) = 1 - f(x),$$

make the structure $\mathcal{F}(k) = (F(k), \oplus, \neg, 0, 1)$ an MV-algebra, where 0 and 1 denote the functions constantly equal to 0 and 1, respectively. Actually, $\mathcal{F}(k)$ is the free MV-algebra over $k$-free generators. Henceforth, $\mathcal{F}(\omega)$ will denote the free MV-algebra over $\omega$-free generators.

Let $A$ be an MV-algebra. Then a non-empty subset $\mathcal{F}$ of $A$ is said to be a filter of $A$ iff: (a) $1 \in \mathcal{F}$, (b) if $x, y \in \mathcal{F}$, then $x \circ y \in \mathcal{F}$, and (c) if $x \in \mathcal{F}$ and $y \geq x$, then $y \in \mathcal{F}$. A filter $\mathcal{F}$ of an MV-algebra $A$ is said to be proper, if $\mathcal{F} \neq A$. A filter $m$ is said to be a maximal filter (or an ultrafilter) if $m \neq A$, and if for any filter $\mathcal{F}$ such that $\mathcal{F} \supseteq m$, then either $\mathcal{F} = A$, or $\mathcal{F} = m$. The set of all ultrafilters of an MV-algebra $A$ will be henceforth denoted by $\mathfrak{M}(A)$, or, when there is no danger of confusion, simply by $\mathfrak{M}$. It is well known (see [4] for instance) that the congruences lattice and the filters lattice of any MV-algebra $A$ are mutually isomorphic, via the isomorphism which associates to every congruence $\theta$ the filter $\{x \in A | (x, 1) \in \theta\}$.

**Definition 2.2**

Let $A$ be an MV-algebra. The $A$ is said to be:

(i) **Simple** iff $A$ is non-trivial, and $\{1\}$ is its only proper filter.

(ii) **Semisimple** iff the intersection of all its maximal filters is $\{1\}$.

Clearly every simple MV-algebra also is semisimple, but not vice-versa. For instance, the standard MV-algebra $[0, 1]_{MV}$ of Example 2.1(1), is simple, whilst the algebra $\mathcal{F}(\omega)$ of Example 2.1(2), is semisimple but not simple. In fact any simple MV-algebra is, up to isomorphism, an MV-subalgebra of $[0, 1]_{MV}$, hence $\mathcal{F}(\omega)$, being not linearly ordered, cannot be simple. On the other hand, if $A$ is any MV-algebra and $m \in \mathfrak{M}(A)$, then the quotient $A/m$ is simple (cf. [4, Theorems 2.4.14, 2.5.7]).

In [14], Mundici introduces the notion of MV-algebraic tensor product (or simply tensor product) between MV-algebras. Let us recall the main steps of this definition: let $A$, $B$ and $C$ be MV-algebras. A **bimorphism** from the direct product $A \times B$ of $A$ and $B$ into $C$, is a map $\beta$ such that:

(i) $\beta(1, 1) = 1$, and for all $a \in A$, and $b \in B$, $\beta(a, 0) = \beta(0, b) = 0$.

(ii) For all $a, a_1, a_2 \in A$, and $b, b_1, b_2 \in B$, and for $\star \in \{\land, \lor\}$, $\beta(a_1 \star a_2, b) = \beta(a_1, b) \star \beta(a_2, b)$, and $\beta(a, b_1 \star b_2) = \beta(a, b_1) \star \beta(a, b_2)$.

(iii) For all $a, a_1, a_2 \in A$, and for all $b, b_1, b_2 \in B$, if $a_1 \circ a_2 = 0$, then $\beta(a_1 \circ a_2, b) = \beta(a_1, b) \circ \beta(a_2, b) = 0$ and $\beta(a_1 \lor a_2, b) = \beta(a_1, b) \lor \beta(a_2, b)$.

The **tensor product** of two MV-algebras $A$ and $B$, is an MV-algebra $A \otimes B$ for which there exists a bimorphism $\beta$ from $A \times B$ into $A \otimes B$ fulfilling the following universal property: for every bimorphism $\beta'$ from $A \times B$ into an MV-algebra $C$, there is a unique homomorphism $\lambda$ from $A \otimes B$ into $C$ such that, for all $(a, b) \in A \times B$, $\lambda(\beta(a, b)) = \beta'(a, b)$.

The tensor product of two MV-algebras $A$ and $B$ always exists and it is unique up to isomorphism (cf. [14, Theorem 3.2]).

We shall henceforth write $a \otimes b$ instead of $\beta(a, b)$. It follows from [14] that the maps $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$, respectively, are embeddings of $A$ and of $B$ into $A \otimes B$. 
In [14], Mundici also proves that in general semisimplicity is not preserved by the tensor product: there exists a semisimple MV-algebra $A$ such that $A \otimes A$ is no longer semisimple (cf. [14, Theorem 3.3]). The following proposition shows that, in some particular cases, semisimplicity is preserved.

**Proposition 2.3**
If $A$ is a semisimple MV-algebra, then $[0, 1]_{MV} \otimes A$ is semisimple. In particular, $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$ is semisimple.

**Proof.** Let $\alpha \otimes a \in [0, 1]_{MV} \otimes A$, and assume $\alpha \otimes a < 1 \otimes 1$. If $\alpha < 1$, then there is a natural number $n$ such that $\alpha^n = 0$ in $[0, 1]_{MV}$. Therefore, in $[0, 1]_{MV} \otimes A$,

$$(\alpha \otimes a)^n \leq (\alpha \otimes 1)^n = 0 \otimes 1 = 0.$$  

Hence the filter generated by $\alpha \otimes a$ is not proper, whence any maximal filter of $[0, 1]_{MV} \otimes A$ entirely consists of elements of the form $1 \otimes a$. Now $A$ is semisimple, whence, if $a < 1$, there exists an $m \in \mathfrak{M}(A)$ such that $a \notin m$. Therefore, letting $m^\times = \{1 \otimes b \mid b \in m\}$, we have that $m^\times$ is an ultrafilter of $[0, 1]_{MV} \otimes A$ such that $1 \otimes a \notin m^\times$. It follows that the intersection of all the ultrafilters of $[0, 1]_{MV} \otimes A$ is the singleton $\{1 \otimes 1\}$, whence $[0, 1]_{MV} \otimes A$ is semisimple. ■

The algebra $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$ will play a central role in this article. We exploit another property of it. Let $\mathcal{F}_{\mathbb{R}}(\omega)$ denote the Lindenbaum algebra of the logic $\mathbb{R}L$ obtained by extending Łukasiewicz language by a constant $\overline{\alpha}$ for every $\alpha \in [0, 1]$, and by adding all the bookkeeping axioms:

$$(\overline{\alpha} \oplus \overline{\beta}) \leftrightarrow \min\{1, \alpha + \beta\} \text{ and } \neg(\overline{\alpha}) \leftrightarrow \overline{1 - \alpha}.$$  

The algebraic counterpart of $\mathbb{R}L$ is the variety generated by $\mathcal{F}_{\mathbb{R}}(\omega)$, whose members will be henceforth called $\mathbb{R}L$-algebras.

**Proposition 2.4**
The algebra $\mathcal{F}_{\mathbb{R}}(\omega)$ is isomorphic to an MV-subalgebra of $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$.

**Proof.** First of all note that $\mathcal{F}(\omega)$ and $[0, 1]_{MV}$ are MV-subalgebras of both $\mathcal{F}_{\mathbb{R}}(\omega)$ and $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$. Moreover, in $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$ we can interpret every constant $\overline{\alpha}$ as $\alpha \otimes [1]$, so obtaining an algebra in the signature of $\mathcal{F}(\omega)$. By the universal property of free algebras, there is a unique homomorphism $h$ from $\mathcal{F}_{\mathbb{R}}(\omega)$ into $[0, 1]_{MV} \otimes \mathcal{F}(\omega)$ (considered as a model of $\mathbb{R}L$, with $\overline{\alpha}$ interpreted as $\alpha \otimes [1]$), such that, for each generator $[p]$, $h([p]) = 1 \otimes [p]$. Clearly, for each $\alpha \in [0, 1]$, $h([\alpha]) = \alpha \otimes [1]$. In order to settle the claim, it is sufficient to show that $h$ is one-one, or equivalently that, if $[\psi] < [1]$, then $h([\psi]) < 1 \otimes [1]$.

