Equivalence of logical consequence relations: an order-theoretic and categorical approach

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Logical Foundations of Rational Interaction

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Equivalences and translations between logical consequence relations abound in logic. The aim of this talk is to propose a uniform treatment of various constructions and concepts connected with the study of logical consequence relations. The approach is of order-theoretic and categorical nature, and provides a roadmap for considering related questions in the future.
References

- Augustus De Morgan and Charles S. Peirce [See e.g., C. Brink, Boolean modules, J. Algebra 71 (1981), no. 2, 291-313.]
- C. Russo, Quantale Modules, with Applications to Logic and Image Processing, Doctoral dissertation, 2007.
The history of these algebras can be traced back to the work of Augustus De Morgan and Charles S. Peirce. It was De Morgan (1847-1864) who started formalising the logic of binary relations as a generalisation of Aristotle’s syllogistic logic. Moreover, he invented relational composition and relational converse. Peirce (1866-1883) gave the first algebraic treatment of the algebra of relations interacting with sets.
Logical Consequence
A consequence relation over the set $M$ is a relation $\vdash \subseteq \wp(M) \times M$ obeying these conditions for all $u \in M$ and for all $X, Y \subseteq M$:

1. $X \vdash u$, whenever $u \in X$ (reflexivity);
2. If $X \vdash u$ and $X \subseteq Y$, then $Y \vdash u$ (monotonicity);
3. If $Y \vdash u$ and $X \vdash v$ for every $v \in Y$, then $X \vdash u$ (transitivity; cut).
Consequence Relations

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Notation:
(i) $u \vdash v$ means: $\{u\} \vdash v$
(ii) $\vdash v$ means: $\emptyset \vdash v$
(iii) $X \vdash Y$ means: $X \vdash v$, for all $v \in Y$
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Consequence Operations

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We use the term consequence operation on the power set $\mathcal{P}(M)$ of a set $M$ for a closure operator on $\mathcal{P}(M)$. That is, a map $\Xi : \mathcal{P}(M) \to \mathcal{P}(M)$ satisfying the following conditions for all $X, Y \subseteq M$:

1. if $X \subseteq Y$, then $\Xi(X) \subseteq \Xi(Y)$;
2. $X \subseteq \Xi(X)$; and
3. $\Xi(\Xi(X)) = \Xi(X)$.
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1. whatever follows from $X$ also follows from any superset of $X$;
2. all members of $X$ are consequences of $X$; and
3. whatever follows from consequences of $X$ also follows from $X$. 
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Given a set $M$, there exists a bijective correspondence between all consequence operations $\Xi$ on $\mathcal{P}(M)$ and all consequence relations $\vdash$ over $M$. More specifically:

- If $\vdash$ is a consequence relation over $M$, then the map $\Xi_+: \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ defined by

$$\Xi_+(X) = \{ u \in M : X \vdash u \}$$

is a consequence operation on $\mathcal{P}(M)$. 
Given a set $M$, there exists a bijective correspondence between all consequence operations $\Xi$ on $\wp(M)$ and all consequence relations $\vdash$ over $M$. More specifically:

- If $\vdash$ is a consequence relation over $M$, then the map $\Xi : \wp(M) \rightarrow \wp(M)$ defined by
  
  $$\Xi : X \mapsto \{ u \in M : X \vdash u \}$$

  is a consequence operation on $\wp(M)$.

- Conversely, if $\Xi$ is a consequence operation on $\wp(M)$, then the relation $\vdash \subseteq \wp(M) \times M$ defined by
  
  $$X \vdash \Xi u \iff u \in \Xi (X)$$

  is a consequence relation over $M$. 

Consequence relations
Given a set $M$, there exists a bijective correspondence between all consequence operations $\Xi$ on $\wp(M)$ and all consequence relations $\vdash$ over $M$. More specifically:

- If $\vdash$ is a consequence relation over $M$, then the map $\Xi \vdash : \wp(M) \to \wp(M)$ defined by

$$\Xi \vdash (X) = \{ u \in M : X \vdash u \}$$

is a consequence operation on $\wp(M)$.

- Conversely, if $\Xi$ is a consequence operation on $\wp(M)$, then the relation $\vdash_{\Xi} \subseteq \wp(M) \times M$ defined by

$$X \vdash_{\Xi} u \iff u \in \Xi(X)$$

is a consequence relation over $M$.

Furthermore, $\Xi_{\vdash_{\Xi}} = \Xi$ and $\vdash_{\Xi_{\vdash}} = \vdash$. 
Let $\vdash$ be a consequence relation over $M$, and let $\Xi$ be the associated consequence operation on $\wp(M)$. $X \subseteq M$ is said to be a $\vdash$-theory if it is closed subset of $M$ under $\Xi$: 

$$X = \Xi(X) = \{ u : u \in M \text{ and } X \vdash u \}$$
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Note that the poset of $\vdash$-theories, denoted by $\text{Th}(\vdash)$ or $\text{Th}(\Xi)$, is a closure system over $M$, that is, a subset of $\wp(M)$ that is closed under arbitrary intersections.
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Note that the poset of $\vdash$-theories, denoted by $\text{Th}(\vdash)$ or $\text{Th}(\Xi)$, is a closure system over $M$, that is, a subset of $\wp(M)$ that is closed under arbitrary intersections.

$\text{Th}(\vdash)$ completely determines $\Xi$ and $\vdash$. Furthermore, there exists a bijective correspondence between closure systems and consequence relations over $M$, and consequence operations on $\wp(M)$.
One of the most distinctive properties of logical consequence is its formal character: what follows logically from a set of premisses can be inferred from the premisses themselves purely by virtue of its logical form.

