THE PROPER ROLE OF MODALITIES IN FITCH’S ARGUMENT

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Fitch’s Paradox

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unknown truth then there is an unknowable truth.

This is paradoxical to the extent that

1. it blunts any distinction between
   (A) an agent A knowing that a sentence \( \varphi \) is true and
   (B) A being in a position to possibly know that \( \varphi \) is true.
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2. and between
   (A) truth of \( \varphi \)
   (B) A knows \( \varphi \).
Fitch’s paradox

- Logic framework
- What is the problem
Outline

1. Fitch’s paradox
   - Logic framework
   - What is the problem

2. Group knowledge and Fitch’s paradox
   - Group of agents
   - Individual agent as group
Fitch’s paradox
  ▶ Logic framework
  ▶ What is the problem

Group knowledge and Fitch’s paradox
  ▶ Group of agents
  ▶ Individual agent as group

The role of modalities
  ▶ Knowability as epistemic possibility
  ▶ Leaving aside the alethic modality
**FORMALIZATION**

Let:
- \( \mathcal{L} = \{ p_1, \ldots, p_n \} \) a non-empty set of propositional variables,
- \( \mathcal{C} = \{ \neg, \Box, F, \land, \lor, \rightarrow \} \) the set of connectives, where
  - \( \Box \) is for 'it is necessary that' and
  - \( F \) is for 'an agent knows that'.

The set of sentences \( \mathcal{E} \mathcal{L} \) of the language \( \mathcal{L} \) is built by induction as follows:

\[
\begin{align*}
\mathcal{E} \mathcal{L}_0 &= \mathcal{L} \\
\mathcal{E} \mathcal{L}_{n+1} &= \mathcal{E} \mathcal{L}_n \cup \{ \neg \varphi, \Box \varphi, F \varphi, (\varphi \star \psi) | \varphi, \psi \in \mathcal{E} \mathcal{L}_n, \star \in \{ \land, \lor, \rightarrow \} \} \\
\mathcal{E} \mathcal{L} &= \bigcup \mathcal{E} \mathcal{L}_n.
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**FORMALIZATION**

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\[ \mathcal{EL}_0 = \mathcal{L} \]

\[ \mathcal{EL}_{n+1} = \mathcal{EL}_n \cup \{\neg \varphi, \square \varphi, F \varphi, (\varphi \star \psi) | \varphi, \psi \in \mathcal{EL}_n, \star \in \{\land, \lor, \rightarrow\}\} \]

\[ \mathcal{EL} = \bigcup \mathcal{EL}_n. \]

Axioms:
- classical tautologies,
- \( K: \square(\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \), \( \diamond = \text{Df} \neg \lozenge \neg \),
- \( K_F: F(\varphi \rightarrow \psi) \rightarrow (F \varphi \rightarrow F \psi) \), \( \hat{F} = \text{Df} \neg F \neg \),
- \( T_F: F \varphi \rightarrow \varphi \).
A semantic view

The relational model $\mathcal{M}$ for the language $\mathcal{L}$ is a quadruple $\mathcal{M} = \langle W, R, R_F, V \rangle$ such that:

- $W$ is a non-empty set of states,
- $R, R_F \subseteq W \times W$, with $R_F$ reflexive,
- $V$ is the set of classical valuations.
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We define:

- $i \models \Box \varphi$ if and only if $\forall j \in W, \langle i, j \rangle \in R$ then $j \models \varphi$.
- $i \models F \varphi$ if and only if $\forall j \in W, \langle i, j \rangle \in R_F$ then $j \models \varphi$.
- $i \models \Diamond \varphi$ if and only if $\exists j \in W, \langle i, j \rangle \in R$ such that $j \models \varphi$.
- $i \models \hat{F} \varphi$ if and only if $\exists j \in W, \langle i, j \rangle \in R_F$ such that $j \models \varphi$. 

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**Paradox**

**Non-omniscience principle (NOP):** $p \land \neg Fp$, ’$p$ is true and an agent does not know $p$’.

**Knowability principle (KP):** $\varphi \rightarrow \Box F\varphi$, ’if $\varphi$ is a true sentence then it is possible to know $\varphi$’.
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If we choose $\varphi = p \land \neg Fp$ this principles are inconsistent.

1. From $(p \land \neg Fp) \rightarrow \Diamond F(p \land \neg Fp)$ is derived $p \rightarrow Fp$, ’if $p$ is true then an agent knows $p$’.

2. From $p \land \neg Fp$ is derived $(p \land \neg Fp) \land \neg \Diamond F(p \land \neg Fp)$, ’$(p \land \neg F)$ is true and it is impossible that an agent knows $(p \land \neg Fp)$’. 

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1. Omniscience in principle. With axiom $Fp \rightarrow p$ we have $p \leftrightarrow Fp$.