**Claim 2.4.1**
\[\mathcal{F}_{\mathbb{R}}(\omega)\] is semisimple.

**Proof of Claim 2.4.1.** Let $[0, 1]_{MV}^{\mathbb{R}}$ be the algebra obtained from $[0, 1]_{MV}$ by adding a constant $\overline{\alpha}$ for every $\alpha \in \mathbb{R}$, to be interpreted as $\alpha$. Then $[0, 1]_{MV}^{\mathbb{R}}$ generates the variety of $\mathbb{R}L$-algebras. Therefore, $\mathcal{F}_{\mathbb{R}}(\omega)$ is the subalgebra of $[0, 1]_{MV}^{(0, 1)_{MV}^{\omega}}$ [where $(0, 1)_{MV}^{\omega}$ denotes the direct product of $\omega$ copies of $[0, 1]_{MV}$ generated by the projections, whence $\mathcal{F}_{\mathbb{R}}(\omega)$ is a subdirect product of simple algebras, and hence it is semisimple].

Since $[\psi] < [1]$, by the semisimplicity of $\mathcal{F}_{\mathbb{R}}(\omega)$, there is a maximal filter $m$ of $\mathcal{F}_{\mathbb{R}}(\omega)$, such that $[\psi] \notin m$. Let $h'$ be the homomorphism associated to $m$. Then $h'$ maps $\mathcal{F}_{\mathbb{R}}(\omega)$ into an algebra isomorphic to $[0, 1]_{MV}$, and $h'([\psi]) < 1$. Hence, there is a homomorphism $h'$ from $\mathcal{F}_{\mathbb{R}}(\omega)$ into $[0, 1]_{MV}^{\mathbb{R}}$ such that $h'([\psi]) < 1$. 
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Let \( h_1 : F_{\mathbb{R}}(\omega) \rightarrow [0,1]_{MV} \otimes F(\omega) \) and \( h_2 : [0,1]_{MV} \otimes F(\omega) \rightarrow [0,1]_{MV} \otimes F(\omega) \) be defined as:

\[
h_1 : [\gamma] \mapsto h'( [\gamma] ) \otimes [1], \quad \text{and} \quad h_2 : \alpha \otimes [\gamma] \mapsto (\alpha \cdot h'( [\gamma] )) \otimes [1].
\]

Both \( h_1 \) and \( h_2 \) are MV-homomorphisms. In fact, let \( g : [0,1]_{MV} \otimes F(\omega) \rightarrow [0,1]_{MV} \) be so defined:\n
\[
g(\alpha \otimes [\gamma]) = h'( [\gamma] ) \cdot \alpha \quad \text{[the definition of} \ g\text{] is meaningful because} \ F(\omega) \text{is an MV-subalgebra of} \ F_{\mathbb{R}}(\omega)). \ \text{Then} \ g \text{is an MV-homomorphism. Indeed the map} \ \beta \text{from} [0,1]_{MV} \times F(\omega) \rightarrow [0,1]_{MV} \text{such that} \ \beta(\alpha, [\gamma]) = \alpha \cdot h'( [\gamma] ) \text{is a bimorphism, whence, by the universal property of tensor product,} \ g \text{is that unique homomorphism such that} g(\alpha \otimes [\gamma]) = \beta(\alpha, [\gamma]). \ \text{Let} \ i \text{be the embedding of} [0,1]_{MV} \ \text{into} [0,1]_{MV} \otimes F(\omega), \ \text{defined by} \ i(\alpha) = \alpha \otimes [1], \ h_1 = i \circ h' \text{and} \ h_2 = i \circ g. \ \text{Then} \ h_1 \text{and} \ h_2 \text{are MV-homomorphisms.}

Now let \( [p] \) be a generator of \( F_{\mathbb{R}}(\omega). \) Then \( h_1([p]) = h'( [p] ) \otimes [1] = h_2(1 \otimes [p]) = h_2(h([p])), \) whence \( h_1 = h_2 \circ h, \) and hence, being \( h_1([\psi]) < 1 \otimes [1], \) also \( h([\psi]) < 1 \otimes [1]. \)

2.3 States on MV-algebras

Let \( A \) be an MV-algebra. A state on \( A \) (cf. [13]) is a map \( s : A \rightarrow [0,1] \) satisfying the following properties:

(i) \( s(1) = 1. \)

(ii) For all \( a, b \in A, \) whenever \( a \otimes b = 0, \) then \( s(a \oplus b) = s(a) + s(b). \)

A state \( s \) is said to be faithful if \( s(\lambda) = 0, \) implies \( \lambda = 0. \)

States constitute a generalization of the notion of probability measure on MV-algebras. Moreover, states are related to de Finetti’s coherence criterion: according to de Finetti (cf. [5]), a probabilistic assessment \( \chi : P(\varphi_1) = \alpha_1, \ldots, P(\varphi_n) = \alpha_n \) of classical events \( \varphi_1, \ldots, \varphi_n \) is said to be coherent iff there is no system of reversible bets on the events which leads to a win independently on the truth of \( \varphi_1, \ldots, \varphi_n. \) In other words, the assessment \( \chi \) is coherent iff for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \) there is a valuation \( V \) such that

\[
\sum_{i=1}^{n} \lambda_i(\alpha_i - V(\varphi_i)) \geq 0.
\]

The celebrated de Finetti’s Theorem states that an assessment \( \chi \) is coherent iff it can be extended to a finitely additive measure on the Boolean algebra of all events.

In [10, Corollary 4.3], Kühr and Mundici extend de Finetti’s coherence criterion to any (algebraizable, cf. [1]) logic \( L_{\Omega} \) whose equivalent algebraic semantics is given by the algebraic variety generated by the algebra \([0,1], \Omega), \) where \( \Omega \) denotes a set of continuous operations on \([0,1]. \) Hence, their result applies to \( \text{Łukasiewicz logic, to every extension of}\ \text{Łukasiewicz logic with rational or real constants (as for instance RPL, cf. [9], and to the logic}\ \mathbb{R}L \text{described in the previous section), and to the PMV}^+ \text{logic (cf. [11, 12]).}

**Theorem 2.5** [10]

Equip the unit interval \([0,1]\) with a set \( \Omega \) of continuous operations, denote by \([0,1]_{\Omega} \) the algebra so obtained, and let \( V([0,1]_{\Omega}) \) be the equational class generated by \([0,1]_{\Omega}. \) For each algebra \( A \in V([0,1]_{\Omega}), \) let \( W \) be the set of homomorphisms from \( A \) into \([0,1]_{\Omega}. \) Then, for all \( A' = \{a_1, \ldots, a_n\} \subseteq A, \) a map \( \chi : A' \rightarrow [0,1] \) is coherent over \( A' \) iff \( \chi \) is extendible to a convex combination of at most \( n+1 \) elements of \( W. \)
In other words, if \( \varphi_1, \ldots, \varphi_n \) are formulas of \( L_\Omega \), then an assessment \( \chi : \varphi_i \mapsto \alpha_i \) (for \( i = 1, \ldots, n \)) is coherent iff there exist at most \( n + 1 \) valuations \( v_1, \ldots, v_{n+1} \) into \([0, 1]_\Omega\) and \( n + 1 \) real numbers \( \lambda_1, \ldots, \lambda_{n+1} \), such that
\[
\sum_{i=1}^{n+1} \lambda_i = 1, \text{ and for all } j \in \{1, \ldots, n\}, \alpha_j = \sum_{i=1}^{n+1} \lambda_i \cdot v_i(\varphi_j).
\]

### 2.4 SMV-algebras

**Definition 2.6**

An SMV-algebra (cf. [7, 8]) is an algebra \((A, \sigma)\) such that:

- \( A \) is an MV-algebra,
- for all \( a, b \in A \), the operation \( \sigma \) satisfies the following equations:
  
  \[
  \begin{align*}
  \sigma(1) &= 0, \\
  \sigma(\neg a) &= \neg \sigma(a), \\
  \sigma(\sigma(a) \oplus \sigma(b)) &= \sigma(a) \oplus \sigma(b), \\
  \sigma(a \oplus b) &= \sigma(a) \oplus \sigma(b \ominus (a \odot b)).
  \end{align*}
  \]

An SMV-algebra is said to be **faithful** if it satisfies the quasi equation: \( \sigma(a) = 0 \) implies \( a = 0 \).