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How can we give a proper account of this feature?

We consider operations \( \sigma \) that act on assertions by uniformly substituting their non-logical components, in such a way as to leave their logical form unchanged. We assume that such actions can be concatenated – \( \tau \circ \sigma \) – and allow for an identity action \( 1 \) that changes nothing at all.
Monoids Acting on Sets

Formally, let $M$ be a nonempty set. A monoid $A = (A, \cdot, 1)$ is said to act on $M$ (and $M$ is said to be an $A$-set) in case an operation $\circ : A \times M \to M$ is defined such that, for all $a, b \in A$ and all $u \in M$,

$$(a \cdot b) \circ u = a \circ (b \circ a); \text{ and } 1 \circ u = u.$$ 

The operation $\circ$ is called scalar product, and the elements in $A$ are called actions. If no confusion arises, we’ll use plain juxtaposition in place of “$\cdot$”.
Let now $M$ be an $A$-set. A consequence relation $\vdash$ over $M$ is said to be **action-invariant** if, for any $a \in A$ and any $X \cup \{u\} \subseteq M$,

whenever $X \vdash u$, then $a \circ X \vdash a \circ u$.

[Here, $a \circ X = \{a \circ v : v \in X\}$.]
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Note that \( \vdash \) is action invariant iff the associated consequence operation \( \Xi \) satisfies the condition

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a \circ \Xi(X) \subseteq \Xi(a \circ X).
\]
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$$\text{whenever } X \vdash u, \text{ then } a \circ X \vdash a \circ u.$$  

[Here, $a \circ X = \{a \circ v : v \in X\}$.]

Note that $\vdash$ is action invariant iff the associated consequence operation $\Xi$ satisfies the condition

$$a \circ \Xi(X) \subseteq \Xi(a \circ X).$$

By extension, we call action-invariant any consequence operation $\Xi$ that satisfies the preceding condition.
A consequence relation $\vdash$ over $M$ is called **finitary**, provided for all $X \cup \{u\} \subseteq M$, if $X \vdash u$, then there is a finite subset $Y$ of $X$ such that $Y \vdash u$. 
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Note that $\vdash$ is finitary iff the associated consequence operation $\Xi$ satisfies a related condition for all $X \subseteq M$ and all $u \in M$: if $u \in \Xi(X)$, then $u \in \Xi(Y)$, for some finite subset $Y$ of $X$.

We use the term **finitary** for any consequence operation that satisfies the preceding condition.
Short Excursion to Universal Algebra
The Formula Algebra of Signature $\mathcal{L}$

$\mathcal{L}$: a language of algebras in logic
For example, $\mathcal{L} = \{\land, \lor, \cdot, \rightarrow, 0, 1\}$

$X$: an infinite countable set (whose members are referred to as variables)
The Formula Algebra of Signature $\mathcal{L}$

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The set $Fm(X)$ of $\mathcal{L}$-formulas over $X$ is defined as follows:

(a) **Inductive beginning**: The constants $0, 1$ and every member of $X$ is a formula.

(b) **Inductive steps**: If $\alpha$ and $\beta$ are formulas, then so are $\alpha \cdot \beta$, $\alpha \land \beta$, $\alpha \lor \beta$ and $\alpha \to \beta$.

(c) All formulas are generated by (a) and (b) above.
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A set $Fm(\mathcal{X})$ of $\mathcal{L}$-formulas over $\mathcal{X}$ is defined as follows:

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(c) All formulas are generated by (a) and (b) above.

An $\mathcal{L}$-algebra $A = \langle A, \land, \lor, \cdot, \to, 0, 1 \rangle$ is any algebra in the preceding signature.
The Formula Algebra of Signature $\mathcal{L}$

$\mathcal{L}$: a language of algebras in logic

For example, $\mathcal{L} = \{\land, \lor, \cdot, \to, 0, 1\}$

$\mathbb{X}$: an infinite countable set (whose members are referred to as variables)

The set $Fm(\mathbb{X})$ of $\mathcal{L}$-formulas over $\mathbb{X}$ is defined as follows:

(a) **Inductive beginning**: The constants $0, 1$ and every member of $\mathbb{X}$ is a formula.

(b) **Inductive steps**: If $\alpha$ and $\beta$ are formulas, then so are $\alpha \cdot \beta$, $\alpha \land \beta$, $\alpha \lor \beta$ and $\alpha \to \beta$.

(c) All formulas are generated by (a) and (b) above.

An $\mathcal{L}$-algebra $\mathbb{A} = \langle A, \land, \lor, \cdot, \to, 0, 1 \rangle$ is any algebra in the preceding signature.

Boolean algebras and Heyting algebras are examples of $\mathcal{L}$-algebras. Another example is the formula algebra $Fm(\mathbb{X})$ of signature $\mathcal{L}$ over $\mathbb{X}$. 
A homomorphism $\varphi : A \to B$ between two $L$-algebras is a map that preserves all operations: that is, for all $a, b \in A$ and all $\star \in \{\wedge, \vee, \cdot, \to\}$, $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(a \star b) = \varphi(a) \star \varphi(b)$. 
A homomorphism $\varphi : A \rightarrow B$ between two $\mathcal{L}$-algebras is a map that preserves all operations: that is, for all $a, b \in A$ and all $\star \in \{\land, \lor, \cdot, \rightarrow\}$, $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(a \star b) = \varphi(a) \star \varphi(b)$.