2. Unknowability as matter of fact.
\( F(p \land \neg Fp) \) IS UNSATISFIABLE

\[ i \vdash F(p \land \neg Fp) \text{ if and only if } \forall j \in W, \langle i, j \rangle \in R_F, j \vdash p \land \neg Fp, \text{ that is if } j \vdash p \text{ and } j \vdash \neg Fp. \]
$F(p \land \neg Fp)$ is unsatisfiable

$i \models F(p \land \neg Fp)$ if and only if $\forall j \in W, \langle i, j \rangle \in R_F, j \models p \land \neg Fp$, that is if $j \models p$ and $j \models \neg Fp$. Since $R_F$ is reflexive, this is true if and only if also $i \models p$ and $i \models \neg Fp$. 

Remark This strictly depends on the reflexivity of $R_F$.

Remark ♦ $F(p \land \neg Fp)$ is inconsistent as well. The only role of alethic modality ♦ is requiring the existence of a state in which $F(p \land \neg Fp)$ is satisfied.
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$i \models F(p \land \neg Fp)$ if and only if $\forall j \in W, \langle i, j \rangle \in R_F, j \models p \land \neg Fp$, that is if $j \models p$ and $j \models \neg Fp$. Since $R_F$ is reflexive, this is true if and only if also $i \models p$ and $i \models \neg Fp$. But $i \models \neg Fp$ if and only if $\exists j \in W, \langle i, j \rangle \in R_F$ such that $j \models \neg p$, that is impossible.

Remark

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$F(p \land \neg Fp)$ is inconsistent as well. The only role of alethic modality $\Box$ is requiring the existence of a state in which $F(p \land \neg Fp)$ is satisfied.
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$i \models F(p \land \neg Fp)$ if and only if $\forall j \in W, \langle i, j \rangle \in R_F, j \models p \land \neg Fp$, that is if $j \models p$ and $j \models \neg Fp$. Since $R_F$ is reflexive, this is true if and only if also $i \models p$ and $i \models \neg Fp$. But $i \models \neg Fp$ if and only if $\exists j \in W, \langle i, j \rangle \in R_F$ such that $j \models \neg p$, that is impossible.
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**Remark**

This strictly depends on the reflexivity of \( R_F \).
\( F(p \land \neg Fp) \) IS UNSATISFIABLE

\[ i \vDash F(p \land \neg Fp) \text{ if and only if } \forall j \in W, \langle i, j \rangle \in R_F, j \vDash p \land \neg Fp, \text{ that is if } j \vDash p \text{ and } j \vDash \neg Fp. \text{ Since } R_F \text{ is reflexive, this is true if and only if also } i \vDash p \text{ and } i \vDash \neg Fp. \text{ But } i \vDash \neg Fp \text{ if and only if } \exists j \in W, \langle i, j \rangle \in R_F \text{ such that } j \vDash \neg p, \text{ that is impossible.} \]

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\( \Diamond F(p \land \neg Fp) \) is inconsistent as well. The only role of alethic modality \( \Diamond \) is requiring the existence of a state in which \( F(p \land \neg Fp) \) is satisfied.
Group knowledge

Following Palczewski [Pal07], we now set Fitch’s paradox into an epistemic multi-agent logic.

Example

A Parisian mathematician knows $q \rightarrow p$, although he does not know $q$. A Japanese mathematician knows $q$, but he does not know $q \rightarrow p$. None of them knows $p$. However there is a sense in which we can say that as a community they know $p$. 
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A Parisian mathematician knows $q \rightarrow p$, although he does not know $q$. A Japanese mathematician knows $q$, but he does not know $q \rightarrow p$. None of them knows $p$. However there is a sense in which we can say that as a community they know $p$.

Group knowability regards the epistemic possibility of accessing information which is not available to the individuals. Our idea consists in formulating KP using the formal concept of “distributed knowledge”.

**Group Knowledge**

We assume the non-empty finite set $G$ of all agents and we add the connectives

- $F_g \varphi$, for 'the agent $g$ knows that $\varphi$', $g \in G$,
- $S\varphi$ for 'somebody in the group $G$ knows that $\varphi$',
- $D\varphi$ for '$\varphi$ is distributively known in the group $G$'.

The set of sentences $\mathcal{ELG}$ of the language $\mathcal{L}$ is built as usual.
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The set of sentences $\mathcal{ELG}$ of the language $\mathcal{L}$ is built as usual. Further axioms:

- $S1$: $S \varphi \leftrightarrow \bigvee_{g \in G} F_g \varphi$
- $D1$: $F_g \varphi \rightarrow D \varphi$
- $K_S$: $S(\varphi \rightarrow \psi) \rightarrow (S \varphi \rightarrow S \psi)$,
- $T_S$: $S \varphi \rightarrow \varphi$,
- $K_D$: $D(\varphi \rightarrow \psi) \rightarrow (D \varphi \rightarrow D \psi)$,
- $T_D$: $D \varphi \rightarrow \varphi$.

