MV-algebras with internal states and probabilistic fuzzy logics

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A B S T R A C T

In this paper we enlarge the language of MV-algebras by a unary operation \( \sigma \) equationally described so as to preserve the basic properties of a state in its original meaning. The resulting class of algebras will be called MV-algebras with internal state (or SMV-algebras for short). After discussing some basic algebraic properties of SMV-algebras, we apply them to the study of the coherence problem for rational assessments on many-valued events. Then we propose an algebraic treatment of the Lebesgue integral and we show that internal states defined on a divisible MV\(_{\Lambda} \)-algebra can be represented by means of this more general notion of integral.

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1. Motivation

States on MV-algebras have been introduced by Mundici in [23] as averaging processes for formulas in Łukasiewicz logic. Moreover, states constitute measures on their associated MV-algebras which generalize the usual probability measures on boolean algebras.\(^1\) States have been deeply investigated by several authors. Many interesting results have been obtained, which connect states with integrals. For instance, we quote the characterization by Kroupa [17] and by Panti [28] of states on semi-simple MV-algebras as integrals with respect to a suitable Borel measure, as well as the work by Marra and Mundici on the Lebesgue state (cf. [20]), and Navara’s paper [26], where it is shown that a huge class of measures on subclasses of (\( \sigma \)-complete) MV-algebras are represented by integrals with respect to classical probability measures.

Finally, states are related to de Finetti’s coherence criterion. According to de Finetti (cf. [7–9]), a probabilistic assessment \( \chi : P(\varphi_1) = \lambda_1, \ldots, P(\varphi_n) = \lambda_n \) of classical events \( \varphi_1, \ldots, \varphi_n \) is said to be coherent iff there is no system of reversible bets on the events which leads to a win independently on the truth of \( \varphi_1, \ldots, \varphi_n \). In other words, the assessment \( \chi \) is coherent iff for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), there is a valuation \( V \) such that

\[
\sum_{i=1}^{n} \lambda_i (V(\varphi_i) - V(\varphi_i)) \geq 0.
\]

The celebrated de Finetti’s Theorem states that an assessment \( \chi \) is coherent iff it can be extended to a finitely additive measure on the boolean algebra of formulas.

\(^0\) The present paper is an extended version of the paper “An algebraic approach to states on MV-algebras” (cf. [13]) appeared in the Proceedings of Eusflat 2007, where some of the results have already been presented.

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\(^1\) While only finite additivity appears in the formal definition of a state (see Definition 2.7), Panti (cf. [28, 1.1]) has recently shown that states on an MV-algebra \( A \) are in one-one correspondence with all regular Borel measures (which are \( \sigma \)-additive!) on the algebra of the maximal ideals of \( A \) (see Theorem 2.11).
In [25], Mundici extends to Łukasiewicz infinite-valued logic de Finetti’s criterion, thus solving a problem of Jeff Paris who firstly studied de Finetti’s coherence for an assessment of formulas of any finite-valued Łukasiewicz logic in [29].

**Theorem 1.1** (Mundici [25]). Let \( \varphi_1, \ldots, \varphi_n \) be formulas of Łukasiewicz logic and \( \alpha_1, \ldots, \alpha_n \in [0,1] \). Then the following are equivalent:

(i) For all \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), there is a valuation \( V \) such that

\[
\sum_{i=1}^{n} \lambda_i (\alpha_i - V(\varphi_i)) \geq 0.
\]

(ii) There is a state \( s \) on the Lindenbaum algebra of Łukasiewicz logic \( \mathcal{F}(k) \) generated by the propositional variables occurring in \( \varphi_1, \ldots, \varphi_n \), such that \( s([\varphi_i]) = \alpha_i \) for all \( i = 1, \ldots, n \), where \([\varphi_i]\) denotes the equivalence class of \( \varphi_i \).

Kühl and Mundici then improve the above stated result in [19]. In fact they show that a map \( s : \{ \varphi_1, \ldots, \varphi_n \} \to [0,1] \) (the \( \varphi_i \)'s being formulas of any \([0,1]\)-valued algebraic logic with continuous connectives) satisfies the de Finetti's coherence criterion iff \( s \) can be extended to a state on the Lindenbaum algebra \( \mathcal{F}(k) \) generated by the propositional variables occurring in \( \varphi_1, \ldots, \varphi_n \), iff \( s \) has an integral representation.

Thus states are also related to probability, and hence to reasoning under uncertainty. Parallel to the investigation of states, various probabilistic logics have been introduced. In particular Hájek (cf. [15]) presents a fuzzy logic (the idea is that an internal state

\[
\text{is equivalent:}
\]

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Thus states are also related to probability, and hence to reasoning under uncertainty. Parallel to the investigation of states, various probabilistic logics have been introduced. In particular Hájek (cf. [15]) presents a fuzzy logic \( \text{FP}(\mathcal{L}) \) with a modality \( P \) (interpreted as \( \text{Probably} \)) which is suitable for the treatment of probability of classical (i.e., \([0,1]\) -valued) events. The axioms of these logics are suggested by the following semantic interpretation: the probability of an event \( \phi \) is interpreted as the truth value of \( \varphi \), finally these formulas are combined by means of the Łukasiewicz connectives. In our language, we can express formulas like:

\[
\text{if it rains, then probably a few people will go to the sea}
\]

formulas of the form \( \neg \varphi \), and with \( \varphi \) replaced by \( = \), plus the axiom \( \sigma(1) = 1 \), but, while a state is a map from an MV-algebra into \([0,1]\), an

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2 We refer the reader to Section 2 for a complete treatment.
internal state is an operation of the algebra. Therefore, if \( \sigma \) is an internal state on an MV-algebra \( A \) and \( a, b \in A \), an expression of the form \( \sigma(a \oplus \sigma(b)) \), which would be meaningless if \( \sigma \) were just a state, denotes a well-defined element of \( A \). Thus not only SMV-algebras allow us to interpret a more powerful logic, but they also constitute a variety of universal algebras which is the equivalent algebraic semantics of the logic SFP(Ł, Ł) in the sense of Blok and Pigozzi [3]. Therefore, when reasoning about the probability of many-valued events, we can safely use an equational logic. Finally, SFP(Ł, Ł) of the form \( x \) and states on MV-algebras. Then in Section 3 we define the notion of internal state of an MV-algebra, and hence we define integral and we show that this notion of integral can be formulated inside any subdirectly irreducible divisible SMV-\( A \). Thus not \( A \) under \( \sigma \) is contained in a non-standard extension of \([0,1]\). Of course, we do not ignore that states in the usual sense have important and deep applications to pure and applied mathematics, which only partially extend to SMV-algebras. This is due to the fact that states use the whole structure of (the unit interval of) the reals, which are only definable in second-order logic. To the contrary, SMV-algebras are equationally definable, whence we cannot expect to be able to define the reals in them: we can only say that in a subdirectly irreducible SMV-algebra \( (A, \sigma) \), the image of \( A \) under \( \sigma \) is contained in a non-standard extension of \([0,1]\).

In any case, in this paper we prove that some important applications of states to integration and to probabilistic coherence, can be extended somehow to SMV-algebras. More precisely, we prove the following:

(a) There is a standard way of obtaining a state over an MV-algebra \( A \) from an SMV-algebra having \( A \) as MV-reduct, and conversely, there is a standard way of obtaining an SMV-algebra from an MV-algebra with a state.

(b) In Section 5.1 we shall reduce the coherence problem to a satisfiability problem in SMV-algebras.

(c) If we add the axioms of divisible \( MVA \)-algebras to SMV-algebra, thus getting the variety of divisible \( MVA \)-algebras, then, in any subdirectly irreducible algebra of such variety, we can define the concept of Lebesgue integral in a very simple and completely algebraic way.

This paper is organized as follows: in the following section we recall some basic definitions and properties of MV-algebras and states on MV-algebras. Then in Section 3 we define the notion of internal state of an MV-algebra, and hence we define the variety \( SMV \) of SMV-algebras. Section 4 is devoted to an algebraic analysis of \( SMV \). In the same section we prove a strong completeness theorem of SFP(Ł, Ł) with respect to the class of SMV-algebras. In Section 5 we relate the two notions of state on an MV-algebra and internal state of an MV-algebra. In particular we present a method for obtaining an SMV-algebra starting from an MV-algebra with a state and vice versa. The results of that section enable us to characterize the coherence of a rational assessment inside the theory of SMV-algebras. In Section 6 we introduce a generalization of Lebesgue integral and we show that this notion of integral can be formulated inside any subdirectly irreducible divisible SMV\( A \)-algebra. We end this paper discussing some open problems and future work.

2. Preliminary notions

An \textit{MV-algebra} is a system \( (A, \oplus, \cdot, 0) \), where \( (A, \oplus, 0) \) is a commutative monoid with neutral element 0, and for each \( x, y \in A \) the following equations hold:

(i) \( (x')' = x \),
(ii) \( x \oplus 1 = 1 \), where \( 1 = 0' \),
(iii) \( x \oplus (y \oplus x')' = y \oplus (y \oplus x')' \).

The class of MV-algebras forms a variety which henceforth will be denoted by \( M\text{V} \). In any MV-algebra one can define further operations as follows:

\[
\begin{align*}
  x \rightarrow y &= (x' \oplus y),
  x \ominus y &= (x \rightarrow y)',
  x \odot y &= (x' \odot y'),
  x \leftarrow y = (x \leftarrow y),
  x \odot y &= (x \odot y'),
  x \ominus y &= (x \ominus y'),
  x \odot (\ominus y) &= \ominus (x \odot y),
  x \ominus (\ominus y) &= x \ominus y,
  x \ominus (\ominus y) &= x \ominus y,
  x \ominus (\ominus y) &= x \ominus y,
  x \ominus (\ominus y) &= x \ominus y.
\end{align*}
\]

Henceforth we shall use the following notation: for every \( x \in A \) and every \( n \in \mathbb{N} \),

\[
\begin{align*}
  nx &= x \oplus \ldots \oplus x, \text{ and } \quad x^n = x \odot \ldots \odot x.
\end{align*}
\]

Any MV-algebra \( A \) can be equipped with an order relation so defined: for all \( x, y \in A \),

\[
x \leq y \quad \text{iff} \quad x \rightarrow y = 1.
\]

An MV-algebra is said to be \textit{linearly ordered} (or an MV-chain) if the order \( \leq \) is linear.
Example 2.1

(1) Consider the real unit interval $[0, 1]$ equipped with (Łukasiewicz) operations so defined: for all $x, y \in [0, 1]$,

$$x \oplus y = \min\{1, x + y\}, \quad \text{and} \quad x' = 1 - x.$$  

The algebra $[0, 1]_{MV} = ([0, 1], \oplus, \cdot, 0)$ is an MV-chain. Chang (cf. [5]) proved that the whole variety $\text{MV}$ is generated by $[0, 1]_{MV}$. $[0, 1]_{MV}$ is called the standard MV-algebra.