The following proposition collects some properties of SMV-algebras. A proof can be found in [8].

**Proposition 2.7**

In any SMV-algebra \((A, \sigma)\), the following conditions hold:

1. \( \sigma(1) = 1 \).
2. If \( a \leq b \), then \( \sigma(a) \leq \sigma(b) \).
3. \( \sigma(a \oplus b) \leq \sigma(a) \oplus \sigma(b) \), and if \( a \odot b = 0 \), then \( \sigma(a \oplus b) = \sigma(a) \oplus \sigma(b) \).
4. \( \sigma(\alpha(a)) = \sigma(a) \).
5. The image \( \sigma(A) \) of \( A \) under \( \sigma \) is the domain of an SMV-subalgebra of \((A, \sigma)\), and if \((A, \sigma)\) is subdirectly irreducible, then \( \sigma(A) \) is an MV-chain.

In [8] we introduce a way to define, starting from an SMV-algebra \((A, \sigma)\), a state on \( A \), and vice-versa, to define an SMV-algebra \((T, \sigma)\) starting from a state on an MV-algebra \( A \), in such a way that \( A \) is an MV-subalgebra of \( T \). We recall the main steps of this construction: (1) let \((A, \sigma)\) be any SMV-algebra and let \( m \) be a maximal filter of \( \sigma(A) \). Then \( \sigma(A)/m \) is a simple MV-algebra and it can be embedded into \([0, 1]_{MV}\) via an embedding \( h \). The map \( h \circ \eta_m \circ \sigma \), where \( \eta_m \) denotes the canonical homomorphism of \( \sigma(A) \) into \( \sigma(A)/m \), is a state on \( A \).

(2) Let \( A \) be an MV-algebra and let \( s \) be a state on \( A \). Let \( T = [0, 1]_{MV} \otimes A \), and let \( \sigma_s : T \to T \) be the map defined by \( \sigma_s(a \otimes s) = a \cdot s(a) \otimes 1 \). Then \( A \) is an MV-subalgebra of \( T \), and \((T, \sigma_s)\) is an SMV-algebra. (Further details can be found in [8, Theorem 5.1, Theorem 5.3].)

We conclude this section introducing a class of SMV-algebras which will play an important role in the remaining of this article.

**Definition 2.8**

An SMV-algebra \((A, \sigma)\) is said to be **\( \sigma \)-simple** if \( A \) is a semisimple MV-algebra, and \( \sigma(A) \) is a simple MV-algebra.
Let $s$ be a state on $\mathcal{F}(\omega)$. Then the SMV-algebra $([0,1]_{MV} \otimes \mathcal{F}(\omega), \sigma_s)$, where $\sigma_s$ is defined by $\sigma_s(\alpha \otimes [\psi]) = \alpha \cdot s([\psi]) \otimes [1]$ is $\sigma$-simple (the semisimplicity of $[0,1]_{MV} \otimes \mathcal{F}(\omega)$ follows from Proposition 2.3).

**Proposition 2.10.**
Let $(A, \sigma)$ be a $\sigma$-simple SMV-algebra. Then for every $a \in A$, the equation $1 \otimes \sigma(a) = \sigma(a) \otimes 1$ holds in $[0,1]_{MV} \otimes A$, where we have identified $\sigma(A)$ with its isomorphic MV-subalgebra of $[0,1]_{MV}$.

**Proof.** Assume, by way of contradiction, that for some $a \in A$, $1 \otimes \sigma(a) \neq \sigma(a) \otimes 1$. By Proposition 2.3, $[0,1]_{MV} \otimes A$ is semisimple, whence there is a maximal MV-congruence $\theta$ which separates $1 \otimes \sigma(a)$ and $\sigma(a) \otimes 1$. The quotient $([0,1]_{MV} \otimes A)/\theta$ is simple, whence it is isomorphic to (and will be identified with) an MV-subalgebra of $[0,1]_{MV}$.

Up to isomorphism, the map $h: a \otimes a \mapsto (a \otimes a)/\theta$, is a homomorphism of $[0,1]_{MV} \otimes A$ into $[0,1]_{MV}$. Let $h'$ be the map on $A$ defined by $h'(a) = h(a \otimes 1)$. The $h'$ is a homomorphism, being the composition of $h$ and of the embedding $i_2: a \mapsto 1 \otimes a$.

Claim 2.10.1
$h(a \otimes a) = a \cdot h'(a) = a \cdot h(a \otimes 1)$.

**Proof of Claim 2.10.1.** The claim can be easily proved if $\alpha$ is a rational number. If $\alpha$ is irrational, it follows from the monotonicity of $h$ and $\otimes$.

Now $h'$ must be identical on the elements of $\sigma(A)$, because $\sigma(A)$ is an MV-subalgebra of $[0,1]_{MV}$. Therefore, from Claim 2.10.1, we obtain:

$$h(\sigma(a) \otimes 1) = 1 \cdot h'(\sigma(a)) = \sigma(a) = \sigma(a) \cdot h'(1) = h(1 \otimes \sigma(a)).$$

\[\square\]

## 3 Comparing the semantics

$SFP(\mathcal{L}, \mathcal{L})$ is the algebraizable logic (in the sense of Blok and Pigozzi, cf. [1]) whose equivalent algebraic semantic is constituted by the variety of SMV-algebras. Thus, in particular the connectives and the constants of $SFP(\mathcal{L}, \mathcal{L})$ are the operations and the constants of SMV-algebras. Moreover, recalling that a valuation of $SFP(\mathcal{L}, \mathcal{L})$ into an SMV-algebra $A$ is a homomorphism from the algebra of formulas of $SFP(\mathcal{L}, \mathcal{L})$ into $A$, we have that the theorems of $SFP(\mathcal{L}, \mathcal{L})$ are precisely those formulas which are evaluated to 1 by any valuation in any SMV-algebra. The set of $SFP(\mathcal{L}, \mathcal{L})$-formulas will be henceforth denoted by $SFP$. Moreover we shall refer to semantics via SMV-algebras as the SMV semantics.

**Definition 3.1**
A formula $\phi \in SFP$ is said to be

(i) an SMV-tautology ($\phi \in SMVTAUT$) iff for every SMV-algebra $(A, \sigma)$ and for every valuation $\nu$ in $(A, \sigma)$, $\nu(\phi) = 1$.

(ii) SMV-1-satisfiable ($\phi \in SMV1SAT$) iff there are an SMV-algebra $(A, \sigma)$, and a valuation $\nu$ in $(A, \sigma)$ such that $\nu(\phi) = 1$.

In [8, Theorem 4.5], it is shown that SMVTAUT coincides with the set of theorems of $SFP(\mathcal{L}, \mathcal{L})$. We now introduce other kinds of semantics for formulas in $SFP$.
3.1 The standard semantics

While the standard model of MV-algebras is \([0, 1]_{MV}\), we cannot consider this model as a standard representative of SMV-algebras, because the only internal state making \([0, 1]_{MV}\) an SMV-algebra is the identity. Our idea is that standard models of SMV-algebras should be MV-algebras of functions on \([0, 1]\), with MV-operations defined pointwise, equipped with an internal state \(\sigma\) representing a sort of integral (hence the image of \(\sigma\) should be a subset of \([0, 1]\), or equivalently a set of constant \([0, 1]\)-valued functions). According to this idea, standard SMV-algebras should be precisely the \(\sigma\)-simple SMV-algebras. Thus we interpret formulas in \(\mathcal{SF}P\) into \(\sigma\)-simple SMV-algebras \((A, \sigma)\). From an interpretation into a \(\sigma\)-simple SMV-algebra, we can obtain interpretations in \([0, 1]\) combining them with a homomorphism \(h\) from \(A\) into \([0, 1]_{MV}\). The concept of 1-satisfiable formula splits into two non-equivalent classes, according as we use \(h\) or not.