A congruence relation of an $\mathcal{L}$-algebra $A = \langle A, \land, \lor, \cdot, \rightarrow, 0, 1 \rangle$ is an equivalence relation $\Theta$ on $A$ satisfying the following substitution property: whenever $a \Theta b$ and $c \Theta d$, then $a \star c \Theta b \star d$, for all the operations $\star \in \{\land, \lor, \cdot, \rightarrow\}$. 
A homomorphism \( \varphi : A \rightarrow B \) between two \( \mathcal{L} \)-algebras is a map that preserves all operations: that is, for all \( a, b \in A \) and all \( \star \in \{\land, \lor, \cdot, \rightarrow\} \),
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Note that if \( \varphi : A \rightarrow B \) is a homomorphism, then \( \text{Ker}(\varphi) = \{(a, b) \in A^2 : \varphi(a) = \varphi(b)\} \) is a congruence relation of \( A \). Moreover, every congruence relation is of this form.
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Subalgebras and Direct Products
A class of $\mathcal{L}$-algebras is called a **variety** provided it is closed under direct products, subalgebras and homomorphic images.
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An $\mathcal{L}$-equation is an ordered pair $(\alpha, \beta)$ of $\mathcal{L}$-formulas over $X$, often written more suggestively as $\alpha \approx \beta$. The set $Fm(X) \times Fm(X)$ of all $\mathcal{L}$-equations over $X$ will be denoted by $Eq(X)$. 
A class of $\mathcal{L}$-algebras is called a **variety** provided it is closed under direct products, subalgebras and homomorphic images.

An $\mathcal{L}$-equation is an ordered pair $(\alpha, \beta)$ of $\mathcal{L}$-formulas over $\mathcal{X}$, often written more suggestively as $\alpha \approx \beta$. The set $Fm(\mathcal{X}) \times Fm(\mathcal{X})$ of all $\mathcal{L}$-equations over $\mathcal{X}$ will be denoted by $Eq(\mathcal{X})$.

**Theorem** [G. Birkhoff, 1935]
A class of $\mathcal{L}$-algebras is an equational class iff it is a variety.
Consequence Operations and Relations on Formula Structures
Consequence relations over $Fm(X), Fm(X)^k (k > 1)$, and sequents.
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The endomorphism monoid \( \mathcal{E}nd \) of \( Fm(X) \) acts on each of the sets above. For example, if \((\alpha, \beta) \in Fm(X)^2\) and \(\sigma \in \mathcal{E}nd\), then
\[
\sigma \circ (\alpha, \beta) = (\sigma(\alpha), \sigma(\beta)).
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The endomorphism monoid $\mathcal{E}nd$ of $Fm(X)$ acts on each of the sets above. For example, if $(\alpha, \beta) \in Fm(X)^2$ and $\sigma \in \mathcal{E}nd$, then 

$$\sigma \circ (\alpha, \beta) = (\sigma(\alpha), \sigma(\beta)).$$

In this setting, we use the term substitution invariant instead of action invariant.
Consequence Relations on $Fm(X)$ and $Fm(X)$

Consequence relations over $Fm(X), Fm(X)^k$ ($k > 1$), and sequents.

The endomorphism monoid $End$ of $Fm(X)$ acts on each of the sets above. For example, if $(\alpha, \beta) \in Fm(X)^2$ and $\sigma \in End$, then $\sigma \circ (\alpha, \beta) = (\sigma(\alpha), \sigma(\beta))$.

In this setting, we use the term substitution invariant instead of action invariant.

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras and $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$. We say that $\varepsilon$ is a $\mathcal{K}$-consequence of $\Sigma \vdash \varepsilon$ if for every $A \in \mathcal{K}$ and every homomorphism $\varphi : Fm(X) \rightarrow A$, if $\Sigma \subseteq \text{Ker}(\varphi)$, then $\varepsilon \in \text{Ker}(\varphi)$. 
Consequence Relations on $Fm(X)$ and $Fm(X)$

Consequence relations over $Fm(X)$, $Fm(X)^k$ ($k > 1$), and sequents.

The endomorphism monoid $\mathcal{E}nd$ of $Fm(X)$ acts on each of the sets above. For example, if $(\alpha, \beta) \in Fm(X)^2$ and $\sigma \in \mathcal{E}nd$, then $\sigma \circ (\alpha, \beta) = (\sigma(\alpha), \sigma(\beta))$.

In this setting, we use the term substitution invariant instead of action invariant.

Let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras and $\Sigma \cup \{\varepsilon\} \subseteq Eq(X)$. We say that $\varepsilon$ is a $\mathcal{K}$-consequence of $\Sigma - \Sigma \models_\mathcal{K} \varepsilon$ if for every $A \in \mathcal{K}$ and every homomorphism $\varphi: Fm(X) \to A$, if $\Sigma \subseteq \text{Ker}(\varphi)$, then $\varepsilon \in \text{Ker}(\varphi)$.

