The complexified Heisenberg group

H.M. Reimann, F. Ricci

1 Introduction

The Heisenberg group is a fundamental model case which shows up in various contexts. It is the simplest nilpotent group, it is the boundary of a pseudoconvex domain which is biholomorphically equivalent to the ball in complex Euclidean space and it is the simplest example of a space with a natural contact structure.
Motivated by these basic aspects we set out to study the simplest nilpotent Lie group with a complex structure. We investigate in this paper the natural mappings and flows on the complexified Heisenberg group. Basic for our approach is the fact, that the complexified Heisenberg group is a nilpotent Lie group of $H-$type and some of the methods used generalize directly to the setting of this bigger class of step two nilpotent groups. The main result is, that orientation preserving generalized contact mappings and contact flows necessarily have to be holomorphic. This is a Liouville type of result: The mappings are characterized by the infinitesimal condition that the tangent mapping preserves the horizontal subbundle of the tangent space. As a consequence of this condition, which - it should be emphasized - is a real condition, we find that the mappings have to be holomorphic. The same is true for the contact flows. We give an independent proof for this, though this could have been derived from the information that the contact mappings must be holomorphic.

2 H-type groups

Let $\mathbf{n}$ be a real nilpotent Lie algebra equipped with a scalar product such that

$$\mathbf{n} = \mathbf{v} \oplus \mathbf{z}$$

decomposes as an orthogonal direct sum of its center $\mathbf{z}$ and a subspace $\mathbf{v}$ with

$$[\mathbf{v}, \mathbf{v}] \subseteq \mathbf{z}.$$

The linear mapping $J : \mathbf{z} \to End(\mathbf{v})$ is defined by

$$\langle J_Z X, X' \rangle = \langle Z, [X, X'] \rangle \qquad \forall\; X\; X' \in \mathbf{v},\; \forall\; Z \in \mathbf{z}$$

The algebra $\mathbf{n}$ - and with it the connected and simply connected Lie group N with Lie algebra $\mathbf{n}$ - is of $H-$type if for every $X \in \mathbf{v}$ of unit length the mapping $\mathrm{ad}(X)$ is an isometry from $(\ker \operatorname{ad}(X))^{\perp}$ onto $\mathbf{z}$. The mapping J_Z satisfies

$$J_Z^t = -J_Z \qquad \forall\, Z \in \mathbf{z}$$

and on $H-$type algebras

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z'\rangle\, \mathrm{I}$$

in particular

$$J_Z^2 = -|Z|^2\, \mathrm{I}$$

In [CDKR] Cowling, Dooley, Korányi and Ricci characterized the $H-$type groups which appear as the $N-$groups in the Iwasawa decomposition of simple Lie groups of real rank one. These algebras are denoted by $\mathbb{R}^a$, $\mathbf{n}_1^a$, $\mathbf{n}_3^a$ and $\mathbf{n}_7^1$ (in the notation of [CDKR]) and correspond to the Iwasawa N$-$groups of $SO_0(1,n), SU(1,n), Sp(1,n)$ and $F_4(-20)$.

In [K] it is shown, that in the Iwasawa decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{n}$ of a simple Lie algebra of real rank one, the nilpotent group $\mathbf{n}$ equipped with the inner product $-\frac{1}{p+4q}B(\ .\ ,\theta .)$, where B is the Killing form, is an $H-$type group ($p = \dim \mathbf{v}, q = \dim \mathbf{z}$).

In [CDKR] it is shown, that these $H-$type algebras satisfy the J^2-condition:
For all $X \in \mathbf{v}$ and $Z, Z' \in \mathbf{z}$ with $\langle Z, Z'\rangle = 0$ there exists $Z'' \in \mathbf{z}$ such that $J_Z J_{Z'} X = J_{Z''} X$. Furthermore, it is shown that this condition characterizes the nilpotent groups $\mathbf{n}$ in the Iwasawa decomposition of the simple real rank one Lie algebras among all $H-$type groups.

