Connections between Qualitative Belief Revision and Conditional Probability

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Will Talk About…

• Qualitative belief revision: AGM
  
  and its connections with

• Probability
  
  – 1-place prob. functions (Kolmogorov)
  
  – Operations that revise 1-place prob. functions (Gärdenfors)
  
  – 2-place prob. functions (Hosiasson-Lindenbaum, Rényi, Popper, Carnap)
AGM Belief Revision

\( K \ a \) \( \star_K : L \rightarrow Th \) \( \star_K(a) \) alias \( K \star a \)

\((\bullet \quad \bullet) \quad \rightarrow \quad \bullet \)

\( L \): propositional language, \( a \in L \)

\( K = Cn(K) \subseteq L \): belief set (alias theory)

\( \star_K \): AGM revision: ‘consistently add \( a \) to \( K \)’
Background: AGM Revision Functions

• Assume some familiarity. Recall only:
  • $K^*a$ defined for every belief set $K$ (even inconsistent) and every proposition $a$ (even inconsistent)
  • $a \in K^*a \subseteq Cn(K \cup \{a\})$
  • But $K^*a$ consistent so long as $a$ is (even when $a$ inconsistent with $K$)
  • So loss: $K \subseteq K^*a$ fails when $a$ consistent but inconsistent with $K$
  • Semantic and syntactic accounts
Enter Probability

\[ K \ast a \]

\[ \ast_K : L \rightarrow Th \]

\[ K \ast a \]

\[ \text{top}(p) \]

\[ p(\cdot) \]

\[ p(\cdot) : \text{one-place prob. function (Kolmogorov)} \]

\[ \text{top}(p) = \{ x : p(x) = 1 \} \]
Background: One-Place Probability Functions

Proper Kolmogorov functions: $p: L \rightarrow [0,1]$ with:

- $p(x) \leq p(y)$ whenever $x \vdash y$
- $p(x \lor y) = p(x) + p(y)$ whenever $x \vdash \neg y$
- $p(x) = 1$ for some formula $x$

plus the Improper one

The unit function: $p(x) = 1$ for all formulae $x$
Back to our Diagram

\[ K \ast a : L \to \text{Th} \]

\[ \text{top}(\rho) = \{ x : \rho(x) = 1 \} \]

\( \rho(\cdot) \): one-place prob. function (Kolmogorov)
$K \ a$ \hspace{1cm} $*_{K}: L \rightarrow Th$ \hspace{1cm} $K*_{a}$

$(\bullet \ \bullet)$ \hspace{4cm} $(\bullet \ \bullet)$

$\text{top}(\rho)$ \hspace{4cm} $\text{top}(\rho)$

$p(\cdot) \ a$ \hspace{1cm} $\#_{p}: L \rightarrow Pr_{1} \ a$ $p\#a$

\textit{P1: } set of all 1-place Kolmogorov functions (proper and improper)

$\#$: probability revision function (Gärdenfors)
Background: Probability Revision Functions

- Can be seen as:
  - 2-place functions $(\cdot, \cdot)$ on $Pr_1 \times L$ into $Pr_1$
  - Indexed 1-place functions $\#_p(\cdot)$ on $L$ into $Pr_1$

- Defined for all functions in $Pr_1$ (even the improper one)

- Satisfying axioms set out by Peter Gärdenfors

*Knowledge in Flux* (1988) ch 5
Since 1988

• Gärdenfors’ paper not given much follow up
  – Careful technical work by Sten and Wlodek in paper of 1989 in *J. Phil. Logic*
  – Since then neglected, apart from tangentially by Horacio Arló Costa and Parikh

• Still lots of interesting things to explore.
  – connections with two-place (alias conditional) probability.

Let’s take a brief look
\( K \diamond a \quad *_K : L \rightarrow Th \quad K * a \)

\( (\cdot \cdot) \quad \top(p) \quad \text{commutes} \quad \top(p) \quad (\cdot \cdot) \)

\( p(\cdot) \diamond a \quad \#_p : L \rightarrow Pr1 \quad p \# a \)

\( \forall p \ \forall \# \ \exists * \ \forall a \quad \text{assuming } L \text{ finite and} \)

\( \forall p \ \forall * \ \exists # \ \forall a \quad K \text{ consistent} \)
\[ \rho_{\cdot\cdot} : \text{two-place probability function} \]

Left projection from \( T \) resp. from \( a \)
Background

• What are two-place (conditional) probability functions?

Two approaches…
Ratio Account

• Defined from one-place functions
  \( p(\cdot,\cdot) \) from \( p(\cdot) \)

• Ratio Definition
  \[ p(x,a) = \frac{p(a \land x)}{p(a)} \] whenever \( p(a) > 0 \), otherwise undefined

• These are partial functions for right argument
  \( p: LxL\{a: p(a) > 0\} \rightarrow [0,1] \)
As Primitive Notion

Functions \( \rho: L^2 \to [0,1] \)

- Full functions, not partial
- Several systems: Hosiasson-Lindenbaum, Rényi, Popper, Carnap, ...
- Can be confusing
- To get a clear picture: they differ in the way in which they treat propositions in the critical zone…
The Critical Zone

• Key concept

• Critical zone for $p: L^2 \rightarrow [0,1]$ is set of all $a \in L$ with $a$ consistent but $p(a, T) = 0$

• Systems agree for values of right argument (the ‘condition’) outside critical zone, disagree inside