**Definition 3.2**
Let \(\phi \in \mathcal{SF}P\). Then \(\phi\) is said to be

(i) a standard tautology \((\phi \in \text{StdTAUT})\) iff for every \(\sigma\)-simple SMV-algebra \((A, \sigma)\), and for every valuation \(v\) in \((A, \sigma)\), \(v(\phi) = 1\).
(ii) Standard-1-satisfiable \((\phi \in \text{Std1SAT})\) iff there is a valuation \(v\) into a \(\sigma\)-simple SMV-algebra \((A, \sigma)\) such that \(v(\phi) = 1\).
(iii) Locally standard positively satisfiable \((\phi \in \text{StdposSAT})\) iff there is a valuation \(v\) into a \(\sigma\)-simple SMV-algebra \((A, \sigma)\) and an MV-homomorphism \(h\) from \(A\) into \([0, 1]_{MV}\), such that \(h(v(\phi)) > 0\).
(iv) Locally standard 1-satisfiable \((\phi \in \text{Std1SAT})\) iff there is a valuation \(v\) into a \(\sigma\)-simple SMV-algebra \((A, \sigma)\) and an MV-homomorphism \(h\) from \(A\) into \([0, 1]_{MV}\), such that \(h(v(\phi)) = 1\).

3.2 The state semantics

Since states are not internal operations, it is not possible to interpret SMV-formulas directly by means of states. Thus, we first associate to any state \(s\) on the countably generated free MV-algebra \(\mathcal{F}(\omega)\) an internal state \(\sigma_s\) on \([0, 1]_{MV} \otimes \mathcal{F}(\omega)\), defined as in Example 2.9, which makes \(([0, 1]_{MV} \otimes \mathcal{F}(\omega), \sigma_s)\) an SMV-algebra, and then we evaluate \(\mathcal{SF}P\)-formulas in it. Moreover \(\sigma_s\) somehow extends \(s\), in the sense that \(\sigma_s(1 \otimes [\phi]) = s([\phi]) \otimes [1]\).

The most natural valuation in \([0, 1]_{MV} \otimes \mathcal{F}(\omega)\), denoted by \(u^s\), is inductively defined as follows:

(i) \(u^s(p) = 1 \otimes [p]\) for \(p\) atomic;
(ii) \(u^s\) commutes with all MV-connectives;
(iii) \(u^s(\sigma(\phi)) = \sigma_s(u^s(\phi))\).

Once again, 1-satisfiability splits into two cases, according as we use a homomorphism \(h\) into \([0, 1]\) or not.

**Definition 3.3**
Let \(\phi \in \mathcal{SF}P\). Then \(\phi\) is said to be

(i) A state tautology \((\phi \in \text{STTAUT})\) iff for every state \(s\) on \(\mathcal{F}(\omega)\), \(u^s(\phi) = 1 \otimes [1]\).
(ii) State-1-satisfiable \((\phi \in \text{ST1SAT})\) iff there is a state \(s\) on \(\mathcal{F}(\omega)\), such that \(u^s(\phi) = 1 \otimes [1]\).
(iii) Locally state-positively satisfiable \((\phi \in \text{STposSAT})\) iff there is a state \(s\) on \(\mathcal{F}(\omega)\) and a homomorphism \(h\) from \([0, 1]_{MV} \otimes \mathcal{F}(\omega)\) into \([0, 1]_{MV}\), such that \(h(u^s(\phi)) > 0\).
(iv) Locally state-1-satisfiable \((\phi \in \text{ST1SAT})\) iff there is a state \(s\) on \(\mathcal{F}(\omega)\) and a homomorphism \(h\) from \([0, 1]_{MV} \otimes \mathcal{F}(\omega)\) into \([0, 1]_{MV}\), such that \(h(u^s(\phi)) = 1\).
3.3 The Kripke semantics

Let $Var$ be the set of propositional variables in $SFP$. A probabilistic Kripke model is a triple $K = (X, e, \mu)$ where $X$ is a finite or countable set of nodes, $e$ is a map associating to every pair $(x, p) \in X \times Var$, an element $e(x, p) \in [0, 1]$ and $\mu : X \to [0, 1]$ is a function such that $\sum_{x \in X} \mu(x) = 1$.

For every $x \in X$ and every $\phi \in SFP$, the truth-value of $\phi$ in $K$ at the node $x$, denoted by $\| \phi \|_{K, x}$, is inductively defined as follows: if $\phi$ is a propositional variable $p$, then $\| p \|_{K, x} = e(x, p)$; $\| 0 \|_{K, x} = 0$; $\| \cdot \|_{K, x}$ commutes with all Łukasiewicz connectives; $\| \sigma(\psi) \|_{K, x} = \sum_{y \in X} \| \psi \|_{K, y} \cdot \mu(y)$. Notice that, for every $\psi \in SFP$, $\| \sigma(\psi) \|_{K, x}$ does not depend on $x$, in the sense that, for all $x, y \in X$, $\| \sigma(\psi) \|_{K, x} = \| \sigma(\psi) \|_{K, y}$.

We say that $\phi$ is valid in a probabilistic Kripke model $K$ (and we write $K \models \phi$), iff for all $x \in X$, $\| \phi \|_{K, x} = 1$.

**Definition 3.4**

Let $\phi \in SFP$. Then $\phi$ is said to be:

(i) a Kripke tautology ($\phi \in KRTAUT$) iff for every probabilistic Kripke model $K = (X, e, \mu)$, $K \models \phi$.

(ii) Kripke 1-satisfiable ($\phi \in KR1SAT$) iff there is a probabilistic Kripke model $K$ such that $K \models \phi$.

(iii) Locally Kripke positively satisfiable ($\phi \in KRposSAT$) iff there is a probabilistic Kripke model $K = (X, e, \mu)$ and node $x \in X$, such that $\| \phi \|_{K, x} > 0$.

(iv) Locally Kripke 1-satisfiable ($\phi \in lKR1SAT$) iff there is a probabilistic Kripke model $K = (X, e, \mu)$ and a node $x \in X$ such that $\| \phi \|_{K, x} = 1$.

**Remark 3.5**

For $MOD \in \{Std, ST, KR\}$, $MOD1SAT \subseteq lMOD1SAT$. We show that all inclusions are proper. We prove the claim for $Std1SAT$ and $lStd1SAT$, the proofs for the other cases being similar (notice in particular that, letting $p$ any propositional variable, then $p \in lST1SAT$, but $p \notin ST1SAT$). Let $\phi = p \land \sigma(\neg p)$. Clearly $\phi \notin Std1SAT$. Now let $A$ be the MV-algebra consisting of all functions from $[0, 1]$ into $[0, 1]$ having only a finite (possibly empty) set of discontinuity points, with MV-operations defined pointwise. Let $\sigma$ be the Lebesgue integral on $A$, let $f$ be the characteristic function of $[0, 1]$, and let $h$ be the MV-homomorphism from $A$ into $[0, 1]_{MV}$ defined, for all $g \in A$, by $h(g) = g(0)$. Finally, let $v$ be a valuation in $(A, \sigma)$ such that $v(p) = f$. Then $h(v(\sigma(\neg p))) = 1$ and $h(v(p)) = 1$, whence $\phi \notin lSTD1SAT$.

**Theorem 3.6**

Let $\phi \in S FP$, and let $\alpha \in [0, 1]$. Then the following are equivalent:

(i) There are $\sigma$-simple SMV-algebra $(A, \sigma_A)$, valuation $v$ into $(A, \sigma_A)$, and homomorphism $h : A \to [0, 1]_{MV}$, such that $h(v(\phi)) = \alpha$.

(ii) There are state $s$ on $\mathcal{F}(\omega)$, and homomorphism $h : [0, 1]_{MV} \otimes \mathcal{F}(\omega) \to [0, 1]_{MV}$, such that $h(u^s(\phi)) = \alpha$.

(iii) There are probabilistic Kripke model $K = (X, e, \mu)$ and a node $x \in X$ such that $\| \phi \|_{K, x} = \alpha$.