(2) Fix a $k \in \mathbb{N}$, and let $F(k)$ be the set of all the McNaughton functions (cf. [6]) from the hypercube $[0, 1]^k$ into $[0, 1]$. In other words, let $F(k)$ be the set of all those functions $f : [0, 1]^k \to [0, 1]$ which are continuous, piecewise linear and such that each piece has integer coefficient. The following pointwise operations defined on $F(k)$:

$$\langle f \oplus g \rangle (x) = \min\{1, f(x) + g(x)\}, \quad \text{and} \quad f' (x) = 1 - f(x),$$

make the structure $\mathcal{F}(k) = (F(k), \oplus, \cdot, 0)$ an MV-algebra, where 0 is the function constantly equal to 0. Actually, $\mathcal{F}(k)$ is the free MV-algebra over $k$-free generators. Henceforth $\mathcal{F}(\omega)$ will denote the free MV-algebra over $\omega$-free generators.

(3) Boolean algebras coincide with MV-algebras satisfying the additional condition of idempotency $x \oplus x = x$. In this sense MV-algebras provide a generalization of boolean algebras. Moreover, in any MV-algebra $A$, the set of idempotent elements $B(A) = \{x \in A : x \oplus x = x\}$ is the domain of the largest boolean subalgebra of $A$, the so-called boolean skeleton of $A$.

McNaughton functions as described in the above example (2) play an important role in the theory of MV-algebras. In fact, to any term $\varphi(x_1, \ldots, x_n)$ of the language of MV-algebras one can easily associate a function $f_\varphi : [0, 1]^k \to [0, 1]$ by stipulating that: for every variable $x_i$, and for every $x \in [0, 1]^k$, $f_\varphi(x) = x_i$, $f_\varphi(x) = 0, f_\varphi(x) = 1 - f_\varphi(x), f_\varphi(x) = \min\{1, f_\varphi(x) + f_\varphi(x)\}$.

For every term $\varphi$ (or equivalently for every Łukasiewicz formula $\varphi$), $f_\varphi$ is called the truth-table of $\varphi$.

**Theorem 2.2** (McNaughton [21]). A function $f : [0, 1]^k \to [0, 1]$ is a truth table of a Łukasiewicz formula in $k$-variables iff $f$ is a McNaughton function.

A filter $\mathfrak{f}$ of an MV-algebra $A$ is a subset of $A$ satisfying the following conditions: (i) $1 \in \mathfrak{f}$, (ii) if $x, y \in \mathfrak{f}$, then $x \oplus y \in \mathfrak{f}$, and (iii) if $x \geq y$ and $y \in \mathfrak{f}$, then $x \in \mathfrak{f}$. A filter $\mathfrak{f}$ of an MV-algebra $A$ is said to be prime if $\mathfrak{f} \neq A$ and, whenever $x, y \in \mathfrak{f}$, then either $x \in \mathfrak{f}$ or $y \in \mathfrak{f}$. We shall henceforth write $\text{Spec}(A)$ to denote the set of all prime filters of an MV-algebra $A$. A filter $\mathfrak{f}$ is maximal (and in this case it will be also called an ultrafilter) if $m \neq \mathfrak{f}$ and for any other filter $\mathfrak{g}$ of $A$ such that $\mathfrak{g} \supseteq \mathfrak{f}$, then either $\mathfrak{g} = A$ or $\mathfrak{g} = \mathfrak{f}$. The set $U$ of all maximal filters of an MV-algebra $A$ will be henceforth denoted by $U(A)$, or, when there is no danger of confusion, by $U$. The set $U$ is non-empty and it can be viewed as a compact Hausdorff space with the so-called spectral topology, whose closed sets are in the form $C_1 = \{u \in U(A) : m \not\in \mathfrak{f}\}$ for any filter $\mathfrak{f}$ of $A$.

**Definition 2.3.** Let $A$ be an MV-algebra. Then $A$ is said to be:

(i) **Simple** if $A$ is non-trivial, and (1) is its only proper filter.

(ii) **Semisimple** if the intersection of all its maximal filters is $\{1\}$.

Clearly every simple MV-algebra also is semisimple, but not vice versa. As a matter of fact notice that the standard MV-algebra $[0, 1]_{MV}$ of Example 2.1 (1) is simple, whilst the algebra $\mathcal{F}(k)$ of (2) is semisimple but not simple. In fact any simple MV-algebra is, up to isomorphism, an MV-subalgebra of $[0, 1]_{MV}$, hence $\mathcal{F}(k)$, being not linearly ordered, cannot be simple. On the other hand, if $A$ is any MV-algebra and $m \in U$, then the quotient $A/m$ is simple.

**Definition 2.4** (Gerla [14]). A divisible MV-algebra (DMV-algebra for short) is a structure $A = (A, \oplus, \cdot, \{\delta_n\}_{n \in \mathbb{N}}, 0)$ where $(A, \oplus, \cdot, 0)$ is an MV-algebra, and for each $n \in \mathbb{N}$, $\delta_n$ is an unary operator satisfying, for each $x \in A$, $x \oplus \delta_n(x) = (n - 1)\delta_n(x)$.

As shown in [14] the variety $\text{DMV}$ of DMV-algebras is generated by the algebra $[0, 1]_{DMV} = ([0, 1], \oplus, \cdot, \{\delta_n\}_{n \in \mathbb{N}}, 0)$, where $([0, 1], \oplus, \cdot, 0)$ is the standard MV-algebra, and for each $x \in [0, 1]$, $\delta_n(x) = \frac{x}{n}$. In any DMV-algebra we can multiply elements by rationals in $[0, 1]$: $0x = 0$, and if $0 < m \leq n$, then $\frac{m}{n}x = m\delta_n(x)$.

In [22] Mundici proved the existence of a categorical equivalence $\mathcal{E}$ between the category of MV-algebras and that of $\ell$-groups with strong order unit. Recall that a lattice-ordered abelian group ($\ell$-group for short) $\mathcal{G}$ is $(\mathbb{G}, \mathcal{G}, \vee)$, or an abelian group $(\mathbb{G}, +, 0, +)$ equipped with a lattice structure $(\mathbb{G}, \wedge, \vee)$ and further satisfying: $x + (y \wedge z) = (x + y) \wedge (x + z)$ for all $x, y, z \in G$. An element $u \in G$ is said a strong order unit for $\mathcal{G}$ if for all $x \in G$, there is an $n \in \mathbb{N}$ such that $nu \geq x$ (where $nu$ stands

---

$^3$ In fact Łukasiewicz logic is algebraizable in the sense of Blok and Pigozzi (cf. [3]) and its equivalent algebraic semantics is the class of MV-algebras (see [27] for a complete treatment of algebraizable many-valued logics). Among others, this means that terms of the language of MV-algebras can be regarded as formulas of the language of Łukasiewicz logic; and vice versa.

$^4$ Spec($A$) usually denotes the set of prime ideals of an MV-algebra $A$ (see, for instance [6]), and here we are using it to denote the set of prime filters. We believe that this small abuse will not cause problems in the understanding of this paper. Actually, ideal and filter are dual notions.

$^5$ Simple and semisimple MV-algebras, respectively, are simple and semisimple algebras in their universal algebraic meaning. In particular it is easy to show that any simple MV-algebra is subdirectly irreducible, and any semisimple MV-algebra is a subdirect product of simple MV-algebras (see [4] for further details).
for $u + \ldots + u$, $n$-times). Since weak units will be never used in this paper, without danger of confusion, we shall henceforth call a strong order unit $u$, a unit for the $\ell$-group $\mathcal{G}$.

An $\ell$-group $\mathcal{G}$ is said to be divisible if for every $x \in \mathcal{G}$ and for every $n \in \mathbb{N}$, there is an $y \in \mathcal{G}$ (usually denoted by $\frac{y}{n}$) such that $ny = x$.

An $\ell$-group $\mathcal{G}$ with a unit $u$, the MV-algebra $\Gamma(\mathcal{G}, u)$ has universe $\{x \in \mathcal{G} : 0 \leq x \leq u\}$, and operations so defined: $x \odot y = u \wedge (x + y)$, and $x' = u - x$. Given an MV-algebra $A$, $\Gamma^{-1}(A)$ will henceforth denote that unique (up to isomorphism) $\ell$-group $\mathcal{G}$ with unit $u$ such that $\Gamma(\mathcal{G}, u) = A$. The existence of such a structure is shown in [22].

In [14] Gerla showed that Mundici’s functor $\Gamma$ can be extended to a categorical equivalence between DMV-algebras and divisible $\ell$-groups with unit, while Theorem 2.2 can be reformulated by saying that the free DMV-algebra over $k$ generators is isomorphic to the algebra of continuous, piecewise linear functions with rational coefficients from $[0,1]^k$ into $[0,1]$.

The next theorem will find use in Section 6.1, and it states that every MV-algebra $A$ can be regarded as an algebra of functions taking value in the unit interval of a totally ordered field.

**Theorem 2.5** ([Di Nola [10]]). Up to isomorphism, every MV-algebra $A$ is an algebra of $[0,1]^*$-valued functions over $\text{Spec}(A)$, where $[0,1]^*$ is an ultrapower of the real unit interval $[0,1]$, only depending on the cardinality of $A$.