$\models_\mathcal{K}$ is a substitution invariant consequence relation over $Eq(X)$. It is finitary whenever $\mathcal{K}$ is a variety.
Equivalence of Two Consequence Relations
The abstract notion of consequence relation we discussed in the previous section is, for many purposes, too fine-grained. Two consequence relations, even on different sets, could count as distinct not because they validate different entailments, but merely in virtue of the fact that they present the same entailments under different guises. There are circumstances under which it may be appropriate to identify such consequence relations with each other. We’ll only consider the case of action-invariant consequence relations.
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Let $M_1$ and $M_2$ be $A$-sets. A mapping $\varphi : M_1 \rightarrow \wp(M_2)$ is said to be action-invariant if $\varphi(a \circ u) = a \circ \varphi(u)$, for all $a \in A$ and all $u \in M_1$. 
Equivalence of Consequence Relations

Let $\vdash_1, \vdash_2$ be action-invariant consequence relations over the sets $M_1, M_2$, respectively. We say that $\vdash_1$ and $\vdash_2$ are equivalent provided there exist action-invariant maps $\tau : M_1 \rightarrow \wp(M_2)$ and $\rho : M_2 \rightarrow \wp(M_1)$ such that the following conditions hold for every $X \cup \{u\} \subseteq M_1$ and for every $Y \cup \{v\} \subseteq M_2$:
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(S1) $X \vdash_1 u$ iff $\tau(X) \vdash_2 \tau(u)$;
(S2) $Y \vdash_2 v$ iff $\rho(Y) \vdash_1 \rho(v)$;
(S3) $v \not\vdash_2 \tau(\rho(v))$;
(S4) $u \not\vdash_1 \rho(\tau(u))$.

The maps $\tau$ and $\rho$ are called transformers.
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Let $\vdash_1, \vdash_2$ be action-invariant consequence relations over the sets $M_1, M_2$, respectively. We say that $\vdash_1$ and $\vdash_2$ are equivalent provided there exist action-invariant maps $\tau : M_1 \to \wp(M_2)$ and $\rho : M_2 \to \wp(M_1)$ such that the following conditions hold for every $X \cup \{u\} \subseteq M_1$ and for every $Y \cup \{v\} \subseteq M_2$:

$(S_1)\ X \vdash_1 u$ iff $\tau(X) \vdash_2 \tau(u)$;

$(S_2)\ Y \vdash_2 v$ iff $\rho(Y) \vdash_1 \rho(v)$;

$(S_3)\ v \not\vdash_2 \tau(\rho(v))$;

$(S_4)\ u \not\vdash_1 \rho(\tau(u))$.

The maps $\tau$ and $\rho$ are called transformers.

The relations $\vdash_1$ and $\vdash_2$ are equivalent iff either $(S_1)$ and $(S_3)$ hold, or else $(S_2)$ and $(S_4)$ hold.
The Diagram of Consequence Operators

There are equivalent characterizations in terms of the corresponding consequent operations (We write $\Xi_1$ for $\Xi_{\vdash 1}$ and $\Xi_2$ for $\Xi_{\vdash 2}$):

$(S_1)\ X \vdash_1 u \iff \tau(X) \vdash_2 \tau(u)$

$(S_3)\ v \not\vdash_2 \tau(\rho(v))$

$(S'_1)\ \Xi_1 = \tau^{-1}\Xi_2\tau$

$(S'_3)\ \Xi_2\tau\rho = \Xi_2$
The Diagram of Consequence Operators

There are equivalent characterizations in terms of the corresponding consequent operations (We write $\Xi_1$ for $\Xi_{\vdash_1}$ and $\Xi_2$ for $\Xi_{\vdash_2}$):

\begin{align*}
(S_1) \quad X \vdash_1 u \text{ iff } \tau (X) \vdash_2 \tau (u) \\
(S_3) \quad v \nvdash_2 \tau (\rho (v)) \\
(S_1') \quad \Xi_1 = \tau^{-1} \Xi_2 \tau \\
(S_3') \quad \Xi_2 \tau \rho = \Xi_2
\end{align*}

\[\begin{array}{ccc}
\phi(M_2) & \xrightarrow{\rho} & \phi(M_1) \\
\downarrow_{\Xi_2} & & \downarrow_{\Xi_1} \\
Th(\Xi_2) & \xleftarrow{\rho^{-1}} & Th(\Xi_1) \\
\end{array}\]

\[\begin{array}{ccc}
\phi(M_1) & \xrightarrow{\tau} & \phi(M_2) \\
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The Diagram of Consequence Operators

There are equivalent characterizations in terms of the corresponding consequent operations (We write $\Xi_1$ for $\Xi_{\vdash 1}$ and $\Xi_2$ for $\Xi_{\vdash 2}$):

\[(S_1) \quad X \vdash_1 u \iff \tau(X) \vdash_2 \tau(u)\]

\[(S_3) \quad v \nmid \vdash_2 \tau(\rho(v))\]

\[(S_1') \quad \Xi_1 = \tau^{-1}\Xi_2\tau\]

\[(S_3') \quad \Xi_2\tau\rho = \Xi_2\]

\[\begin{align*}
\varphi(M_2) & \xrightarrow{\rho} \varphi(M_1) & \xrightarrow{\tau} \varphi(M_2) \\
\downarrow^{\Xi_2} & \downarrow^{\Xi_1} & \downarrow^{\Xi_2} \\
Th(\Xi_2) & \xleftarrow{\rho^{-1}} Th(\Xi_1) & \xleftarrow{\tau^{-1}} Th(\Xi_2)
\end{align*}\]

\[\tau^{-1} \upharpoonright Th(\Xi_2) \quad \text{is the inverse of} \quad \rho^{-1} \upharpoonright Th(\Xi_1)\]

(i.e., $\rho^{-1}\tau^{-1}\Xi_2 = \Xi_2$ and $\tau^{-1}\rho^{-1}\Xi_1 = \Xi_1$)
A consequence relation $\vdash$ over $Fm$ is called **algebraizable** provided there exists a class $\mathcal{K}$ of $\mathcal{L}$-algebras such that $\vdash$ and $|=_{\mathcal{K}}$ are equivalent.
A consequence relation $\vdash$ over $\mathcal{F}m$ is called algebraizable provided there exists a class $\mathcal{K}$ of $\mathcal{L}$-algebras such that $\vdash$ and $|=_{\mathcal{K}}$ are equivalent.