**Proof.** (i) $\Rightarrow$ (ii). Let $(A, \sigma_A)$, $h$ and $v$ be as in (i). We define $s : \mathcal{F}(\omega) \to [0, 1]$, letting, for every $[\psi] \in \mathcal{F}(\omega)$, $s([\psi]) = \sigma_A(v(\psi))$. $s$ is a state on $\mathcal{F}(\omega)$, whence the map

$$
\sigma_s : [0, 1]_{MV} \otimes \mathcal{F}(\omega) \to [0, 1]_{MV} \otimes \mathcal{F}(\omega)
$$

defined as in Example 2.9, makes the algebra $([0, 1]_{MV} \otimes \mathcal{F}(\omega), \sigma_s)$ an SMV-algebra (cf. [8, Theorem 5.3]).

Now let $\sigma' : [0, 1]_{MV} \otimes A \to [0, 1]_{MV} \otimes A$ be a map so defined: for all $\alpha \otimes a \in [0, 1]_{MV} \otimes A$, $\sigma'(\alpha \otimes a) = \alpha \cdot \sigma_A(a)$. Then $([0, 1]_{MV} \otimes A, \sigma')$ is an SMV-algebra (cf. [8, Theorems 5.2, 5.3]).
Consider the map \( v_0 : [0, 1]_{MV} \otimes \mathcal{F}(\omega) \rightarrow [0, 1]_{MV} \otimes A \) such that, for all \( \alpha \otimes [\psi] \in [0, 1]_{MV} \otimes \mathcal{F}(\omega) \), \( v_0(\alpha \otimes [\psi]) = \alpha \otimes v(\psi) \).

**Claim 3.6.1**
The map \( v_0 \) is an SMV-homomorphism.

**Proof of Claim 3.6.1.** The map \( v_0' : [0, 1]_{MV} \times \mathcal{F}(\omega) \rightarrow [0, 1]_{MV} \otimes A \), defined by \( v_0'(\alpha, [\psi]) = \alpha \otimes v(\psi) \) is a bimorphism, whence there is a unique homomorphism \( v_0'' \) from \( [0, 1]_{MV} \otimes \mathcal{F}(\omega) \) into \([0, 1]_{MV} \otimes A \) such that \( v_0'(\alpha, [\psi]) = v_0''(\alpha \otimes [\psi]) \). Then \( v_0 = v_0'' \) is an MV-homomorphism.

Now we prove that \( v_0 \) is compatible with \( \sigma \):

\[
v_0(\sigma \cdot (\alpha \otimes [\psi])) = v_0(\alpha \cdot s([\psi]) \otimes [1])
= \alpha \cdot s([\psi]) \otimes 1
= \alpha \cdot \sigma_\omega(\nu(\psi)) \otimes 1
= \sigma'(\alpha \otimes \nu(\psi))
= \sigma'(v_0(\alpha \otimes [\psi])).
\]

This settles Claim 3.6.1.

**Claim 3.6.2**
For all \( \psi \), \( v_0 \circ u^s(\psi) = 1 \otimes \nu(\psi) \).

**Proof of Claim 3.6.2.** The proof is by induction on \( \psi \). If \( \psi \) is \( \sigma \)-free, then \( v_0(u^s(\psi)) = v_0(1 \otimes [\psi]) = 1 \otimes \nu(\psi) \). If \( \psi = \sigma(\gamma) \), then \( v_0(u^s(\sigma(\gamma))) = \sigma'(v_0(u^s(\gamma))) \). By the inductive hypothesis, \( v_0(u^s(\gamma)) = 1 \otimes \nu(\gamma) \), whence (also using Proposition 2.10) \( \sigma'(1 \otimes \nu(\gamma)) = \sigma_\omega(\nu(\gamma)) \otimes 1 = 1 \otimes \sigma_\omega(\nu(\gamma)) = 1 \otimes v(\sigma(\gamma)) \). Then Claim 3.6.2 holds.

Finally let \( h^* \) be the map from \([0, 1]_{MV} \otimes A \) into \([0, 1]_{MV} \), defined as follows: for all \( \alpha \otimes a \in [0, 1]_{MV} \otimes A \), \( h^*(\alpha \otimes a) = \alpha \cdot h(a) \). From [8, Theorem 5.3], it follows that \( h^* \) is a well-defined MV-homomorphism. Moreover, from Claim 3.6.2,

\[
h^*(v_0(u^s(\phi))) = h^*(1 \otimes \nu(\phi)) = h(\nu(\phi)) = \alpha.
\]

We have shown that the following diagram commutes.

\[
\begin{array}{ccc}
SFP & \xrightarrow{u^s} & ([0, 1]_{MV} \otimes \mathcal{F}(\omega), \sigma_s) \\
\downarrow v & & \downarrow v_0 \\
(A, \sigma_A) & \xrightarrow{1 \otimes v} & ([0, 1]_{MV} \otimes A, \sigma') \\
\downarrow h & & \downarrow h^* \\
[0, 1]_{MV} & & [0, 1]_{MV}
\end{array}
\]

(where \( h \) and \( h^* \) are MV-homomorphisms). Then (ii) holds.

(ii)\(\Rightarrow\)(i). Let \( s \) be a state on \( \mathcal{F}(\omega) \), and let \( h \) be a homomorphism from \([0, 1]_{MV} \otimes \mathcal{F}(\omega) \) into \([0, 1]_{MV} \), such that \( h(u^s(\phi)) = \alpha \). Let \( \sigma_s \) be the internal state on \([0, 1]_{MV} \otimes \mathcal{F}(\omega) \) defined as in the definition of state semantics. Then \(([0, 1]_{MV} \otimes \mathcal{F}(\omega), \sigma_s) \) is a \( \sigma \)-simple SMV-algebra. Taking \( v = u^s \), we immediately have \( h(\nu(\phi)) = \alpha \).

(ii)\(\Rightarrow\)(iii). Let \( s \) be a state on \( \mathcal{F}(\omega) \), and let \( h \) be a homomorphism from \([0, 1]_{MV} \otimes \mathcal{F}(\omega) \) into \([0, 1]_{MV} \) such that \( h(u^s(\phi)) = \alpha \). For every subformula \( \delta \) of the form \( \sigma(\gamma) \), let \( \alpha_\gamma = h(u^s(\sigma(\gamma))) \). Let us
write every subformula $\delta$ of $\phi$ as an MV-combination of (propositional variables and) formulas of the form $\sigma(\gamma)$, and let $\delta^*$ be the result of substituting in $\delta$ each subformula $\sigma(\gamma)$ by $\overline{\alpha \gamma}$. Then $\delta^*$ (whence, in particular, $\phi^*$) is a formula of $\text{RL}$.

Let $\sigma(\gamma_1), \ldots, \sigma(\gamma_r)$ be a list of all subformulas of $\phi$ beginning with $\sigma$. Then the assessment $\sigma(\gamma_i^*) = \alpha \gamma_i$ ($i = 1, \ldots, r$), is coherent, whence, from Theorem 2.5, there are (at most) $r + 1$ valuations $v_1, \ldots, v_{r+1}$ of $\text{RL}$ and (at most) $r + 1$ positive real numbers $\lambda_1, \ldots, \lambda_{r+1}$ such that $\sum_{i=1}^{r+1} \lambda_i = 1$, and $\alpha \gamma_i = \sum_{i=1}^{r+1} \lambda_i v_i(\gamma_i^*)$ for all $j = 1, \ldots, r + 1$.

Let $v$ be the valuation defined by $v(p) = h(u^x(p)) = h(1 \otimes [p])$, and set $X = \{v, v_1, \ldots, v_{r+1}\}$. Also define $\mu : X \to [0, 1]$ by $\mu(v_i) = \lambda_i$ ($i = 1, \ldots, r + 1$), and $\mu(v) = 0$. Finally let, for every $x \in X$, and for every propositional variable $p$, $e(x, p) = x(p)$. Then $K = (X, e, \mu)$ is a probabilistic Kripke model, such that $\|\phi\|_{K, x} = h(u^x(\phi)) = \alpha$.