The previous theorem can be refined when we restrict to semisimple MV-algebras. The theorem is shown in Belluce’s paper [2], but it can be also derived from Chang’s completeness theorem (cf. [5]).

**Theorem 2.6** (Belluce [2], Chang [5]). Up to isomorphism every semisimple MV-algebra $A$ is an algebra of $[0,1]$-valued continuous functions defined on the compact Hausdorff space $U(A)$ with the spectral topology. Moreover, for all $m, n \in U(A)$ such that $m \neq n$, there exists an $f \in A$ such that $f(m) \neq f(n)$.

### 2.1. States on MV-algebras

In order to generalize probability measures to MV-algebras, Mundici introduced in [23] the notion of state on MV-algebras.

**Definition 2.7.** [23] Let $A$ be an MV-algebra. Then a map $s : A \rightarrow [0,1]$ is a state on $A$ if the following are satisfied:

(i) $s(1) = 1$,
(ii) whenever $x \odot y = 0$, then $s(x \oplus y) = s(x) + s(y)$.

A state is said to be faithful if $s(x) = 0$ implies $x = 0$.

The following proposition collects some properties of states. The easy proof is left to the reader.

**Proposition 2.8.** Let $s$ be a state on an MV-algebra $A$. Then the following properties hold for any $x, y \in A$:

(i) $s(0) = 0$.
(ii) $s(x') = 1 - s(x)$.
(iii) if $x \leq y$, then $s(x) \leq s(y)$.
(iv) $s(x) + s(y) = s(x \oplus y) + s(x \odot y)$.
(v) $s(x) + s(y) = s(x \lor y) + s(x \land y)$.
(vi) $s(x \lor y) \leq s(x \oplus y) \leq s(x) + s(y)$.

Hence any state is monotone, subadditive, and moreover (by (v)), the restriction of $s$ to the boolean skeleton of $A$ is a finitely additive probability measure.

By a state on an $\ell$-group $\mathcal{G}$ with a unit $u$ we mean a normalized positive homomorphism $h : G \rightarrow \mathcal{G}$. Precisely a state $h$ on $\mathcal{G}$ has to satisfy: for each $x, y \in G$, $h(x + y) = h(x) + h(y)$, and $h(x) \geq 0$ whenever $x \geq 0$, and $h(u) = 1$. The equivalence between MV-algebras and $\ell$-groups with unit has the following counterpart for states.

**Proposition 2.9.** [23]

(1) Let $\mathcal{G} = (G, \odot, \land, \lor, 0)$ be an $\ell$-group with a unit $u$, let $h$ be a state on $\mathcal{G}$, and let $s$ be the restriction of $h$ to $\Gamma(\mathcal{G}, u)$. Then $s$ is a state on the MV-algebra $\Gamma(\mathcal{G}, u)$.

(2) Any state $s$ on an MV-algebra $A$ can be extended to a state on the $\ell$-group corresponding to $A$.

In the light of Theorems 2.5 and 2.6 it is natural to ask if states correspond to integrals. Using Theorem 2.6, and [18, 4.4.10], Kroupa proved:

**Theorem 2.10** (Kroupa [17, 18]). Let $s$ be a state on a semisimple MV-algebra $A$. Then there is a unique Borel probability measure $\mu$ on $U(A)$ with the spectral topology such that for any $f \in A$,

$$\int f \, d\mu = s(f).$$
Theorem 2.11 (Panti [28]). Let $A$ be an MV-algebra, and let $P(U(A))$ be the set of all regular Borel probability measures on $U(A)$ (with the spectral topology). Then the states on $A$ are in one–one correspondence with the elements of $P(U(A))$.

3. MV-algebras with an internal state

Definition 3.1. An MV-algebra with internal state (SMV-algebra for short) is a structure $(A, \sigma) = (A, \oplus, *, 0, \sigma)$, where $(A, \oplus, *, 0)$ is an MV-algebra, and $\sigma$ is an unary operator on $A$ satisfying, for each $x, y \in A$:

\begin{align*}
(\sigma 1) \quad & \sigma(0) = 0, \\
(\sigma 2) \quad & \sigma(x^+) = (\sigma(x))^+, \\
(\sigma 3) \quad & \sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \oplus (x \circ y)), \\
(\sigma 4) \quad & \sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y).
\end{align*}

An SMV-algebra $(A, \sigma)$ is said to be faithfull if it satisfies the quasi-equation: $\sigma(x) = 0$ implies $x = 0$.

Clearly the class of SMV-algebras constitutes a variety which will be henceforth denoted by $\text{SMV}$.

Example 3.2

(a) We start from a trivial example. Let $A$ be any MV-algebra and $\sigma$ be the identity on $A$. Then $(A, \sigma)$ is an SMV-algebra.

(b) A slightly less trivial example. Let $\sigma$ be an idempotent endomorphism of an MV-algebra $A$ (for example, we may take $A$ to be a non-trivial ultrapower of the standard MV-algebra $\{0, 1\}_{\text{MV}}$ and $\sigma$ to be the standard part function). Then $(A, \sigma)$ is an SMV-algebra.

(c) This is a sufficiently general example for our purposes. Let $A$ be the MV-algebra of all continuous and piecewise linear functions with real coefficients from $[0, 1]^n$ into $[0, 1]$. Then $A$, with the pointwise application of MV-algebraic $\oplus$ and $\ast$, forms an MV-algebra. Now let $f \in A$, $\sigma(f)$ be the function from $[0, 1]^n$ to $[0, 1]$ which is constantly equal to

$$
\int_{[0,1]^n} f(x) \, dx.
$$

Then $(A, \sigma)$ is an SMV-algebra. As will be clear from the results of the next section, $(A, \sigma)$ is simple, whence it is sub-directly reducible, but is not totally ordered. Although rather general, this algebra is faithful: it satisfies the quasi equation $\sigma(x) = 0$ implies $x = 0$, which is not valid in general.

Lemma 3.3. In any SMV-algebra $(A, \sigma)$ the following properties hold:

(a) $\sigma(1) = 1$.

(b) If $x \leq y$, then $\sigma(x) \leq \sigma(y)$.

(c) $\sigma(x \ast y) \leq \sigma(x) \ast \sigma(y)$, and if $x \lor y = 0$, then $\sigma(x \ast y) = \sigma(x) \ast \sigma(y)$.

(d) $\sigma(x \ast y) \geq \sigma(x) \ast \sigma(y)$, and if $y \leq x$, then $\sigma(x \lor y) = \sigma(x) \lor \sigma(y)$.

(e) Letting $d(x, y) = (x \lor y) \ast (y \lor x)$, we have $d(\sigma(x), \sigma(y)) \leq \sigma(d(x, y))$.

(f) $\sigma(x) \ast \sigma(y) \leq \sigma(x \lor y)$. Thus if $x \lor y = 0$, then $\sigma(x) \ast \sigma(y) = 0$.

(g) $\sigma(\sigma(x)) = \sigma(x)$.

(h) The image $\sigma(A)$ of $A$ under $\sigma$ is the domain of an MV-subalgebra of $A$.

Proof

(a) By $(\sigma 1)$ and $(\sigma 2)$.

(b) If $x \leq y$, then $y = x \oplus (y \ominus x)$, and hence $\sigma(y) = \sigma(x \oplus (y \ominus x))$. Since $x \ominus (y \ominus x) = 0$, by $(\sigma 3)$ we get $\sigma(y) = \sigma(x \oplus y) = \sigma(x) \oplus \sigma(y) \geq \sigma(x)$.

(c) By $(b)$, $\sigma(y) \geq \sigma(x \oplus (y \ominus x))$, whence $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \circ y)) \leq \sigma(x) \oplus \sigma(y)$. If $x \circ y = 0$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \circ y)) = \sigma(x) \oplus \sigma(y)$.

(d) Using $(\sigma 2)$, $(c)$ and the order-reversing property of $\ast$, we obtain: $\sigma(x \oplus y) = \sigma((x^\prime \oplus y^\prime)^\prime) = (\sigma(x^\prime \oplus y^\prime))^\prime \geq (\sigma(x^\prime))^\prime \ast \sigma(y^\prime)^\prime = ((\sigma(x))^\prime \ast \sigma(y))^\prime = \sigma(x) \ast \sigma(y)$. Moreover, if $y \leq x$, then $x^\prime \circ y = 0$. Hence again by $(c)$, $\sigma(x \oplus y) = (\sigma(x^\prime \oplus y^\prime))^\prime = (\sigma(x^\prime))^\prime \ast \sigma(y)^\prime = \sigma(x) \ast \sigma(y)$.

\* Panti’s theorem refers to the space of maximal ideals, but his result can be equivalently expressed in terms of maximal filters.
Lemma 4.2.
Proof. Let \( r \). Thus \( \sigma(x) \circ \sigma(y) = \sigma(x \circ y) \). If \( x \circ y = 0 \), then \( 0 = \sigma(x \circ y) \geq \sigma(x) \circ \sigma(y). \) Therefore, \( \sigma(x \circ y) = 0 \).

Theorem 4.1.
Conversely, given a \( (A, \sigma) \), we define
\[
\theta_1 = \{ (x, y) : (d(x, y))^\ast \in \bar{f} \}.
\]

Theorem 4.3
Now we are ready to prove the main result of this section.

Proof. The above defined maps are mutually inverse isomorphisms between the lattice of congruences of an SMV-algebra \( (A, \sigma) \), and the lattice of \( \bar{f} \)-filters of \( (A, \sigma) \).

Lemma 4.2.
Let \( (A, \sigma) \) be an SMV-algebra. Then the \( \bar{f} \)-filter \( \bar{f}_{\sigma(x)} \) generated by a single element \( \sigma(x) \in \sigma(A) \), is \( \bar{f}_{\sigma(x)} = \{ y \in A : \exists n \in \mathbb{N} (y \geq \sigma(x)^n) \} \).

Proof. Given \( \bar{f} \)

(a) If \( (A, \sigma) \) is a subdirectly irreducible SMV-algebra, then \( \sigma(A) \) is linearly ordered.
(b) If \( (A, \sigma) \) is faithful, then \( (A, \sigma) \) is subdirectly irreducible iff \( \sigma(A) \) is subdirectly irreducible (as an MV-algebra).