When the preceding holds, we say that the class $\mathcal{K}$ is an equivalent algebraic semantics for the consequence relation $\vdash$.
A consequence relation \( \vdash \) over \( Fm \) is called **algebraizable** provided there exists a class \( \mathcal{K} \) of \( L \)-algebras such that \( \vdash \) and \( \models_{\mathcal{K}} \) are equivalent.

When the preceding holds, we say that the class \( \mathcal{K} \) is an **equivalent algebraic semantics** for the consequence relation \( \vdash \).

**BA**: the variety of Boolean algebras

**CL**: classical propositional calculus

\[
\tau : Fm \rightarrow \wp(Eq), \text{ defined by } \tau(\alpha) = \{ \alpha \approx 1 \}
\]

\[
\rho : Eq \rightarrow \wp(Fm), \text{ defined by } \rho(\gamma \approx \delta) = \{(\gamma \rightarrow \delta) \land (\delta \rightarrow \gamma)\}
\]
A consequence relation $\vdash$ over $Fm$ is called **algebraizable** provided there exists a class $\mathcal{K}$ of $\mathcal{L}$-algebras such that $\vdash$ and $\models_\mathcal{K}$ are equivalent.

When the preceding holds, we say that the class $\mathcal{K}$ is an **equivalent algebraic semantics** for the consequence relation $\vdash$.

$\mathcal{B}A$: the variety of Boolean algebras

$\mathcal{C}L$: classical propositional calculus

$\tau: Fm \to \varphi(Eq)$, defined by $\tau(\alpha) = \{\alpha \approx 1\}$

$\rho: Eq \to \varphi(Fm)$, defined by

$\rho(\gamma \approx \delta) = \{ (\gamma \to \delta) \land (\delta \to \gamma) \}$

One can check that $\vdash_{\mathcal{C}L}$ and $\models_{\mathcal{B}A}$ are equivalent via $\tau$ and $\rho$. Thus, $\vdash_{\mathcal{C}L}$ is algebraizable.
The Blok-Pigozzi Theorem

We have seen that if two action-invariant consequence relations are equivalent, then the lattices of their theories are isomorphic. Is the converse true?
The Blok-Pigozzi Theorem

We have seen that if two action-invariant consequence relations are equivalent, then the lattices of their theories are isomorphic. Is the converse true?

**Theorem** (Wim Blok and Don Pigozzi; 1989)
A substitution-invariant consequence relation $\vdash$ over $Fm(X)$ is algebraizable – with equivalent algebraic semantics a class $\mathcal{K}$ of $\mathcal{L}$-algebras – if and only if there exists a lattice isomorphism between $Th(\vdash)$ and $Th(\models_\mathcal{K})$ that commutes with inverse substitutions.

[The latter condition means that if $\varphi: Th(\vdash) \to Th(\models_\mathcal{K})$ is the lattice isomorphism in question, then for all $\sigma \in End(Fm(X))$ and all $Y \in Th(\vdash)$, $\varphi(\sigma^{-1}(Y)) = \sigma^{-1}(\varphi(Y))$.]
The isomorphism $\varphi$ above is **induced by the equivalence**, in the sense that the diagram below commutes:

\[
\begin{array}{cccc}
\varphi(Eq(X)) & \xrightarrow{\rho} & \varphi(Fm(X)) & \xrightarrow{\tau} & \varphi(Eq(X)) \\
\downarrow^{\Xi \models \kappa} & & \downarrow^{\Xi \vdash} & & \downarrow^{\Xi \models \kappa} \\
Th(\Xi \models \kappa) & \xrightarrow{\varphi} & Th(\Xi \vdash) & \xrightarrow{\varphi^{-1}} & Th(\Xi \models \kappa)
\end{array}
\]
The isomorphism $\varphi$ above is induced by the equivalence, in the sense that the diagram below commutes:

\[
\begin{array}{ccc}
\phi(Eq(X)) & \xrightarrow{\rho} & \phi(Fm(X)) \\
\downarrow & & \downarrow \\
\Xi \models \mathcal{K} & & \Xi \models \mathcal{K} \\
\end{array}
\begin{array}{ccc}
\theta \models \Xi & \xrightarrow{\tau} & \phi(Eq(X)) \\
\downarrow & & \downarrow \\
\Xi \models \mathcal{K} & & \Xi \models \mathcal{K} \\
\end{array}
\begin{array}{ccc}
Th(\Xi \models \mathcal{K}) & \xrightarrow{\phi} & Th(\Xi \models \mathcal{K}) \\
\downarrow & & \downarrow \\
\Xi \models \mathcal{K} & & \Xi \models \mathcal{K} \\
\end{array}
\begin{array}{ccc}
\Xi \models \mathcal{K} & \xrightarrow{\theta^{-1}} & Th(\Xi \models \mathcal{K}) \\
\downarrow & & \downarrow \\
\Xi \models \mathcal{K} & & \Xi \models \mathcal{K} \\
\end{array}
\]

A Short List of Generalizations: W. Blok and D. Pigozzi, 1989 (finite dimensional systems); J. Rebagliato and V. Verdú, 1993 (associative sequents); A. Pynko, 1999 (finite-dimensional sequents for finitary substitution invariant consequence relations); J. Raftery, 2006 (associative sequents).
Modules over Quantales
Let $M$ be an $A$-set. The natural action of $A$ on $M$ extends to an action of the corresponding powersets. More specifically, for $B \subseteq A$ and $X \subseteq M$, we define

$$B \circ X = \{ b \circ x : b \in B, x \in X \}.$$
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$\wp(A)$ is a ringlike object – in which set-union plays the role of addition and complex product serves as multiplication.
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$\varphi(A)$ is a ringlike object – in which set-union plays the role of addition and complex product serves as multiplication. On the other hand, $\varphi(M)$ is a structure corresponding to an abelian group, with set-union playing again the role of addition. The aforementioned action of power sets possesses the critical property of being residuated, which, in this instance, means that it preserves arbitrary unions in each coordinate.
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The preceding considerations lead to the general concept of a (left) module, to be defined shortly.
A quantale is an algebraic structure $A = \langle A, \vee, \cdot, 1 \rangle$ such that:

(Q1) $\langle A, \vee \rangle$ is a complete join semilattice (and, hence, a complete lattice);
(Q2) $\langle A, \cdot, 1 \rangle$ is a monoid;
(Q3) For all $x \in A$, $\{y_i\}_{i \in I} \subseteq A$,

$$x \cdot \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \cdot y_i)$$

and

$$\left( \bigvee_{i \in I} y_i \right) \cdot x = \bigvee_{i \in I} (y_i \cdot x).$$
A quantale is an algebraic structure $A = \langle A, \vee, \cdot, 1 \rangle$ such that:

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$$x \cdot \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \cdot y_i)$$

and

$$\bigvee_{i \in I} (y_i \cdot x) = (\bigvee_{i \in I} y_i) \cdot x.$$ 

The multiplication of a quantale is an example of a binary residuated map, which will be defined shortly.
Let $P$ and $Q$ be posets. A map $\varphi : P \rightarrow Q$ is said to be \textit{residuated} provided there exists a map $\varphi_* : Q \rightarrow P$ such that

$$\varphi(x) \leq y \iff x \leq \varphi_*(y),$$

for all $x \in P$ and $y \in Q$. We refer to $\varphi_*$ as the \textit{residual} of $\varphi$. 
Let $P$ and $Q$ be posets. A map $\varphi : P \to Q$ is said to be residuated provided there exists a map $\varphi^*_\ast : Q \to P$ such that

$$\varphi(x) \leq y \iff x \leq \varphi^*_\ast(y),$$

for all $x \in P$ and $y \in Q$. We refer to $\varphi^*_\ast$ as the residual of $\varphi$. [If $P$ and $Q$ are complete, then $\varphi$ is residuated iff it preserving all joins.]
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Let $P$, $Q$ and $R$ be posets. A map $\circ : P \times Q \to R$ is said to be residuated provided there exist maps $\backslash_\circ : P \times R \to Q$ and $\slash_\circ : R \times Q \to P$ such that

$$x \circ y \leq z \iff x \leq z \slash_\circ y \iff y \leq x \backslash_\circ z,$$

for all $x \in P, y \in Q, z \in R$. 
Let $P$ and $Q$ be posets. A map $\varphi : P \to Q$ is said to be **residuated** provided there exists a map $\varphi^* : Q \to P$ such that
\[
\varphi(x) \leq y \iff x \leq \varphi^*(y),
\]
for all $x \in P$ and $y \in Q$. We refer to $\varphi^*$ as the **residual** of $\varphi$. [If $P$ and $Q$ are complete, then $\varphi$ is residuated iff it preserving all joins.]

Let $P$, $Q$ and $R$ be posets. A map $\circ : P \times Q \to R$ is said to be **residuated** provided there exist maps $\backslash_\circ : P \times R \to Q$ and $/\circ : R \times Q \to P$ such that
\[
x \circ y \leq z \iff x \leq z /\circ y \iff y \leq x \backslash_\circ z,
\]
for all $x \in P, y \in Q, z \in R$. [Again, if $P$, $Q$ and $R$ are complete, then $\circ$ is residuated iff it preserves all joins in each coordinate.]
Let $P$ and $Q$ be posets. A map $\varphi : P \to Q$ is said to be **residuated** provided there exists a map $\varphi^* : Q \to P$ such that

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Let $P$, $Q$, and $R$ be posets. A map $\circ : P \times Q \to R$ is said to be **residuated** provided there exist maps $\backslash \circ : P \times R \to Q$ and $/ \circ : R \times Q \to P$ such that

$$x \circ y \leq z \iff x \leq z / \circ y \iff y \leq x \backslash \circ z,$$

for all $x \in P, y \in Q, z \in R$. [Again, if $P$, $Q$, and $R$ are complete, then $\circ$ is residuated iff it preserves all joins in each coordinate.]

We refer to the operations $\backslash \circ$ and $/ \circ$ as the **left residual** and **right residual** of $\circ$, respectively.
Let $A$ be a quantale and let $P$ a complete join-semilattice. A (left) module action of $A$ on $P$ is a map $\circ : A \times P \to P$ satisfying the following conditions, for all $x \in P$ and for all $a, b \in A$:

1. $1 \circ x = x$
2. $a \circ (b \circ x) = ab \circ x$

3. $\circ$ is residuated; equivalently, it satisfies the following distributive law, for all $a \in A$, $\{u_i\}_{i \in I} \subseteq P$:

$$a \circ \bigvee_{i \in I} u_i = \bigvee_{i \in I} (a \circ u_i).$$
Let $A$ be a quantale and let $P$ a complete join-semilattice. A (left) module action of $A$ on $P$ is a map $\circ : A \times P \to P$ satisfying the following conditions, for all $x \in P$ and for all $a, b \in A$:

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$$a \circ \bigvee_{i \in I} u_i = \bigvee_{i \in I} (a \circ u_i).$$

We will refer to $P = \langle P, \circ \rangle$ is a (left) $A$-module and denote the residuals of $\circ$ by $\setminus$ and $/$. Note that $A$ is an $A$-module with respect to its multiplication.
Consequence Operations in Modules
Let $P$ and $Q$ be $A$ modules. A map $\tau : P \rightarrow Q$ is called a **module morphism** provided it is residuated and action-invariant. Of course, the latter condition means that $a \circ \tau(x) = \tau(a \circ x)$, for all $a \in A$ and all $x \in P$. 
Let $P$ and $Q$ be $A$ modules. A map $\tau : P \to Q$ is called a module morphism provided it is residuated and action-invariant. Of course, the latter condition means that $a \circ \tau(x) = \tau(a \circ x)$, for all $a \in A$ and all $x \in P$.