(iii) $\Rightarrow$ (i). Let $K = (X, e, \mu)$ be a probabilistic Kripke model, and let $x \in X$ be such that $\|\phi\|_{K, x} = \alpha$. Let $A = [0, 1]_{MV}^X$ be the MV-algebra of all functions from $X$ to $[0, 1]_{MV}$ with MV-operations defined pointwise. Moreover, for every $a = (a_x | x \in X) \in A$, let $\sigma_A(a)$ be the function constantly equal to $\sum_{x \in X} a_x \cdot e(x)$. Then $(A, \sigma_A)$ is a $\sigma$-simple SMV-algebra.

Finally, let $v$ be the $(A, \sigma_A)$-valuation defined, for every propositional variable $p$ and for every $x \in X$, by $v(p)_x = e(x, p)$. By induction on $\phi$ we can see that for every subformula $\gamma$ of $\phi$ and for every $x \in X$, $v(\gamma)_x = \|\gamma\|_{K, x}$. Let us prove the induction step corresponding to $\sigma$:

$$v(\sigma(\psi))_x = \sigma_A(v(\psi))_x$$

$$= \sum_{y \in X} v(\psi)_y \cdot \mu(y)$$

$$= \sum_{y \in X} \|\psi\|_{K, y} \cdot \mu(y)$$

$$= \|\sigma(\psi)\|_{K, x}.$$  

Finally, define $h : A \to [0, 1]_{MV}$ by $h(a)_x = \alpha_x$. Then, $h$ is a homomorphism, and $h(v(\phi)) = w(\phi)_x = \|\phi\|_{K, x} = \alpha$.

**Corollary 3.7**

Let $\phi \in \text{SFP}$. Then the following are equivalent:

(i) $\phi \in \text{Std TAUT}$.
(ii) $\phi \in \text{ST TAUT}$.
(iii) $\phi \in \text{KRT TAUT}$.
(iv) $\neg \phi \notin \text{ISdtposSAT}$.
(v) $\neg \phi \notin \text{IST posSAT}$.
(vi) $\neg \phi \notin \text{IKR posSAT}$.

**Proof.** The claim follows from Theorem 3.6, from the fact that $\phi \in \text{MODTAUT}$ iff $\neg \phi \notin \text{lMOD posSAT}$, for MOD being either Std, ST or KR, and from the fact that if either $\nu$ is a valuation into a $\sigma$-simple SMV-algebra $(A, \sigma)$ such that $\nu(\psi) < 1$, or $s$ is a state on $\mathcal{F}_\omega$ such that $u^x(\psi) < 1$, then there is a homomorphism $h$ from $A$ (from $[0, 1]_{MV} \otimes \mathcal{F}_\omega$, respectively) into $[0, 1]_{MV}$ such that $h(\nu(\psi)) < 1$ ($h(u^x(\psi)) < 1$, respectively).

**Corollary 3.8**

Let $\phi \in \text{SFP}$. Then the following are equivalent:

(i) $\phi \in \text{LStd1SAT}$.
(ii) $\phi \in \text{LST1SAT}$.
(iii) $\phi \in \text{LKR1SAT}$.
For 1-satisfiability we have a different result: the state semantics is different from the other three, as e.g. the formula $p$ is not in $ST1SAT$, but it clearly belongs to $Std1SAT$, $SMV1SAT$ and $KR1SAT$. However, SMV semantics coincides with standard semantics and with Kripke semantics as regards to 1-satisfiability.

**Theorem 3.9**

Let $\phi \in SFP$. Then the followings are equivalent:

(i) $\phi \in Std1SAT$.
(ii) $\phi \in KR1SAT$.
(iii) $\phi \in SMV1SAT$.

**Proof.**

(i)$\Rightarrow$(ii). Let $(A, \sigma)$ be a $\sigma$-simple SMV-algebra and let $v_0$ be a valuation in $(A, \sigma)$ such that $v_0(\phi) = 1$. Let $s$ be the state on $F(\omega)$ defined as follows: for every $[\gamma] \in F(\omega), s([\gamma]) = \sigma(v_0([\gamma]))$.

Up to isomorphism, we can (and we will) identify $\sigma(A)$ with its isomorphic subalgebra of $[0,1]_{MV}$. Thus, $\sigma$ may be regarded as a state on $A$.

Let, for every subformula of $\phi$ of the form $\sigma([\gamma]), \alpha_\gamma = s([\gamma])$, and let us write every subformula $\delta$ of $\phi$ as an MV-combination of propositional variables and formulas of the form $\sigma([\gamma])$. Let $\delta^*$ be the result of substituting in $\delta$ each subformula $\sigma([\gamma])$ by $\overline{\sigma}_\gamma$. Then $\delta^*$ is a formula of $\mathbb{R}$.

As in the proof of Theorem 3.6, let $\sigma([\gamma_1]), \ldots, \sigma([\gamma_r])$ be a list of all subformulas of $\phi$ which begin with $\sigma$. Then the assessment consisting of all conditions $\sigma([\gamma_i]) = \alpha_{\gamma_i}$ (with $i = 1, \ldots, r$), is coherent, whence, by Theorem 2.5, there are (at most) $r+1$ valuations $v_1, \ldots, v_{r+1}$ of $\mathbb{R}$ in $[0,1]_{MV}$ and (at most) $r+1$ positive real numbers $\lambda_1, \ldots, \lambda_{r+1}$ such that $\lambda_1 + \cdots + \lambda_{r+1} = 1$ and $\alpha_{\gamma_j} = \sum_{i=1}^{r+1} \lambda_i v_i([\gamma_j])$ for $j = 1, \ldots, r$.

Note that $\alpha_\phi = \sigma(v_0(\phi)) = \sigma(1) = 1$. Therefore, since $\lambda_i > 0$ for $i = 1, \ldots, r+1$, we obtain $v_0(\phi^*) = \cdots = v_{r+1}(\phi^*) = 1$.

Now let $X = \{v_0, v_1, \ldots, v_{r+1}\}, \mu(v_i) = \lambda_i$ for $i = 1, \ldots, r+1$, and $\mu(v_0) = 0$. Finally, let for every propositional variable $p$ and for $i = 0, \ldots, r+1$, $e(v_i, p) = v_i(p)$.

Then $K = (X, e, \mu)$ is a Kripke model. Moreover by induction on $\delta$, we see that for every SFP$(\mathbb{L}, p)$ subformula $\delta$ of $\phi$ and for $j = 0, \ldots, r+1$, we have $\|\delta\|_{K, v_j} = v_j(\delta^*)$. The only non-trivial step corresponding to $\sigma$ is proved as follows: since $\sigma(\delta^*) = \alpha_\delta$, for $j = 0, \ldots, r+1$ we have:

$$\|\sigma(\delta)\|_{K, v_j} = \sum_{i=0}^{r+1} \mu(v_i) \cdot \|\delta\|_{K, v_i} = \sum_{i=0}^{r+1} \lambda_i v_i(\delta^*) = \alpha_\delta = v_j(\delta^*)$$

Thus, for $j = 0, \ldots, r+1$, $\|\phi\|_{K, v_j} = v_j(\phi^*) = 1$. It follows that $K \models \phi$.

(ii)$\Rightarrow$(i). Let $K = (X, e, \mu)$ be a Kripke model such that $(X, e, \mu) \models \phi$. Let $A = [0,1]_{MV}$ and let for each $a = \langle a_y : y \in X \rangle \in [0,1]_{MV}, \sigma_\mu(a)$ be the constant function from $X$ into $[0,1]$, defined, for every $y \in X$, by

$$(\sigma_\mu(a))_y = \sum_{z \in X} \mu(z) \cdot a_z$$

Then $(A, \sigma_\mu)$ is a $\sigma$-simple SMV-algebra. Finally, let $v$ be the valuation in $(A, \sigma_\mu)$ defined, for every propositional variable $p$ and for every $y \in X$, by $v(p)_y = e(y, p)$. Then for all $y \in X$ we have $v(\phi)_y = \|\phi\|_{K, y} = 1$. Hence $v(\phi) = 1$, as desired.

(i)$\Rightarrow$(iii) is trivial.
(iii)⇒(i). Let \((A, \sigma)\) be an SMV-algebra, and let \(v\) be a valuation into \((A, \sigma)\) such that \(v(\phi) = 1\).