Proof

(a) Let \( \bar{b} \) be the smallest non-trivial \( \sigma \)-filter of \( (A, \sigma) \) and let \( x \in \bar{b} \setminus \{ 1 \} \). Suppose by contradiction that \( \sigma(A) \) is not linearly ordered, and let \( \sigma(a), \sigma(b) \in \sigma(A) \) be such that \( \sigma(a) \not\geq \sigma(b) \) and \( \sigma(b) \not\geq \sigma(a) \). Then the filters \( \bar{f}_{\sigma(a) \rightarrow \sigma(b)} \) and \( \bar{f}_{\sigma(b) \rightarrow \sigma(a)} \), generated by \( \sigma(a) \rightarrow \sigma(b) \) and \( \sigma(b) \rightarrow \sigma(a) \) respectively, are non-trivial. Hence they both contain \( \bar{b} \). In particular \( x \in \bar{f}_{\sigma(a) \rightarrow \sigma(b)} \) and \( x \in \bar{f}_{\sigma(b) \rightarrow \sigma(a)} \). Since \( \sigma(a) \rightarrow \sigma(b) \in \sigma(A) \), and \( \sigma(b) \rightarrow \sigma(a) \in \sigma(A) \), by Lemma 4.2 there is an \( n \in \mathbb{N} \) such that \( x \geq (\sigma(a) \rightarrow \sigma(b))^n \) and \( x \geq (\sigma(b) \rightarrow \sigma(a))^n \). Therefore, \( x \geq (\sigma(a) \rightarrow \sigma(b))^n \vee (\sigma(b) \rightarrow \sigma(a))^n = 1 \).

Hence \( n = 1 \) which is a contradiction.

(b) If \( (A, \sigma) \) is faithful, then by definition \( \sigma(x) = 0 \) implies \( x = 0 \), and hence \( \sigma(x) = 1 \) implies \( x = 1 \). It follows that the intersection of a non-trivial \( \sigma \)-filter \( \bar{b} \) of \( (A, \sigma) \) with \( \sigma(A) \), is a non-trivial \( \sigma \)-filter of \( A \). Moreover, every filter of \( \sigma(A) \) is closed under \( \sigma \). Then every MV-filter of \( \sigma(A) \) is indeed a \( \sigma \)-filter. Hence, if \( \bar{b} \) is a minimal \( \sigma \)-filter of \( (A, \sigma) \), \( \bar{b} \cap \sigma(A) \) is a minimal non-trivial \( \sigma \)-filter of \( \sigma(A) \). In fact if \( \bar{f} \) is another non-trivial filter of \( \sigma(A) \), the \( \sigma \)-filter \( \bar{f} \) of \( (A, \sigma) \) generated by \( \bar{f} \) contains \( \bar{b} \), and
\[
\bar{f} = \bar{f} \cap \sigma(A) \supseteq \bar{b} \cap \sigma(A).
\]
Hence $\mathfrak{h} \cap \sigma(A)$ is minimal. Therefore, if $(A, \sigma)$ is subdirectly irreducible, then so is $\sigma(A)$.

Conversely, if $\mathfrak{h}$ is the minimal non-trivial filter of $\sigma(A)$, then the $\sigma$-filter $f$ of $(A, \sigma)$ generated by $\mathfrak{h}$ is the minimal non-trivial $\sigma$-filter of $(A, \sigma)$. In fact if $\mathfrak{g}$ is another non-trivial $\sigma$-filter of $(A, \sigma)$, then $\mathfrak{g} \cap \sigma(A) \supseteq f \cap \sigma(A) = \mathfrak{h}$. Then $\mathfrak{g}$ contains the $\sigma$-filter generated by $\mathfrak{h}$, that is, $f \subseteq \mathfrak{g}$, i.e., $f$ is minimal. Thus $(A, \sigma)$ is subdirectly irreducible. $\Box$

**Remark 4.4.** The variety $\text{SMV}$ is not generated by its linearly ordered elements. To see this, one just notes that the equation $\sigma(x \lor y) = \sigma(x) \lor \sigma(y)$ is valid in any linearly ordered SMV-algebra, but does not hold in general. In fact take, in Example 3.2 (c), $f(x) = x$, and $g(x) = 1 - x$. Then $\sigma(f) = \sigma(g) = \frac{1}{2}$, and $\sigma(f \lor g) = \frac{1}{2} > \sigma(f) \lor \sigma(g) = \frac{1}{2}$.

Now we are ready to prove that the modal logic $\text{SFP}(L, L)$ introduced in the first section, is strongly complete with respect to the variety of SMV-algebras. Moreover, we shall show that an $\text{SFP}(L, L)$-formula $\phi$ is derivable in $\text{SFP}(L, L)$ from a (countable) set $\Gamma$ of formulas (and we write $\Gamma \vdash_{\text{SFP}} \phi$) iff $\phi$ holds in every SMV-algebra $(A, \sigma)$ being a model of $\Gamma$, and such that $\sigma(A)$ is a totally ordered MV-algebra.

**Theorem 4.5.** Let $\Gamma \cup \{\phi\}$ be a set of $\text{SFP}(L, L)$ sentences. The following are equivalent:

1. $\Gamma \vdash_{\text{SFP}} \phi$.
2. For every valuation $v$ into any SMV-algebra $(A, \sigma)$, if $v(\psi) = 1$ for every $\psi \in \Gamma$, then $v(\phi) = 1$.
3. For every valuation $v$ into any SMV-algebra $(A, \sigma)$ such that $\sigma(A)$ is totally ordered, if $v(\psi) = 1$ for every $\psi \in \Gamma$, then $v(\phi) = 1$.

**Proof**

(1) $\Rightarrow$ (3). The claim is proved by an easy induction on the length of the derivation of $\phi$ from $\Gamma$.

(3) $\Rightarrow$ (2). Arguing by way of contradiction, suppose that (2) does not hold. Thus for some SMV-algebra $(A, \sigma)$ and for some valuation $v$ into $(A, \sigma)$ we have $v(\psi) = 1$ for every $\psi \in \Gamma$ and $v(\phi) < 1$. Let us represent $(A, \sigma)$ as a subdirect product of subdirectly irreducible factors $(A_i, \sigma_i)_{i \in I}$. By Theorem 4.3, each algebra $(A_i, \sigma_i)$ is a totally ordered MV-algebra. Moreover, for some $i \in I$ we have $(v(\psi))_i < 1$ and $(v(\psi))_i = 1$ for all $\psi \in \Gamma$. Thus letting for every formula $\gamma$, $v(\gamma) = (v(\gamma))_i$, we have that $\mathfrak{v}$ is a valuation into $(A_i, \sigma_i)$ which contradicts claim (3).

(2) $\Rightarrow$ (1). Once again, arguing by way of contradiction, assume that $\phi$ is not derivable from $\Gamma$. Consider the Lindenbaum algebra $(A, \sigma)$ of the theory $\Gamma$ over $\text{SFP}(L, L)$. Then $(A, \sigma)$ is an SMV-algebra. For every formula $\gamma$, let $[\gamma]$ denote its equivalence class modulo provable equivalence in $\text{SFP}(L, L)$ plus $\Gamma$. Then $[\phi] < 1$, as $\phi$ is not derivable from $\Gamma$, and for $\phi \in \Gamma$, $[\phi] = 1$. Thus letting for every formula $\gamma$, $v(\gamma) = [\gamma]$, we have that $v$ is a valuation in $(A, \sigma)$ which contradicts (2). $\Box$

5. SMV-algebras and states on MV-algebras

In this section we relate the two notions of SMV-algebras and states. Among others, we shall show that, starting from an SMV-algebra $(A, \sigma)$, one can define a state $s$ on the MV-algebra $A$. Vice versa starting from a state $s$ on an MV-algebra $A$, we shall build an MV-algebra $T$ containing $A$ as MV-subalgebra, and an internal state $\sigma$ on $T$.

Let us start with an SMV-algebra $(A, \sigma)$. By Lemma 3.3 (b), $\sigma(A) \oplus \sigma^*$, where $\oplus$ and $\oplus^*$ respectively denote the restrictions of the MV-algebraic operations of $A$ to $\sigma(A)$ is an MV-subalgebra of $A$. If $m$ is a maximal filter on $\sigma(A)$, then the quotient MV-algebra $\sigma(A)/m$ is simple, and hence it is embeddable into the standard MV-algebra $[0, 1]_{\text{MV}}$ (recall Section 2 and see [6] for further details). Call $i : \sigma(A)/m \rightarrow [0, 1]_{\text{MV}}$ such an embedding, and let $\eta_\sigma : A \rightarrow \sigma(A)/m$ be the canonical MV-homomorphism induced by the ultrafilter $m$. Finally, let us call $s$ that map obtained by the composition $i \circ \eta_\sigma : A \rightarrow [0, 1]_{\text{MV}}$.

Then $s$ is a state on $A$ as the following theorem shows:

**Theorem 5.1.** Let $(A, \sigma)$ be any SMV-algebra, and let $s : A \rightarrow [0, 1]_{\text{MV}}$ be defined as above. Then $s$ is a state on $A$.

**Proof.** Given that $\sigma(1) = 1$ and $\eta_\sigma$ preserve 1, it is clear that $s(1) = 1$. To show that $s$ is additive, let $x, y \in A$ be such that $x \odot y = 0$. Thus by Lemma 3.3 (c), one has $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$. Moreover, by the same lemma (f), $\sigma(x) \odot \sigma(y) = 0$, thus $s(x) \odot s(y) = 0$. Hence $s(x \odot y) = s(x) \odot s(y) = s(x) + s(y) - s(x) \odot s(y) = s(x) + s(y)$. $\Box$

Conversely, we shall obtain an SMV-algebra from an MV-algebra equipped with a state. To this purpose, recall that the tensor product $A_1 \otimes A_2$ of two MV-algebras $A_1$ and $A_2$ is an MV-algebra (unique up to isomorphism) such that there is a universal bimorphism $\beta$ from the cartesian product $A_1 \times A_2$ into $A_1 \otimes A_2$ (cf. [24, Definition 2.1] for the concept of bimorphism). Universal means that for every (other) bimorphism $\beta' : A_1 \times A_2 \rightarrow B$ ($B$ being an MV-algebra) there exists a unique homomorphism $\lambda : A_1 \otimes A_2 \rightarrow B$ such that $\beta' = \lambda \circ \beta$.