For a given quantale $A$, $A - Mod$ will denote the category of $A$-modules and $A$-module morphisms.
A closure operator on a complete join-semilattice $P$ is a map $\xi : P \rightarrow P$ with the usual properties of being isotone, enlarging ($x \leq \xi(x)$), and idempotent. It is completely determined by its image $P_\xi$, which is a closure system (that is, a subset of $P$ that is closed under arbitrary meets in $P$.)
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If $P$ is an $A$-module, we use the term action-invariant consequence operation for a closure operator $\xi$ on $P$ that satisfies $a \circ \xi(u) \leq \xi(a \circ u)$, for all $a \in A$ and $u \in P$. 
**Lemma**

Let $P$ be an $A$-module and let $\xi$ be an action invariant consequence operation on $A$.

1. $P_\xi$ is an $A$-module with respect to the scalar multiplication $\circ_\xi : A \times P_\xi \rightarrow P_\xi$ – defined by $a \circ_\xi \xi(u) = \xi(a \circ u)$, for all $a \in A$ and all $u \in P$.

2. The map $\xi : P \rightarrow P_\xi$, with the module structure of $P_\xi$ defined by (1) above, is a module morphism.

3. Every epimorphic image of $P$ in $A - Mod$ is isomorphic to an $A$-module of the form $P_\xi$, for some action invariant consequence operation $\xi$ on $P$.

Note: The epimorphisms in $A - Mod$ are the surjective maps [José Gil-Férez, 2008]
An $A$-module $P$ is called *cyclic*, if there exists an element $u \in P$, such that $P = A \circ u$. 
An $A$-module $P$ is called cyclic, if there exists an element $u \in P$, such that $P = A \circ u$.

**Example:** Let $\mathcal{E}$ denote the endomorphism monoid of $\text{Fm}(X)$. Then $\varphi(\text{Fm}(X))$ is a cyclic $\varphi(\mathcal{E})$-module. Indeed, just let $u = X$ or $u = p$, where $p$ is any fixed variable.
An $A$-module $P$ is called **cyclic**, if there exists an element $u \in P$, such that $P = A \circ u$.

**Example:** Let $E$ denote the endomorphism monoid of $\text{Fm}(X)$. Then $\varphi(\text{Fm}(X))$ is a cyclic $\varphi(E)$-module. Indeed, just let $u = X$ or $u = p$, where $p$ is any fixed variable.

**Proposition**

Every cyclic $A$-module is isomorphic to a module of the form $A_\xi$, for some action-invariant consequence operation $\xi$ on $A$. 
Equivalence in the Setting of Modules
The preceding discussion shows that the action-invariant consequence operations on an $A$-module $P$ correspond bijectively to the epimorphic images of $P$. Thus, these operations may be identified with objects of the category $A-\text{Mod}$. Not surprisingly then, we stipulate that they are equivalent if the $A$-modules associated with them are isomorphic.
The preceding discussion shows that the action-invariant consequence operations on an $A$-module $P$ correspond bijectively to the epimorphic images of $P$. Thus, these operations may be identified with objects of the category $A - Mod$. Not surprisingly then, we stipulate that they are equivalent if the $A$-modules associated with them are isomorphic.

**Definition**

Let $\xi$ and $\zeta$ be consequence operations on the $A$-modules $P$ and $Q$, respectively.

- We say that $\xi$ and $\zeta$ are equivalent, if there exists a module isomorphism $\varphi : P_{\xi} \rightarrow Q_{\zeta}$. We refer to the isomorphism $\varphi$ as an equivalence between $\xi$ and $\zeta$. 
**Induced Equivalences**

**Definition**

Let \( \varphi : P_\xi \to Q_\eta \) be an equivalence between \( \xi \) and \( \zeta \). We say that the equivalence \( \varphi \) is **induced** by the module morphisms \( \tau : P \to Q \) and \( \rho : Q \to P \), if \( \varphi \xi = \zeta \tau \) and \( \varphi^{-1} \zeta = \xi \rho \). In this case we will say that \( \xi \) and \( \zeta \) are equivalent **via** \( \tau \) and \( \rho \).
The Fundamental Categorical Property
Assume that $P$ and $Q$ are $A$-modules, $\xi$ and $\zeta$ are action-invariant consequence operations on $P$ and $Q$ respectively, and $\varphi : P_\xi \to Q_\zeta$ is an isomorphism of modules. We wish to find a module morphism $\tau : P \to Q$ that induces $\varphi$, that is, it completes the square below.

\[
P \xrightarrow{\tau} Q \quad (S)
\]

\[
P \xrightarrow{\xi} P_\xi \xrightarrow{\varphi} Q_\zeta \xrightarrow{\zeta} Q
\]
Assume that $P$ and $Q$ are $A$-modules, $\xi$ and $\zeta$ are action-invariant consequence operations on $P$ and $Q$ respectively, and $\varphi : P_\xi \rightarrow Q_\zeta$ is an isomorphism of modules. We wish to find a module morphism $\tau : P \rightarrow Q$ that induces $\varphi$, that is, it completes the square below.