Let \(m = \mathcal{M}(\sigma(A))\), and let \(n\) be the filter of \(A\), generated by \(m\). Then \(n\) is an MV-filter, closed under \(\sigma\) (i.e., \(n\) is a \(\sigma\)-filter in the terminology of [7, 8]). Indeed, if \(b \in n\), then there is a \(\sigma(a) \in m\) such that \(\sigma(a) \leq b\), whence \(\sigma(a) \leq \sigma(b)\) (recall that any internal state is monotone and idempotent, Proposition 2.3), and \(\sigma(b) \in n\).

Now let \((A_n, \sigma_n)\) be the quotient of \((A, \sigma)\) modulo \(n\). Then \(\sigma_n(A_n)\) is a simple MV-algebra, because it is isomorphic to \(\sigma(A)\) modulo \(m\), which is a maximal filter.

Finally, let \(p\) denote the intersection of all the maximal filters of \(A_n\). If \(a \in p\), then, for every \(k \in \mathbb{N}\), \(\sigma^{-1}(a) \leq a\), whence \(\sigma^{-1}(\sigma_n(a)) \leq \sigma_n(a)\), and \(\sigma_n(a) \in p\). Thus \(p\) is a \(\sigma\)-filter. Moreover if \(a \in p\), then \(\sigma_n(a) = 1\) because \(\sigma_n(A)\) is simple. Therefore, the quotient \((A_p, \sigma_p)\) of \((A_n, \sigma_n)\) modulo \(p\) is a \(\sigma\)-simple SMV-algebra, which is a quotient of \((A, \sigma)\) modulo the composition of the homomorphisms induced by the filters \(n\) and \(p\). Call \(h\) such a composition. Then \(h \circ v\) is a valuation into a \(\sigma\)-simple SMV-algebra such that \(h \circ v(\phi) = 1\).

**Corollary 3.10**

Let \(\phi \in \mathcal{SF}_P\), and let \(\sigma(\gamma_1), \ldots, \sigma(\gamma_r)\) be an enumeration of its subformulas which begin with \(\sigma\).

Then:

1. For every Kripke model \(K = (X, e, \mu)\) and for every \(x \in X\), there exists a Kripke model \(K' = (X', e', \mu')\), with \(|X'| = r + 2\), and a node \(x' \in X'\), such that \(\|\phi\|_{K,x} = \|\phi\|_{K',x'}\).
2. \(\phi \in \text{KR1SAT}\) iff there exists a Kripke model \(K'\) with at most \(r + 2\) nodes, such that \(K' \models \phi\).
3. The following are equivalent:
   1. \(\phi \in \text{KR1SAT}\);
   2. \(\sigma(\phi) \in \text{KR1SAT}\);
   3. \(\sigma(\phi) \in \text{KR1SAT}\).
4. \(\phi \in \text{KR1SAT}\) iff there exist a \(\sigma\)-simple and faithful SMV-algebra \((A, \sigma)\), and a valuation \(v\) into \((A, \sigma)\), such that \(v(\phi) = 1\).

**Proof.**

1. For the non-trivial direction, let \(K = (X, e, \mu)\) be a Kripke model, and let \(x\) be an element of \(X\) such that \(\|\phi\|_{K,x} = \sigma \in [0, 1]\). By Theorem 3.6, (iii)⇒(ii), there exist a state \(s\) on \(\mathcal{F}(\omega)\), and an MV-homomorphism \(h : [0, 1]_{\text{MV}} \otimes \mathcal{F}(\omega) \to [0, 1]_{\text{MV}}\), such that \(h(\mu(\phi)) = \sigma\). Moreover, inspection on the proof of Theorem 3.6, (ii)⇒(iii), shows that \(h(\mu(\phi)) = \sigma\) implies the existence of a Kripke model \(K' = (X', e', \mu')\), with at most \(r + 2\) nodes, such that for some \(x' \in X'\), \(\|\phi\|_{K',x'} = \sigma\).
2. For the non-trivial direction, by Theorem 3.9, \(\phi \in \text{KR1SAT}\) is equivalent to \(\phi \in \text{Std1SAT}\). Inspection on the proof of Theorem 3.9, (i)⇒(ii), shows that \(\phi \in \text{Std1SAT}\), implies the existence of a Kripke model \(K = (X, e, \mu)\) with at most \(r + 2\) nodes, such that for every \(x \in X\), \(e(x, \phi) = 1\).
3. The direction (3a)⇒(3b) is clear. To prove (3b)⇒(3a), let \(K = (X, e, \mu)\) be a Kripke model such that, for some node \(y \in X\), \(\|\sigma(\phi)\|_{K,y} = \sum_{x \in X} \mu(x) \cdot \|\phi\|_{K,x} = 1\). Let \(X' = \{x \in X | \mu(x) > 0\}\), and let \(K' = (X', e', \mu')\) be the Kripke model resulting from \(K\), restricting the nodes to be in \(X'\). Then, for all \(x' \in X\), \(\mu'(x') > 0\), and \(\sum_{x' \in X'} \mu'(x') \cdot \|\phi\|_{K',x'} = 1\). This is only possible if \(\|\phi\|_{K',x'} = 1\) for every \(x' \in X'\), whence \(\phi \in \text{KR1SAT}\).
   Finally note that (3a)⇒(3c) holds because \(\sigma(1) = 1\) is always satisfied by any state \(\sigma\), and (3c)⇒(3b) easily follows from Remark 3.5.
4. Let \(K = (X, e, \mu)\) be a Kripke model such that, for every \(x \in X\), \(\|\phi\|_{K,x} = 1\). As in the previous proof of (3b)⇒(3a), let \(K' = (X', e', \mu')\) be the Kripke model obtained deleting from \(X\) all those nodes \(x_i\) such that \(\mu(x_i) = 0\). Then \(K' \models \phi\). Theorem 3.9, (ii)⇒(i), states that \(K' \models \phi\) implies the
existence of a \( \sigma \)-simple SMV-algebra \((A, \sigma)\), and a valuation \(v\) into \((A, \sigma)\), such that \(v(\phi) = 1\). Moreover, inspection on the proof of Theorem 3.9 shows that \(\sigma(a) = 0\) iff \(a = 0\), whence \((A, \sigma)\) is faithful.

4 Complexity issues

In this section, we prove a PSPACE containment result for SMV1SAT and for all the classes MOD1TAUT, MOD1SAT and IMOD1SAT, where MOD \(\in\{\text{Std, ST, KR}\}\).

Let \(\phi\) be in \(SF\), and let \(\psi_1, \ldots, \psi_t\) be an enumeration of all its subformulas and, among \(\psi_1, \ldots, \psi_t\), let \(\sigma(\gamma_i), \ldots, \sigma(\gamma_r)\) be all those formulas, beginning with \(\sigma\).

For \(i = 1, \ldots, r\), and \(j = 1, \ldots, r+2\), let \(x_{\psi_i}^j\) and \(x_{\psi_i}^j\) be mutually distinct variables, and let \(\phi_\mathbb{F}\) be the formula in the language of ordered fields, expressing the following:

(a) For \(i = 1, \ldots, t\) and \(j = 1, \ldots, r+2\), \(0 \leq x_{\psi_i}^j \leq 1\).
(b) For \(j = 1, \ldots, r+2\), \(z_j \geq 0\), and \(z_1 + \cdots + z_{r+2} = 1\).
(c) For \(i, h, k = 1, \ldots, t\) and \(j = 1, \ldots, r+2\), if \(\psi_i = \psi_h \oplus \psi_k\), then either \(x_{\psi_i}^j + x_{\psi_h}^j \geq 1\) and \(x_{\psi_i}^j = 1\), or \(x_{\psi_i}^j + x_{\psi_h}^j < 1\) and \(x_{\psi_i}^j = x_{\psi_i}^j + x_{\psi_h}^j\).
(d) For \(i, h = 1, \ldots, t\) and \(j = 1, \ldots, r+2\), if \(\psi_i = \neg \psi_h\), then \(x_{\psi_i}^j = 1 - x_{\psi_h}^j\).
(e) For \(i = 1, \ldots, r\) and \(j = 1, \ldots, r+2\),

\[
x_{\psi_i}^j = \sum_{m=1}^{r+1} z_m^m x_{\psi_i}^m,
\]

where \(h_i\) and \(k_i\) are such that \(\sigma(\gamma_i) = \psi_{h_i}\), and \(\gamma_i = \psi_{k_i}\)

**Theorem 4.1**

Let \(q\) be such that \(\phi = \psi_q\), and let, for \(i = 1, \ldots, t\), \(\exists x_{\psi_i}^r\) be an abbreviation for \(\exists x_{\psi_i}^1 \cdots \exists x_{\psi_i}^{r+2}\) and \(\exists z^r\) be an abbreviation for \(\exists z_1 \cdots \exists z_{r+2}\). Then:

(1) \(\phi \in \text{KR1TAUT} = \text{STD1TAUT} = \text{ST1TAUT}\) iff the formula

\[
\phi^{\text{TAUT}} = \exists z^r \exists x_{\psi_1}^r \cdots \exists x_{\psi_t}^r \left( \phi_\mathbb{F} \land \bigvee_{j=1}^{r+2} x_{\psi_q}^j < 1 \right)
\]

is false in the real-ordered field \(\mathbb{R}\).