In the following we shall only consider tensor products of the form $T = [0, 1]_{\text{MV}} \otimes A, [0, 1]_{\text{MV}}$ and $A$ being the standard MV-algebra and an MV-algebra, respectively. Henceforth, for $a \in [0, 1]$ and $a \in A$, we shall denote $\beta(a, a)$ by $a \odot a$.

**Proposition 5.2.** Let $T = [0, 1]_{\text{MV}} \otimes A$. Then the following conditions hold for any $a, a_1, a_2 \in [0, 1]$ and any $a, a_1, a_2 \in A$.
(a) \( (x_1 \otimes x_2) \otimes 1 = (x_1 \otimes 1) \otimes (x_2 \otimes 1), \) and \( 1 \otimes (a_1 \otimes a_2) = (1 \otimes a_1) \otimes (1 \otimes a_2). \)
(b) \((x \otimes 1)^t = (1 - x) \otimes 1, \) and \((1 \otimes a)^t = 1 \otimes a^t.\)
(c) The maps \( x \rightarrow (x \otimes 1) \) and \( a \rightarrow (1 \otimes a) \) are, respectively, embeddings of \([0, 1]_{\text{MV}}\) and \( A \) into \( T.\)
(d) If \( x_1 \otimes x_2 = 0, \) then \((x_1 + x_2) \otimes a = (x_1 \otimes 0) \otimes (x_2 \otimes a), \) and if \( a_1 \otimes a_2 = 0, \) then \( x \otimes (a_1 \otimes a_2) = (x \otimes a_1) \otimes (x \otimes a_2). \)
(e) \( x \otimes (a_1 \otimes a_2) = (x \otimes a_1) \otimes (x \otimes a_2), \) and \((x_1 \otimes x_2) \otimes a = (x_1 \otimes a) \otimes (x_2 \otimes a). \)
(f) \( 1 \otimes 1 \) is the top element of \( T, \) while for every \( a \in A \) and every \( x \in [0, 1], \) \( 0 \otimes a \) and \( x \otimes 0 \) coincide with the bottom element of \( T.\)

**Proof.** The proof follows from the tensor product construction of \([0, 1]_{\text{MV}} \otimes A\) (cf. [24, Section 3]).

Due to **Proposition 5.2** (c), for any \( x \in [0, 1] \) and without any danger of confusion, we shall sometimes denote \( x \otimes 1 \) by \( x.\)

Let now \( s : A \rightarrow [0, 1] \) be a state, and let \( T = [0, 1]_{\text{MV}} \otimes A \) be the MV-algebra defined as above. Then consider the unary operation \( \sigma : T \rightarrow T \) to be so defined: for each \( x \otimes a \in T, \)

\[ \sigma(x \otimes a) = x : s(a). \]

Notice that \( \sigma \) maps \( T \) into \( T, \) and hence \( \sigma \) is a unary operation on \( T. \) Moreover:

**Theorem 5.3.** Let \( s, T \) and \( \sigma \) be defined as above. Then \( \sigma \) is well defined, and \((T, \sigma)\) is an SMV-algebra.

**Proof.** Let \((\varphi, u) = \Gamma^{-1}(A)\) be the \( \varphi \)-group with unit \( u \) corresponding to \( A \) and let \( h \) be the state on \( \varphi \) as in **Proposition 2.9** (1). In turn, let \( \varphi^d \) the divisible extension of \( \varphi, \) and let \( h^d : \varphi^d \rightarrow \mathbb{R} \) be the unique state extending \( h. \) Finally, let \( \Lambda^d = \Gamma(\varphi^d, u) \) and let \( s^d \) be the restriction of \( h^d \) to \( \Lambda^d. \) By **Proposition 2.9** (2) \( s^d : \Lambda^d \rightarrow [0, 1] \) is a state on \( \Lambda^d \) extending \( s. \) Let now \( T^d \) be the MV-algebra \([0, 1]_{\text{MV}} \otimes \varphi \otimes \mathbb{Q} \) on \( \Lambda^d. \) Then

**Claim 1**

The map \( \lambda : q \otimes a \rightarrow qa \) is a homomorphism from \( T^d \) into \( A.\)

As a matter of fact, let \( \beta : ([0, 1] \otimes \mathbb{Q}) \times A \rightarrow A^d \) be defined as: \( \beta(q, a) = qa. \) Then \( \beta \) enjoys the following properties for each \( q, q_1, q_2 \in [0, 1] \otimes \mathbb{Q}, \) and each \( a, a_1, a_2 \in A: \)

(i) \( \beta(1, 1) = 1. \) In fact \( \beta(1, 1) = 1 \cdot 1 = 1. \)

(ii) \( \beta(0, a) = \beta(x, 0) = 0. \)

(iii) \( \beta(q, a_1 \otimes a_2) = \beta(q, a_1) \otimes \beta(q, a_2). \) In fact \( \beta(q, a_1 \otimes a_2) = q(a_1 \otimes a_2) = qa_1 \otimes qa_2. \) Analogously it can be shown that \( \beta(q, a_1 \otimes a_2) = \beta(q, a_1) \otimes \beta(q, a_2), \) \( \beta(q \otimes q_2, a) = \beta(q_1, a) \otimes \beta(q_2, a), \) and \( \beta(q_1 \otimes q_2, a = \beta(q_1, a) \otimes \beta(q_2, a). \)

(iv) If \( a_1 \otimes a_2 = 0, \) then \( \beta(q, a_1) \otimes \beta(q, a_2) = 0. \) In fact \( \beta(q, a_1) \otimes \beta(q, a_2) = qa_1 \otimes qa_2 = a_1 \otimes a_2 = 0. \) Moreover, since \( -\) distributes on \( \otimes, \) one also has \( \beta(q, a_1 \otimes a_2) = \beta(q, a_1) \otimes \beta(q, a_2). \)

(v) If \( q_1 \otimes q_2 = 0, \) then \( \beta(q_1, a) \otimes \beta(q_2, a) = 0, \) and \( \beta(q_1 \otimes q_2, a = \beta(q_1, a) \otimes \beta(q_2, a). \) This can be easily shown using the same argument of \( (iv). \)

This means that \( \beta \) is a bimorphism (in the sense of [24]). Therefore, there is a homomorphism \( \lambda' \) such that \( \lambda'(q \otimes a) = qa \) for all \( q \in [0, 1] \otimes \mathbb{Q}, \) and for all \( a \in A. \) Thus \( \lambda' = \lambda \) and \( \lambda \) is a homomorphism as required. The claim is settled.

Turning back to the proof of **Theorem 5.3**, we can define an internal state \( \sigma^d \) in \( T^d, \) as follows: \( \sigma^d(q \otimes a) = s^d(\lambda(q \otimes a)). \)

Note that \( \sigma^d \) is well defined. In fact, if \( q \otimes a = r \otimes b, \) then \( \lambda(q \otimes a) = \lambda(r \otimes b). \) Therefore, \( \sigma^d(q \otimes a) = s^d(\lambda(q \otimes a)) = s^d(\lambda(1 \otimes 1)) = s^d(1) = s(1) = 1. \) Moreover, if \( q \otimes a = r \otimes b = 0, \) then \( \lambda(q \otimes a) = \lambda(r \otimes b) = 0, \) and hence \( \sigma^d(q \otimes a) = s^d(\lambda(q \otimes a)) = s^d(0) = 0. \)

(c) The verification of the axioms of SMV-algebras is almost immediate once \( (T^d, \sigma^d) \) is an SMV-algebra, and \( \sigma : T \rightarrow T \)

extends \( \sigma^d \) by continuity. \( \square \)

### 5.1. Characterizing coherence within SMV-algebras

**Theorem 5.1** and **Theorem 5.3**, allow us to treat the coherence problem for a rational assessment of formulas of Lukasiewicz logic inside the theory of SMV-algebras.

The main result of this section states that, if we restrict to rational assessments, then the coherence problem can be equationally characterized in the theory of SMV-algebras.
First of all recall that Łukasiewicz logic is algebraizable in the sense of Blok and Pigozzi (cf. [3]), whence the formulas \( \varphi_1, \ldots, \varphi_n \) can be regarded as terms in the language of MV-algebras. Let us now assume that all the \( x_i \) are rational numbers, say \( x_i = \frac{m_i}{n_i} \). Moreover, let \( x_1, \ldots, x_n \) be fresh variables, and consider for each \( i = 1, \ldots, n \), the equations:

\[ e_i : (m_i - 1)x_i = x'_i, \quad \text{and} \quad \delta_i : \sigma_i = n_ix_i. \]

Then we can prove the following:

**Theorem 5.4.** Let \( \chi : P(\varphi_i) = \frac{m_i}{n_i} \) be a rational assessment of the Łukasiewicz formulas \( \varphi_1, \ldots, \varphi_n \). Then the following are equivalent:

- (a) \( \chi \) is coherent.
- (b) The equations \( e_i \) and \( \delta_i \) (for \( i = 1, \ldots, n \)) are satisfied in some non-trivial SMV-algebra.

**Proof.** By Theorem 1.1 it is sufficient to prove that (b) is equivalent to the existence of a state \( s \) on \( \mathcal{F}(k) \) such that, for all \( i = 1, \ldots, n \), \( s(\varphi_i) = \frac{n_i}{m_i} \).

(a) \( \Rightarrow \) (b). Let \( s \) be a state on \( \mathcal{F}(\omega) \) extending \( \chi \). Recalling the tensor product construction (see Theorem 5.3), let \( T = [0, 1]_{\text{MV}} \otimes \mathcal{F}(\omega) \), and let \( \sigma : T \rightarrow T \) be defined for \( x \otimes [\psi] \in T \), by \( \sigma(x \otimes [\psi]) = x \cdot s([\psi]) \). Given that \( 1 \otimes [\varphi_i] \in T \), and \( \sigma(1 \otimes [\varphi_i]) = s([\varphi_i]) \) (for each \( i = 1, \ldots, n \)) it is clear that \( \sigma \) extends \( s \) (up to isomorphism).