\[ \begin{array}{ccc}
P_\xi & \xrightarrow{\varphi} & Q_\zeta \\
\downarrow^{\xi} & & \downarrow^{\zeta} \\
P & \xrightarrow{\tau} & Q \\
\end{array} \] (S)

**Theorem**

The objects $P$ of the category $A - Mod$ for which every square of type (S) can be completed are precisely
Assume that $P$ and $Q$ are $A$-modules, $\xi$ and $\zeta$ are action-invariant consequence operations on $P$ and $Q$ respectively, and $\varphi : P_\xi \to Q_\zeta$ is an isomorphism of modules. We wish to find a module morphism $\tau : P \to Q$ that induces $\varphi$, that is, it completes the square below.

\[
\begin{array}{ccc}
P & \xrightarrow{\tau} & Q \\
\downarrow{\xi} & & \downarrow{\zeta} \\
\varphi : P_\xi & \Rightarrow & Q_\zeta
\end{array}
\]

**Theorem**

The objects $P$ of the category $A - Mod$ for which every square of type (S) can be completed are precisely the projective objects of $A - Mod$. 
An object $P$ of $A - Mod$ is called **projective**, if whenever there are modules $Q$ and $R$ and module morphisms $\psi: Q \rightarrow R$ and $\chi: P \rightarrow R$, with $\psi$ an epimorphism, then there exists a morphism $\varphi: P \rightarrow Q$, such that $\chi = \psi \varphi$.

\[
\begin{array}{ccc}
P & \xrightarrow{\varphi} & Q \\
\downarrow \chi & & \downarrow \psi \\
& R & \\
\end{array}
\]
Characterization of Cyclic Projective Modules

**Theorem**
For an $A$-module $P = \langle P, \circ \rangle$, the following statements are equivalent.

1. $P$ is cyclic and a projective object in $A - \mathcal{M}od$.
2. There exist elements $b \in A$ and $u \in P$ such that $b \circ u = u$ and $[(a \circ u) \circ u] b = ab$, for all $a \in A$. 
Characterization of Cyclic Projective Modules

Theorem
For an $A$-module $P = \langle P, \circ \rangle$, the following statements are equivalent.

(1) $P$ is cyclic and a projective object in $A - Mod$.

(2) There exist elements $b \in A$ and $u \in P$ such that $b \circ u = u$ and $[(a \circ u)/u]b = ab$, for all $a \in A$.

Example: $\varphi(\text{Fm}(X))$ is a projective cyclic $\varphi(\mathcal{E})$-module, where $\mathcal{E}$ denote the endomorphism monoid of $\text{Fm}(X)$.

To verify cyclicity and Condition 2, choose $u = \{p\}$, where $p$ is any fixed variable, and $b = \{\kappa_p\}$, where $\kappa_p$ is the substitution that sends all variables to $p$. 
Example: $\varphi(\text{Eq})$ is a projective cyclic $\varphi(\Sigma)$-module.

To verify cyclicity and Condition 2 of the preceding theorem, choose a partition $\{X_1, X_2\}$ of $X$ with $X_1, X_2$ infinite, and choose $p \in X_1, q \in X_2$. Then set $u = \{p \approx q\}, b = \{p \approx q\} / X_1 \times X_2$. 
Example: \( \varphi(\mathcal{E}q) \) is a projective cyclic \( \varphi(\Sigma) \)-module.

To verify cyclicity and Condition 2 of the preceding theorem, choose a partition \( \{X_1, X_2\} \) of \( X \) with \( X_1, X_2 \) infinite, and choose \( p \in X_1, q \in X_2 \). Then set 
\[
u = \{p \approx q\}, \quad b = \{p \approx q\}/_{\times}X_1 \times X_2.\]

**Co-products in** \( A - \text{Mod} \) **exist.** The underlying algebra of the co-product \( \bigsqcup P_i \), of \( (P_i | i \in I) \) in \( A - \text{Mod} \), is the direct product \( \prod P_i \). For each \( i \in I \), the associated embedding \( \varphi_i : P_i \rightarrow \prod P_i \) sends an element \( x \in P_i \) to the element whose \( ith \) coordinate is \( x \) and whose all other coordinates are the least elements of the corresponding factors.
**Example:** $\mathcal{P}(\text{Eq})$ is a projective cyclic $\mathcal{P}(\Sigma)$-module.

To verify cyclicity and Condition 2 of the preceding theorem, choose a partition $\{X_1, X_2\}$ of $X$ with $X_1, X_2$ infinite, and choose $p \in X_1, q \in X_2$. Then set $u = \{p \approx q\}$, $b = \{p \approx q\}/_o X_1 \times X_2$.

**Co-products in $A - Mod$ exist.** The underlying algebra of the co-product $\bigsqcup P_i$, of $(P_i | i \in I)$ in $A - Mod$, is the direct product $\prod P_i$. For each $i \in I$, the associated embedding $\varphi_i : P_i \rightarrow \prod P_i$ sends an element $x \in P_i$ to the element whose $i$th coordinate is $x$ and whose all other coordinates are the least elements of the corresponding factors.

The co-product of a family of projective objects in $A - Mod$ is projective.
Example

Let $\mathcal{S}eq$ denote a set of sequents closed under type. Then $\varphi(\mathcal{S}eq)$ is in general a non-cyclic $\varphi(\mathcal{E})$-module. However, it is a co-product of cyclic projective modules. Thus, it is a projective $\varphi(\mathcal{E})$-module.