(2) \(\phi \in \text{IKR1SAT} = \text{ISTD1SAT} = \text{IST1SAT}\) iff the formula

\[
\phi^{\text{ISAT}} = \exists z^r \exists x_{\psi_1}^r \cdots \exists x_{\psi_t}^r \left( \phi_\mathbb{F} \land \bigvee_{j=1}^{r+2} x_{\psi_q}^j = 1 \right)
\]

is true in the real-ordered field \(\mathbb{R}\).

**Proof.**

(1) For the left-to-right direction, argue contrapositively. Suppose that \(\phi^{\text{TAUT}}\) is true in the real field.

Then there is a valuation \(v\) in the real field making \(\phi_\mathbb{F} \land \bigvee_{j=1}^{r+2} x_{\psi_q}^j < 1\) true. Let \(X = \{x_1, \ldots, x_{r+2}\}\).
be a set with \( r + 2 \) elements, and let for \( j = 1, \ldots, r + 2 \) and for every propositional variable \( p \), \( e(x_j, p) = v(x^j_\phi) \) and \( \mu(x_j) = v(z_j) \). Then \( K = (X, e, \mu) \) is a Kripke model. We prove that for every subformula \( \psi_h \) of \( \phi \) and for \( j = 1, \ldots, r + 2 \), \( \| \psi_h \|_{K, x_j} = v(x^j_{\psi_h}) \). The proof is by induction on \( \psi_h \). For \( \psi_h \) atomic, the claim follows from the definition of \( e \), and the steps corresponding to Łukasiewicz connectives are trivial. For the step corresponding to \( \sigma \), assume \( \psi_h = \sigma(\psi_k) \). From the induction hypothesis and from the definition of \( \phi^{T\!A\!U\!T} \), we get:

\[
\| \psi_h \|_{K, x_j} = \| \sigma(\psi_k) \|_{K, x_j} = \sum_{i=1}^{r+1} \mu(x_i) \| \psi_k \|_{K, x_j} = \sum_{i=1}^{r+1} v(z_i) v(x^i_{\psi_k}) = v(x^j_{\psi_k}).
\]

Since for some \( j \), \( v(x^j_{\phi}) = v(x^j_{\psi_q}) < 1 \), for this \( j \) it also holds \( \| \psi_h \|_{K, x_j} < 1 \), and hence \( \phi \notin KR1TAUT \).

Conversely, let \( K = (X, e, \mu) \) be a (finite or countable) probabilistic Kripke model with \( X = \{x_1, \ldots, x_n, \ldots\} \) such that \( \| \phi \|_{K, x} = \alpha < 1 \) for some \( x \in X \). From Corollary 3.10 (1), there exists a Kripke model \( K' = (X', e', \mu') \) with \( X' = \{x'_1, \ldots, x'_{r+2}\} \), and a node \( x'_j \in X' \) such that \( \| \phi \|_{K', x'_j} = \alpha < 1 \). Let \( v \) be a valuation into the real field \( \mathbb{R} \) such that, for \( i = 1, \ldots, t \) and \( j = 1, \ldots, r + 2 \), \( v(x'^j_i) = \| \psi_i \|_{K', x'_j} \) and \( v(z_j) = \mu'(x'_j) \). Then, conditions (a), (b), (c), (d) and (e) preceding Theorem 4.1 hold. Moreover \( v(x^j_{\psi_q}) < 1 \), and \( \phi^{T\!A\!U\!T} \) holds in \( \mathbb{R} \). This settles part (1).

(2) The right-to-left direction is proved essentially by the same argument as in part (1). For the other direction, let \( K = (X, e, \mu) \) be a Kripke model and let \( x \in X \) be such that \( \| \phi \|_{K, x} = 1 \). From Corollary 3.10 (1) there exists a Kripke model \( K' = (X', e', \mu') \) with \( X' = \{x'_1, \ldots, x'_{r+2}\} \), and a node \( x'_j \in X' \) such that \( \| \phi \|_{K', x'_j} = 1 \). Let \( v \) be a valuation into \( \mathbb{R} \) such that: for \( i = 1, \ldots, t \) and \( j = 1, \ldots, r + 2 \), \( v(x'^j_i) = \| \psi_i \|_{K', x'_j} \) and \( v(z_j) = \mu'(x'_j) \). Then, \( v(\sum_{i=1}^{r+2} x'^j_i = 1) = 1 \) iff there exists a \( j \in \{1, \ldots, r+2\} \) such that \( v(x'^j_j) = 1 \) iff there exists a \( j \in \{1, \ldots, r+2\} \) such that \( \| \psi_q \|_{K', k_j} = 1 \), whence, letting \( j = l \), (2) holds.

**Corollary 4.2**

The sets \( KR1TAUT = Std1TAUT = ST1TAUT \), \( lKR1SAT = lStd1SAT = lST1SAT \) and \( KR1SAT = Std1SAT = SMV1SAT \) are in PSPACE.

**Proof.** By Theorem 4.1, checking that a formula \( \phi \in SFP(\mathbb{L}, \mathbb{L}) \) is in one of the above sets, reduces to check the truth (or the falsity) of an existential formula of field theory (either \( \phi^{T\!A\!U\!T} \), or \( \phi^{l\!S\!A\!T} \)), in the real field. Moreover, \( \phi^{T\!A\!U\!T} \) and \( \phi^{l\!S\!A\!T} \) can be computed in polynomial time from \( \phi \), whence our claim follows from Canny’s Theorem (cf. [2, Theorem 3.3]) stating that the existential theory of the reals can be decided in PSPACE.

Finally, Corollary 3.10 (3), \( \phi \in KR1SAT \) iff \( \sigma(\phi) \in lKR1SAT \), and \( \sigma(\phi) \) can be obviously computed from \( \phi \) in polynomial time. Thus \( KR1SAT = ST1SAT = lStd1SAT \) are in PSPACE.

**5 Concluding remarks**

In this article, we introduced several semantics for the probabilistic logic \( SFP(\mathbb{L}, \mathbb{L}) \), namely the SMV-semantics, the standard semantics, the state semantics and the Kripke semantics. For each kind
of semantics, we investigated the sets of 1-tautologies, and 1-satisfiable, positively satisfiable and locally-satisfiable formulas.

As for 1-tautologies, we proved that those of the standard semantics, state semantics and Kripke semantics coincide, but we do not know yet, if any of them is equivalent to the SMV semantics, whence, since the set of theorems of $SFP(Ł,Ł)$ coincides with the set of SMV-tautologies, the problem left open in [6] to establish the completeness of the weaker logic $FP(Ł,Ł)$ with respect to Kripke semantics, is still open.

As for the sets of locally satisfiable formulas we proved that those of standard semantics, state semantics and Kripke semantics coincide, while, as regards 1-satisfiable formulas, those of standard semantics, Kripke semantics and SMV semantics coincide.

In the final part of this article, we proved that all the above discussed sets of formulas are in PSPACE. The problem of establishing an NP-containment for them remains open.

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**References**


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