• Also agree with ratio definition of $p(x | a)$ when that is defined i.e. when $p(x, T) > 0$

• We give a modular presentation
Basic System (van Fraassen 1976, 1995)

Axioms describe behaviour throughout $L^2$

- Extensionality: $p(x,a) = p(x,b)$ when $Cn(a) = Cn(b)$
- Left projection $p_a$ of $p(\cdot,\cdot)$ from right value $a$ is a Kolmogorov function (proper or unit)
- Product: $p(x \land y,a) = p(x,a) \cdot p(y,a \land x)$

Properties of vF

- Consistent with $p_a$ unit function for all $a$
- Implies: when $a$ is inconsistent, $p_a$ is the unit function
The Positive Case

Two cases left unconstrained

• When $p(a,T) > 0$
• *Critical zone*: a consistent but $p(a,T) = 0$

First case general agreement

• (Positive case): When $p(a,T) > 0$, $p_a$ is a proper one-place probability function with $p_a(a) = 1$
• What should we do in second case?
Four Options for Critical Zone

• **Say nothing more** (Popper)

• **Zone empty**: only inconsistent propositions have probability zero (Carnap)

• **In zone**: $p_a$ is the unit function: $p(x,a) = 1$ for all $x$ (Kolmogorov)

• **In zone**: $p_a$ is a proper one-place probability function (Hosiasson-Lindenbaum)

  (Rényi 1955 in effect gives scheme with parameter, can be instantiated to either Popper or HL)
Bird’s Eye View of Classes
(mostly Leblanc, Roeper 1989, 1999)

- $vF = \text{Popper} \cup \{1(\cdot,\cdot)\}$
- Popper
- Unit $\cup$ HL
- Carnap $= \text{Unit} \cap$ HL
Back to our Diagram

$K \ a \ 
\begin{array}{c}
\bullet \\
\bullet
\end{array} 
\xrightarrow{\mathbf{\ast}_k: L \to Th} 
\begin{array}{c}
\bullet \\
\bullet
\end{array} 

\begin{array}{c}
p(\cdot) \\
\bullet
\end{array} 
\xrightarrow{\#_p: L \to Pr_1} 
\begin{array}{c}
p(a) \\
\bullet
\end{array} 

p_T(x) = p(x, \top) 
\quad p_a(x) = p(x, a)
\[ K \quad a \quad \ast_K : L \to Th \quad K \ast a \]

\[ \bullet \quad \bullet \quad \bullet \]

\[ p(\cdot) \quad a \quad \#_p : L \to Pr_1 \quad p\#a \]

\[ p_T(x) = p(x, T) \quad \text{commutes} \quad p_a(x) = p(x, a) \]

\[ \forall P \exists \# \forall a \quad \forall \# \exists P \forall a \]

\[ (\text{same assumptions}) \]

\[ p(\cdot, \cdot) \in \text{Prob}2 \]
$K \quad a$  \quad $*_K : L \to Th$  \quad $K*a$

$\pi(x, a, b) = p(x, a \land b)$  \quad $p'(x, a) = \pi(x, a, \top)$

- $\pi(\cdot, \cdot, \cdot) \in \text{Prob3}$

Commutes one way: $p' = p$
Can/Should We Go Further?

Idea: *conditionalize the conditionalization*

Can be conceived as:

- **3-place functions** $L^3$ into $[0,1]$
  - Defined from 2-place ones
  - Primitive 3-place
- **Operations on probability functions**
  - 2-place plus proposition into another 2-place
  - Iterate the operation taking 1-place plus proposition to 1-place
\[ K \to a \]
\[ \ast_K : L \to Th \]
\[ K \ast a \]

\[ p(\cdot) \to a \]
\[ \#_p : L \to Pr_1 \]
\[ p\#a \]

\[ \pi(x,a,b) = p(x,a \land b) \]
aggregate

\[ \pi(\cdot,\cdot,\cdot) \in Pr_3 \]
Aggregative vs Revisionary Conceptions

Aggregative

• Dfn: \( \pi(x, a, b) = p(x, a \land b) \)
• Property: \( \pi(x, a, \lnot a) = p(x, a \land \lnot a) = 1 \)

Revisionary

• Dfn: \( \pi(x, a, b) = ( (p \# a) \# b )(x) \)
• Property: \( \pi(x, a, \lnot a) = ( (p \# a) \# \lnot a )(x) \)
  
  can be \( \neq 1 \) when \( \lnot a \) consistent
General Moral of the Story

• Intimate relations between
  – AGM qualitative belief revision
  – probability (1-place, 2-place, prob revision functions)

• Qualitative belief revision helps see conceptual options in probabilistic context (and vice versa)

• Delicate, intricate: still much to explore, for both belief revision and probability
More Specifically

• AGM revision essentially qualitative part of revision of 1-place prob functions (Gärdenfors)
• Also of two-place prob. functions (HL axioms)
• Several different kinds of two-place probability, arising from natural options in the critical zone. HL appears to be the most useful.
• Three-place probability may be understood in several ways, including aggregative vs revisionary ones
References

Peppas, Pavlos ‘Belief revision’ chapter 8 of *Handbook of Knowledge Representation* (van Harmelen et al eds, Amsterdam: Elsevier, 2007)

Gärdenfors, Peter *Knowledge in Flux* (1988 reprint 2008)


Makinson ‘Conditional Probability in the Light of Qualitative Belief Change’ (Pontignano proceedings, david.makinson@gmail.com)