Let now \( v \) be an evaluation on \( (T, \sigma) \) such that \( v(x_i) = \frac{m_i}{n_i} \) for each \( i = 1, \ldots, n \) (notice that \( \frac{1}{m_i} = \frac{1}{m_i} \otimes [1] \in T \), whence \( v \) is an evaluation on \( (T, \sigma) \)). Then \( v \) satisfies the equations \( e_i \) because:

\[ (m_i - 1)v(x_i) = \frac{m_i - 1}{m_i} = 1 - \frac{1}{m_i} = v(x'_i). \]

Moreover \( v \) satisfies the equations \( \delta_i \) because:

\[ \sigma(1 \otimes [\varphi_i]) = 1 \cdot s([\varphi_i]) = s([\varphi_i]) = \frac{n_i}{m_i} = n_i v(x_i). \]

Thus the equations \( e_i \) and \( \delta_i \) are satisfied in a non-trivial SMV-algebra as required.

(b) \( \Rightarrow \) (a). Let now \( (A, \sigma) \) be an SMV-algebra, and let \( v \) be an evaluation on \( (A, \sigma) \) satisfying the equations \( e_i \) and \( \delta_i \) for each \( i = 1, \ldots, n \). Without loss of generality we may assume the MV-algebra \( A \) to be finitely (or even countably) generated, so that there is an epimorphism \( h_v : \mathcal{F}(\omega) \rightarrow A \) such that \( h_v([\psi]) = v([\psi]) \) for every propositional variable \( x \). Then:

\[ (m_i - 1)h_v(x_i) = (h_v(x'_i)), \quad \text{and} \quad \sigma(h_v([\varphi_i])) = n_i h_v([x_i]). \]

As in the proof of Theorem 5.1, let \( m \) be a maximal MV-filter of \( \sigma(A) \), and define, for each \( [\psi] \in \mathcal{F}(\omega) \), \( s([\psi]) = \sigma(h_v([\psi]))/m \). Since quotients preserve identities, one has

\[ s([\varphi_i]) = (n_i h_v([x_i]))/m, \quad \text{and} \quad (m_i - 1)/(h_v([x_i]) = (h_v([x_i])/m). \]

Hence the MV-homomorphism \( h_v : \sigma(A)/m \rightarrow [0, 1]_{\text{MV}} \) maps \( h_v([x_i]) \) in \( \frac{m_i}{n_i} \), and \( s([\varphi_i]) \) in \( \frac{n_i}{m_i} \). Now there remains to be proved that \( s \) is a state. First of all it is clear that \( s(1) = 1 \). As to additivity, let \( [\psi_1], [\psi_2] \in \mathcal{F}(\omega) \) such that \( [\psi_1] \cap [\psi_2] = 0 \). Then:

\[ s([\psi_1] \cap [\psi_2]) = (s(h_v([\psi_1])) \cap h_v([\psi_2]))/m = (s(h_v([\psi_1])) \cap s([\psi_2]))/m = s([\psi_1]) \cap s([\psi_2]). \]

(where the last equality follows from the fact that, if \( [\psi_1] \cap [\psi_2] = 0 \), then \( h_v([\psi_1]) \cap h_v([\psi_2]) = 0 \) in \( A \), and so \( s([\psi_1]) \cap s([\psi_2]) = (\sigma(h_v([\psi_1]))) \cap \sigma(h_v([\psi_2]))/m = 0/m = 0 \), and \( s([\psi_1]) \cap s([\psi_2]) = s([\psi_1]) \cap s([\psi_2]) \)). Hence \( s \) is a state on \( \mathcal{F}(\omega) \), and it extends the assessment \( \chi \). Thus \( \chi \) is coherent. \( \square \)

6. **Lebesgue integral on MV-algebras**

In this section we propose an algebraic treatment of the Lebesgue integral. This generalization is obtained as follows:

(a) Instead of the real field, we consider a divisible and totally ordered \( \ell \)-group \( \mathbb{R} = (\mathbb{R}, \leq, +, -) \) with a unit \( u \). Therefore, the structure we are considering need not have a multiplication, and need not be complete with respect to the order. Anyway, if we interpret \( u \) as 1, then we have a copy of rational numbers in \( \mathbb{R} \); the rational \( \frac{m}{n} \) is identified by \( \pm(\frac{m}{n}) \). Moreover, in \( \mathbb{R} \) we can define multiplication by a rational number: for each \( x \in \mathbb{R} \), and for each \( \frac{m}{n} \in \mathbb{Q} \), \( \pm(\frac{m}{n})x \) can be identified by \( \pm(\frac{m}{n})x \). Hence we may assume without loss of generality that the ordered group \( (\mathbb{Q}, \leq, +, -) \) is an ordered subgroup of \( \mathbb{R} \). In particular, the unit \( u \) will be henceforth denoted by 1. Moreover, \( \mathbb{R} \) can be regarded as a vector space over the rational field \( (\mathbb{Q}, +, -) \), \( 0, 1 \).

(b) Instead of the usual measure on the reals, we have a \( \mathbb{R} \)-valued measure \( \mu \) from a Boolean algebra \( B \), that is, a map \( \mu : \mathcal{B} \rightarrow [0, 1] \) such that \( \mu(1) = 1 \), and, if \( a \wedge b = 0 \), then \( \mu(a \vee b) = \mu(a) + \mu(b) \). By Stone representation theorem, \( B \) can be identified with the family of clopen subsets of the topological space \( U(B) \), of ultrafilters of \( B \), equipped with the spectral topology (in our picture the elements of \( B \) represent \( \mu \)-measurable subsets of \( U(B) \)). We shall use the
notation \( m, n, \ldots \) to denote arbitrary ultrafilters of \( U(B) \). When \( B \) is clear from the context, we shall write \( U \) instead of \( U(B) \).

An element \( x \in G \) is said to be bounded if there is a rational \( q > 0 \) such that \( |x| \leq q \) (\( |x| \) standing for \( x \lor \neg x \)). Every bounded element \( x \in G \) has a standard part \( \text{st}(x) \) defined by \( \text{st}(x) = \sup \{ q \in \mathbb{Q} : q < x \} = \inf \{ q \in \mathbb{Q} : x \leq q \} \), where infima and suprema refer to the reals.

To each \( G \cap [0,1] \)-valued measure on \( B \) we can associate a \( [0,1] \cap \mathbb{R} \)-valued measure \( \mu_{st} \) letting \( \mu_{st}(b) = \text{st}(\mu(b)) \).

To simplify our description, we restrict our attention to the set of functions \( f \) which are bounded, that is, there is \( q \in \mathbb{Q} \), \( q > 0 \), such that for all \( m \in U \), \( |f(m)| \leq q \).

**Definition 6.1.** Let \( U \in \mathcal{U} \) be a divisible \( \ell \)-group with a unit, let \( B \) be a boolean algebra, and let \( U \) be the set of ultrafilters of \( B \).

Then a function \( h : U \rightarrow G \cap [0,1] \) is said to be basic if there are a partition \( X_1, \ldots, X_n \) of \( U \), with \( \chi_1, \ldots, \chi_n \in B \), and mutually distinct rationals \( q_1, \ldots, q_n \), such that for \( i = 1, \ldots, n \) and for \( m \in X_i \), one has \( h(m) = q_i \). Then we define the integral of \( h \) as

\[
I(h) = \text{st} \left( \sum_{i=1}^{n} \mu(X_i)q_i \right) = \sum_{i=1}^{n} \mu(X_i)q_i.
\]

By definition \( I(h) \) is a real number. The second equality of (1) follows from the fact that the standard part function \( \text{st} : G \to \mathbb{R} \) is linear.

Now let \( f : U \rightarrow G \cap [0,1] \) be a bounded function and let \( F^+ \) (\( F^+ \), respectively) denote the set of all basic functions \( h : U \rightarrow G \cap [0,1] \) such that \( h(m) \leq f(m) \) for all \( m \in U \) \( (h(m) \geq f(m) \) for all \( m \in U \), respectively), and let \( I^+(f) = \sup \{ I(h) : h \in F^+ \} \) and \( I^-(f) = \inf \{ I(h) : h \in F^- \} \).

**Definition 6.2.** We say that \( f \) is Lebesgue-integrable iff \( I^-(f) = I^+(f) \). In this case we define the integral

\[
\int f \, d\mu
\]

to be the common value \( I^-(f) = I^+(f) \).

**Remark 6.3.** If \( f \) is Lebesgue integrable, then \( \int f \, d\mu \) is a real number, but possibly not an element of \( G \). The Lebesgue integral is a linear and weakly monotonic functional, in the sense that for every \( q, r \in \mathbb{Q} \) and for every pair \( f, g \) of integrable functions, we have that \( qf + rg \) is integrable and \( \int (qf + rg) \, d\mu = q \int f \, d\mu + r \int g \, d\mu \), and if \( f \leq g \) for all \( m \in U \), then \( \int f \, d\mu \leq \int g \, d\mu \).

**Definition 6.4.** A function \( f : U \rightarrow G \cap [0,1] \) is said to be measurable if it is bounded and for every \( q \in \mathbb{Q} \), the sets \( U_{f < q} = \{ m \in U : f(m) < q \} \) and \( U_{f = q} = \{ m \in U : f(m) = q \} \) are measurable (that is, they are elements of the boolean algebra \( B \)).

Since measurable sets are closed under the boolean operations, it follows that if \( f \) is measurable, then for all \( r, q \in \mathbb{Q} \), the set \( U_{f < q} \cap U_{f > r} = \{ m \in U : q < f(m) < r \} \) is measurable.

**Lemma 6.5.** With the same terminology of Definitions 6.1 and 6.4, every measurable function \( f \) is Lebesgue integrable.