To the usual ND rules we add the following closure over implication

\[
\frac{\varphi_1 \land \ldots \land \varphi_n \rightarrow \psi}{S \varphi_1 \land \ldots \land S \varphi_n \rightarrow D \psi} \quad RD
\]
A semantic view

Let be

- $G \subseteq N$
- $\mathcal{M} = \langle W, R, R^1_F, \ldots, R^n_F, V \rangle$ a relational model.

**Somebody knows**

\[ i \models S \varphi \text{ if and only if } \exists g \in G, i \models F_g \varphi \]

Somebody knows $\varphi$ if at least an agent in the group $G$ knows $\varphi$. 
A SEMANTIC VIEW

Let be

- $G \subseteq N$
- $\mathcal{M} = \langle W, R, R_1^F, \ldots, R_n^F, V \rangle$ a relational model.

**SOMEBODY KNOWS**

$$i \models S\varphi \text{ if and only if } \exists g \in G, i \models F_g\varphi$$

Somebody knows $\varphi$ if at least an agent in the group $G$ knows $\varphi$.

**DISTRIBUTED KNOWLEDGE**

$$i \models D\varphi \text{ if and only if } \forall j \in W, \langle i, j \rangle \in \bigcap_{g \in G} R_g^F, j \models \varphi$$

$\varphi$ is distributively known by a group if either someone of its members knows $\varphi$ or if $\varphi$ is in the deductive closure of a group member’s knowledge.
Fitch’s argument for a group of agents

**Non-omniscience principle (NOP$_s$):** $p \land \neg Sp$, that is ‘$p$ and nobody in the group $G$ knows that $p$’.

**Knowability principle ($KP_D$):** $\varphi \rightarrow \diamond D\varphi$, that is ‘if $\varphi$ then it is possible that $\varphi$ is distributively known by the group’.
**Fitch’s argument for a group of agents**

**Non-omniscience principle (NOPs)**: \( p \land \neg Sp \), that is ‘\( p \) and nobody in the group \( G \) knows that \( p \)’.

**Knowability principle (KP\( D \))**: \( \varphi \rightarrow \Diamond D \varphi \), that is ‘if \( \varphi \) then it is possible that \( \varphi \) is distributively known by the group’.

In this logic, if we choose \( \varphi = p \land \neg Sp \) the principles are consistent. From \( p \land \neg Sp \) and \( (p \land \neg Sp) \rightarrow \Diamond D(p \land \neg Sp) \) is derived \( \Diamond D(p \land \neg Sp) \) without contradiction.
**$D(p \land \neg Sp)$ is satisfiable**

We assume $G = \{1, 2\}$.

1. $i \vdash \neg F_1 p$
2. $i \vdash F_1 q$
3. $i \vdash \neg F_2 p$
4. $i \vdash F_2(q \rightarrow p)$
5. $i \vdash \neg Sp$, from 1 and 3
6. $i \vdash Dp$, from 2 and 4
7. $j, k \vdash \neg Sp$
8. $i \vdash F_{1,2} \neg Sp$, from 7
9. $i \vdash D \neg Sp$, from 8
10. $i \vdash Dp \land D \neg Sp$, from 6 and 9
11. $i \vdash D(p \land \neg Sp)$, from 10
The main reason why the paradox does not arise is that distributed knowledge formalizes the epistemic possibility of an omniscient multi-agent.
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**Remark**

\(\Diamond D(p \land \neg Sp)\) is consistent as well. Also in this case the only role of alethic modality \(\Diamond\) is requiring the existence of a state in which \(D(p \land \neg Sp)\) is satisfied.
Imagine an agent as a group whose members are the “same agent” at different temporal-epistemic states. The idea of distributed knowledge applies to this ideal agent as well.
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**Example**

Freud does not know how many cigars he smoked in his life, but at every moment he knows whether he was smoking a cigar. He did not count them, so he does not know the total their number, but Freud as a group can distributively know it.
Individual agents as groups

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Example

Freud does not know how many cigars he smoked in his life, but at every moment he knows whether he was smoking a cigar. He did not count them, so he does not know the total their number, but Freud as a group can distributively know it.

Since \( \not \exists D_G \varphi \rightarrow K_g \varphi \) for any \( g \in G \) then clearly Freud will not come to actually know the number of cigars he smoked.
Conclusion

1. Knowability should be formalized only in terms of epistemic possibility.

2. \( \varphi \) is knowable actually means that \( \varphi \) is conceivable given the information state of an agent.

To formalize this we can dispense with the alethic modality.
Reformulating *KP*

What happens to the single-agent case if we take this view on the role of modalities in Fitch’s paradox is that we can reformulate the KP as

\[ \varphi \rightarrow \hat{F}\varphi. \]
Reformulating KP

What happens to the single-agent case if we take this view on the role of modalities in Fitch’s paradox is that we can reformulate the KP as

$$\varphi \rightarrow \hat{F}\varphi.$$  

In this situation we have the welcome consequence to the effect that the NOP is consistent with the KP.

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