**Proof.** It suffices to show that for every rational \( \varepsilon > 0 \) there are \( h \in F^- \) and \( k \in F^+ \) such that \( I(k) - I(h) < \varepsilon \). Let \( q \in \mathbb{Q} \) be such that for all \( m \in U \) we have \( -q < f(m) < q \). Let \( n \) be a natural number such that \( \frac{2q}{n} < \varepsilon \) and let for \( i = 0, \ldots, n \), \( a_i = -\frac{q}{2^n} + \frac{i}{2^n} \). Define \( h(m) \) and \( k(m) \) as follows: let \( m \in U \) be given and let \( i(m) \) be the unique integer \( i \) with \( 0 \leq i < n \) such that \( a_i < f(m) < a_{i+1} \). Define now \( h(m) = a_i \) and \( k(m) = a_{i+1} \), it follows that \( h(m) \leq f(m) \leq k(m) \). Moreover, for every \( m \in U \), we have \( k(m) - h(m) = \frac{2q}{n} \). Thus

\[
I(k) - I(h) = \frac{2q}{n} \sum_{i=0}^{n-1} \mu(U_{f \in [a_i, a_{i+1})}) = \frac{2q}{n} < \varepsilon. \quad \square
\]

We want to introduce a completely algebraic treatment of Lebesgue integration of bounded functions. For simplicity, we assume that our functions are \([0,1] \cap G\)-valued. This is not a strong restriction: modulo a linear transformation, every bounded function from \( U \) into \([0,1] \cap G\) can be transformed into a function from \( U \) into \([0,1] \cap G\). In our picture, an element \( a \) of \([0,1] \cap G\) is represented by the function which is constantly equal to \( a \).

We are going to prove that all the structures we need, that is, the boolean algebra \( B \), the set \( U \) of its ultrafilters, the divisible and totally ordered group \( G \) with unit \( u \), the measure \( \mu \), the functions from \( U \) into \([0,1] \cap G\) and their integrals, can be reduced to a unique type of algebraic structure, namely, to divisible SMV\(\Lambda\)-algebras, which will be introduced below.

### 6.1. Integral representation of divisible SMV\(\Lambda\)-algebras

**Definition 6.6.** An SMV\(\Lambda\)-algebra is an algebra \((A, \oplus, *, \Delta, 0, 1) \) whose \( \Delta \)-free reduct is an MV-algebra, and \( \Delta \) (cf. [1]) is a unary operator on \( A \) satisfying the following identities:
Any MV_\alpha-algebra A can be regarded as an algebra of functions from a compact Hausdorff space into the unit interval [0,1]^* of a hyperreal field. The representation is as follows: the set \Delta(A) is the domain of a subalgebra of A which is a boolean algebra. Now take its dual space (U, T) (which is a compact Hausdorff space), where \( U = U(\Delta(A)) \) is the set of ultrafilters of \( \Delta(A) \) and \( T \) is the topology generated by all sets of the form \( C_a = \{ m \in U : a \in m \} \) for \( a \in \Delta(A) \). For every \( m \in U \) there is a unique ultrafilter \( m' \) of \( A \) extending \( m \), namely \( m' = \{ x \in A : \Delta(x) \in m \} \). Then consider the quotient \( A/m' \) of \( A \) modulo \( m' \). We can construct an extension \([0,1]^*\) of the standard MV-algebra \([0,1]_{MV}\) such that for every ultrafilter \( m' \) of \( A, A/m' \) embeds into \([0,1]^*\). One may prove this using the fact that the class of MV-chains has the amalgamation property which, in turns, follows from the amalgamation property of totally ordered abelian groups (cf. [30, Corollary 2.2]), via Mundici’s functor \( T \) (cf. [22]).

We can associate to each \( a \in A \) the function \( f_a \) on \( U \) defined for \( m \in U \) by \( f_a(m) = a/m' \) (the equivalence class of \( a \) modulo the congruence determined by the unique ultrafilter \( m' \) of \( A \) extending \( m \)). Operations on these functions are defined componentwise. The elements of \( B = \Delta(A) \) correspond to the \( \{0,1\} \)-valued functions.

With respect to Di Nola’s representation, we have the following advantages: (1) the elements of \( B \) are precisely those of the form \( \Delta(x) \), whence we have a very simple way to express them; (2) in the case of MV_\alpha-algebras, the topological space \((U, T)\) is compact and totally disconnected.

In any MV-algebra we can simulate sum, because \( \oplus \) is a truncated sum, but we cannot simulate rationals and multiplication by a rational. To allow multiplication by rationals we shall use DMV-algebras and their states.

\begin{itemize}
  \item[(1)] \( (A, \oplus, *, 0, \Delta, \{\delta_n\}_{n \in \mathbb{N}}, \sigma) \) is an MV_\alpha-algebra.
  \item[(2)] \( (A, \oplus, *, 0, \delta_n)_{n \in \mathbb{N}} \) is a DMV-algebra.
  \item[(3)] \( (A, \oplus, *, \sigma, 0) \) is an SMV-algebra satisfying the following equation:
\end{itemize}
\[ \sigma(\Delta(\sigma(x))) = \Delta(\sigma(x)). \]

**Lemma 6.8.** Let \( A \) be an SDMV_\alpha-algebra. Then:

(a) If \( q \) is a rational in \([0,1]\) and \( f \in A \), then \( \sigma(qf) = q\sigma(f) \).
(b) The set \( \sigma(A) = \{ \sigma(x) : x \in A \} \) is (the domain of) a divisible MV_\alpha-subalgebra of \( A \) which is closed under \( \sigma \).

**Proof**

(a) Let \( q = 0 \) or \( q = 1 \), the claim is obvious (note that \( \sigma(0) = 0 \) follows from \((\sigma 1)\) and \((\sigma 2)\)). Now suppose \( q = \frac{m}{n} \) with \( 0 < m < n \). Then using \((\sigma 2)\) and the fact that for \( i + j \leq n \) we have \((i)\delta_n(x) \oplus (j)\delta_n(x) = 0 \), we get that \( \sigma(x) \oplus \sigma(\delta_n(x)) = \sigma(x \oplus \delta_n(x)) = \sigma((n - 1)\delta_n(x)) = (n - 1)\sigma(\delta_n(x)) \). Thus \( \sigma(\delta_n(x)) = \delta_n(\sigma(x)) \), whence \( \sigma(m)\delta_n(x) = (m)\delta_n(\sigma(x)) \), as desired.

(b) We already know that \( \sigma(A) \) is closed under \( \oplus, * \) and \( \sigma \) (recall Lemma 3.3). Moreover we have just proved that \( \sigma(\delta_n(x)) = \delta_n(\sigma(x)) \), and hence \( \sigma(A) \) is closed under \( \delta_n \). Now there remains to be proved that \( \sigma(A) \) is closed under \( \Delta \).

\[ \text{Axiom \((\sigma 5)\) states that if } a \in \sigma(A), \text{ say } a = \sigma(x) \text{ for some } x, \text{ then } \Delta(a) = \sigma(\Delta(a)), \text{ whence } \Delta(a) \in \sigma(A). \]

In our representation of MV_\alpha-algebras as algebras of functions, an element \( f \) of \( \Delta(A) \) represents the characteristic function of the set \( Z_f \) of all \( m \in U \) such that \( f(m) = 1 \). Our idea is that, if \( f \in \Delta(A) \), then \( \sigma(f) \) should represent the measure \( \mu(Z_f) \) of \( Z_f \) and if \( f \) is an arbitrary element of \( A \), then \( \sigma(f) \) should represent \( \int f d\mu \). When trying to formalize this idea, we meet a problem: in general, \( \sigma(A) \) is not totally ordered, whereas the set of integrals, being a set of reals, is totally ordered. Even worse, in Di Nola’s representation of \( A \), the elements of \( \sigma(A) \) need not be constant. We shall show that these problems do not occur if \( A \) is subdirectly irreducible.

We start from the following:

**Definition 6.9.** A \((\sigma, \Delta)\)-filter of an SDMV_\alpha-algebra is a filter of its MV-reduct which is closed under \( \sigma \) and \( \Delta \).

Let \( (A, \sigma) \) be an SDM\( V_\alpha \)-algebra. Then:
Lemma 6.10

(1) The maps $0 \mapsto f_0$, associating to each congruence $\theta$ the set $f_\theta = \{ x \in A : (x, 1) \in \theta \}$ and $f_i \mapsto f_0$ mapping each $\{ \sigma, A \}$-filter $i$ into $\theta_i = \{ (x, y) \in A \times A : x \rightarrow y \in f_i \}$ are mutually inverse homomorphisms between the congruence lattice and the $\{ \sigma, A \}$-filter lattice of $(A, \sigma)$.

(2) The $\{ \sigma, A \}$-filter generated by an element $\sigma(a) \in \sigma(A)$ is the set $\{ x : \Delta(\sigma(a)) \leq x \}$.

Proof

(1) By Theorem 4.1, it is sufficient to prove that a $\sigma$-filter $i$ is a $\{ \sigma, A \}$-filter iff $\theta_i$ is a congruence of $(A, \sigma)$.

($\Rightarrow$): Suppose that $i$ is a $\{ \sigma, A \}$-filter. If $(x, y) \in \theta_i$, then for every $n \in \mathbb{N}$, $\delta_n(x) \rightarrow \delta_n(y) \supseteq x \rightarrow y \in i$. Thus $\theta_i$ is compatible with $\delta_n$ for each $n \in \mathbb{N}$. Moreover, if $(x, y) \in \theta_i$, then $\Delta(x \rightarrow y) \in f_i$, as $i$ is closed under $\Delta$. Since $\Delta x \rightarrow x y \subseteq \Delta(x \rightarrow y)$, $\Delta x \rightarrow x y \in f_i$, and $\Delta x \Delta y \in \theta_i$. Thus $\theta_i$ is also compatible with $\Delta$ and is a congruence of $(A, \sigma)$.

($\Leftarrow$): Suppose that $\theta_i$ is a congruence of $(A, \sigma)$. Then $\theta_i$ is the SMV-reduct of $(A, \sigma)$, whence $i$ is a filter closed under $\sigma$. Finally, if $x \in f_i$, then $(x, 1) \in \theta_i$ and $(\Delta x, 1) \in \theta_i$ as $\theta_i$ is compatible with $\Delta$. Thus $\Delta x \in f_i$ and $i$ is closed under $\Delta$.

(2) Let $h = \{ x : \Delta(\sigma(a)) \leq x \}$, then the $\{ \sigma, A \}$-filter generated by $\sigma(a)$ must contain $\Delta(\sigma(a))$, and hence it must contain $h$.

For the opposite direction, it suffices to show that if $h$ is a filter containing $\sigma(a)$ and closed under $\Delta$ and under $\sigma$. That $\sigma(a) \in h$ follows from the condition $\Delta(x) \leq x$. That $h$ is upwards closed is trivial, and that $h$ is closed under $\sigma$ follows from the fact that $\Delta(x) \circ \Delta(x) = \Delta(x)$. Closure under $\Delta$ follows from the condition $\Delta(\Delta(x)) = \Delta(x)$, and closure under $\sigma$ follows from condition ($\sigma$5). □

As usual, we shall interpret the elements of $A$ as functions from the set $U$ of ultrafilters of $\Delta(A)$ into some non-standard interval $[0, 1]$. Note that all MV-operations, as well as $A$ and the operations $\delta_n$, are defined componentwise, while $\sigma$ is not, because a congruence of the underlying MV$_A$-algebra need not be a congruence of $A$.

Lemma 6.11. Let $(A, \sigma)$ be a subdirectly irreducible SDMV$_A$-algebra. Then:

(1) $\sigma(A)$ is linearly ordered.

(2) Let $\mathcal{G}$ be the unique totally ordered abelian group with unit 1 such that the MV-reduct of $A$ is isomorphic to $\Gamma(\mathcal{G}, 1)$. Then the map $\mu$ on $\Delta(A)$ defined, for $\Delta(x) \in \Delta(A)$, by $\mu(\Delta(x)) = \sigma(\Delta(x))$, is a measure on $\Delta(A)$ taking values in $G \cap [0, 1]$.

(3) For every element $f \in A$ (represented as a function from the set of maximal $\Delta$-filters of the MV$_A$-reduct of $A$), $st(\sigma(f))$ is constant.

(4) Every $f \in A$ is a measurable function, whence it is Lebesgue integrable (in the sense of Definition 6.2).

Proof

(1) Let $f$ be the smallest non-trivial $\{ \sigma, A \}$-filter of $(A, \sigma)$. Let $c \in f$, and $c < 1$. Suppose by contradiction that $\sigma(a) \in f$ and $\sigma(b) \in f$ are incomparable with respect to the order. Then by Lemma 6.10, the filter generated by $\sigma(a) \rightarrow \sigma(b)$ is $f = \{ x : \Delta(\sigma(a) \rightarrow \sigma(b)) \leq x \}$. Moreover, such filter is non-trivial, and hence $c \in f$, and $\Delta(\sigma(a) \rightarrow \sigma(b)) \leq c$. Similarly we can prove that $\Delta(\sigma(b) \rightarrow \sigma(a)) \leq c$. Hence $1 = \Delta(\sigma(a) \rightarrow \sigma(b)) \vee \Delta(\sigma(b) \rightarrow \sigma(a)) \leq c$, which is a contradiction.

(2) This follows easily from ($\sigma$1) and ($\sigma$3).

(3) We have $\sigma(1) = 1$, $\sigma(0) = 0$ and for $0 < m < n$, $\sigma(m) = (m)\delta_n(x)$. It follows immediately that for every rational $q \in [0, 1]$, $\sigma(q) = q$, and hence $q \in \sigma(A)$. Since $\sigma(A)$ is linearly ordered, for every $f \in A$ and for every $q \in [0, 1]$ we either have that $q \leq \sigma(f)$, or $\sigma(f) \leq q$. Thus if we interpret $q$ as the constant function $q(m)$ on $U$ which is equal to $q$ on each $m \in U$, we either have that for all $m \in U$, $q = q(m) \leq \sigma(f)(m)$, or for all $m \in U$, $\sigma(f)(m) \leq q(m) = q$. Thus $st(\sigma(f)(m))$ is constantly equal to $sup(q \in [0, 1] : q \leq \sigma(f)) = inf(q \in [0, 1] : \sigma(f) \leq q)$.

(4) Let $q \in [0, 1]$. Then $U_{f \mapsto q} = \Delta(f \rightarrow q) \wedge (\Delta(q \rightarrow f))$, and $U_{f \mapsto q} = \Delta(f \rightarrow q) \wedge (\Delta(q \rightarrow f))$. Since $\Delta(A)$ is closed under all MV-operations, we have that $U_{f \mapsto q}$ and $U_{f \mapsto q}$ belong to $\Delta(A)$, the algebra of measurable sets. □

Theorem 6.12. Under the assumptions of Lemma 6.11, we have $\int f d\mu = st(\sigma(f))$.

Proof. By Lemma 6.11, ($4$), $\int f d\mu$ exists, whence we only have to prove that $\int f d\mu = st(\sigma(f))$. It suffices to prove that for every (arbitrarily small) positive real number $\varepsilon$, there are $h \in F^-$ and $k \in F^+$ such that $l(h) \leq \sigma(f) \leq l(k)$ and $l(k) - l(h) < \varepsilon$. Now let $\varepsilon > 0$ be given, and let $n \in \omega$ be such that $\frac{1}{n} < \varepsilon$. Let

$$h = \Delta(f) \oplus \bigoplus_{i=0}^{n-1} \frac{i}{n} \left( \Delta \left( \frac{i}{n} f \right) \wedge \left( \Delta \left( \frac{i+1}{n} f \right) \right) \right)$$
and
\[ k = \Delta(f) \oplus \left( \frac{n-i+1}{n} \left( \Delta\left( \frac{i}{n} \rightarrow f \right) \land \left( \Delta\left( \frac{i+1}{n} \rightarrow f \right) \right) \right) \right). \]

Note that for all \( m \in U \), the following conditions hold:

1. If \( f(m) = 1 \), then \( h(m) = k(m) = 1 \).
2. If \( \frac{i}{n} \leq f(m) < \frac{i+1}{n} \) (\( i = 0, \ldots, n-1 \)), then \( h(m) = \frac{i}{n} \) and \( k(m) = \frac{i+1}{n} \). Thus \( h \leq f \leq k \), and hence \( \sigma(h) \leq \sigma(f) \leq \sigma(k) \).

Moreover,
\[ l(k) - l(h) = \frac{n-1}{n} \mu_{st}\left( U_{f(i/n)} \right) = \frac{1}{n} < e. \]

To settle our claim, it suffices to prove that \( st(\sigma(h)) = l(h) \) and \( st(\sigma(k)) = l(k) \). Now let for \( i = 0, \ldots, n-1 \), \( t_i = (\Delta\left( \frac{i}{n} \rightarrow f \right) \land \left( \Delta\left( \frac{i+1}{n} \rightarrow f \right) \right) \) and let \( t_n = \Delta(f) \). Then for \( i, j = 0, \ldots, n \), if \( i \neq j \), then \( t_i \circ t_j = 0 \) (because if \( i < n \), then \( t_i \) is the characteristic function of \( U_{f(i/n)} \) and \( t_n \) is the characteristic function of \( U_{f(1/n)} \)). Thus by Lemma 3.3, (b) and (c),
\[ \sigma(h) = \sum_{i=0}^{n-1} \frac{i}{n} \sigma(t_i) \quad \text{and} \quad \sigma(k) = \sum_{i=0}^{n-1} \frac{i+1}{n} \sigma(t_i), \]
and hence
\[ st(\sigma(h)) = \sum_{i=0}^{n-1} \frac{i}{n} (st(\sigma(t_i))) = \sum_{i=0}^{n-1} \frac{i}{n} (\mu_{st}\left( U_{f(i/n)} \right)) = l(h) \]
and
\[ st(\sigma(k)) = \sum_{i=0}^{n-1} \frac{i+1}{n} (st(\sigma(t_i))) = \sum_{i=0}^{n-1} \frac{i+1}{n} (\mu_{st}\left( U_{f(i/n)} \right)) = l(k). \]

7. Conclusions and open problems

In this paper we have introduced a new approach to states on MV-algebras. This approach allows us to treat a variant of the concept of state within the framework of universal algebra. We have thus introduced the variety of SMV-algebras, which is the equivalent algebraic semantics of a variant of a probabilistic logic introduced by Flaminio and Godo in [12]. Then we have started a general investigation of the variety of SMV-algebras. Several interesting problems remain still open, namely:

(a) Is the variety of SMV-algebras generated by all SMV-algebras of the form \((A, \sigma)\) where \(A\) is a semisimple MV-algebra and \(\sigma(A)\) is a simple MV-algebra?

(b) Is the variety of SMV-algebras generated by all algebras of the form \((\{0, 1\}_{MV} \otimes A, \sigma)\) where \(A\) is any MV-algebra and there is a state \(s\) on \(A\) such that for all \(a \in A\) and for all \(x \in \{0, 1\}, \sigma(x \otimes a) = ss(a)\)?

(c) What is the computational complexity of the coherence problem for probabilistic assessments of MV-valued events? It follows from [16,25] that the coherence problem for many valued events is in PSPACE. We wonder if it is possible to improve this bound showing, e.g., that the problem is in NP.

(d) What is the computational complexity of the equational logic of SMV-algebras?

(e) What is the computational complexity of the satisfiability problem of equations of SMV-algebras?

(f) Same problems as in (d) and (e) but for the variety generated by all SMV-algebras of the form \((A, \sigma)\) with \(A\) a semisimple MV-algebra and \(\sigma(A)\) simple.

(g) Is there any connection between the semantics of SMV-algebras and the Kripke semantics for probabilistic logics introduced in [12,16]?

(h) Try to establish an equivalence of categories between the category of MV-algebras with a state, with morphisms the state preserving MV-homomorphisms, and a suitably defined full subcategory of the category of SMV-algebras, with morphisms the SMV-homomorphisms.

(i) Give a characterization of some remarkable classes of SMV-algebras, like finite SMV-algebras, simple SMV-algebras, and subdirectly irreducible SMV-algebras.

We plan to investigate all these problems in a future paper.

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