In $H-$type Lie algebras, the exponential mapping is a bijection of $\mathbf{n}$ onto the connected and simply connected Lie group N with Lie algebra $\mathbf{n}$. The elements $n \in N$ are parametrized by $(X, Z) \in \mathbf{v} \oplus \mathbf{z} = \mathbf{n}$:

$$n = \exp(X + Z).$$

The Baker-Campbell-Hausdorff formula then determines the multiplication law on N:

$$(X, Z)(X', Z') = (X + X', Z + Z' + \tfrac{1}{2}[X, X']) \qquad \forall\, (X, Z), (X', Z') \in N.$$

The isometric automorphisms of $H-$type algebras have been determined by Riehm [R]. The full automorphism group of $H-$type algebras was determined by L. Saal [S].

3 The complexification

The Lie algebra $\mathbf{n}_0$ of the Heisenberg group is the 3–dimensional real Lie algebra generated by the vectors W_1, W_2, Z with commutator relations

$$\begin{aligned} [W_1, W_2] &= Z \\ [W_1, Z] &= [W_2, Z] = 0. \end{aligned}$$

The Heisenberg group N_0 is the connected nilpotent Lie group with Lie algebra $\mathbf{n}_0$. If we assume that W_1, W_2, Z form an orthonormal basis, $\mathbf{n}_0$ clearly is a $H-$type algebra. The complexified Lie algebra $\mathbf{n} = (\mathbf{n}_0)^{\mathbf{C}}$ is obtained by taking W_1, W_2, Z to be the generators of a complex Lie algebra with the same bracket relations as above.
The complexified Heisenberg group is the (uniquely defined) connected complex Lie group N with Lie algebra $\mathbf{n}$. As a model for N we use the space $\mathbf{C}^3 = \{(w_1, w_2, z)\}$ with the multiplication law derived from the Baker-Campbell-Hausdorff formula:

$$(w_1, w_2, z)(w_1', w_2', z') = (w_1 + w_1', w_2 + w_2', z + z' + \frac{1}{2}(w_1 w_2' - w_2 w_1')).$$

The Lie algebra is isomorphic to the space of left invariant holomorphic vectorfields in $T^{1,0}\mathbf{C}^3$ (the holomorphic tangent space of $\mathbf{C}^3$). Under this isomorphism W_1, W_2 and Z are mapped onto the vectorfields

$$\begin{aligned} W_1 &= \frac{\partial}{\partial w_1} - \frac{1}{2} w_2 \frac{\partial}{\partial z} \\ W_2 &= \frac{\partial}{\partial w_2} + \frac{1}{2} w_1 \frac{\partial}{\partial z} \\ Z &= \frac{\partial}{\partial z}. \end{aligned}$$

The dual left invariant forms are dw_1, dw_2 and

$$\vartheta = dz + \frac{1}{2} w_2\, dw_1 - \frac{1}{2} w_2\, dw_2.$$

We will also look at $\mathbf{n}$ as a real Lie algebra $\mathbf{n_R}$ with complex structure J derived from multiplication by i. This algebra is then generated as a vector space by

$$X_1, X_2, X_3 = JX_1,\ X_4 = JX_2,\ Z_1,\ Z_2 = JZ_1$$

and the bracket relations are

$$\begin{aligned} Z_1 &= [X_1, X_2] = -[X_3, X_4] \\ Z_2 &= [X_1, X_4] = -[X_2, X_3] \end{aligned}$$

with the remaining undefined brackets equal to zero. The bracket is of course $J-$invariant:

$$[JX, Y] = J[X, Y] \qquad X, Y \in \mathbf{n_R}.$$

The Lie algebra $\mathbf{n_R}$ is of $H-$type. To verify this, set

$$\mathbf{n_R} = \mathbf{v} + \mathbf{z}$$

with $\mathbf{v} = \mathrm{span}\{X_1, X_2, X_3, X_4\}$ and $\mathbf{z} = \mathrm{span}\{Z_1, Z_2\}$ such that one has

$$\begin{aligned}
[\mathbf{v},\mathbf{v}] &= \mathbf{z} \\
[\mathbf{v},\mathbf{z}] &= [\mathbf{z},\mathbf{z}] = 0.
\end{aligned}$$

Define the scalar product on $\mathbf{n}_\mathbf{R}$ in such a way, that $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ is an orthonormal system and consider the mappings $J_i : \mathbf{v} \to \mathbf{v} \;\; i = 1,2$ defined by the relations

$$(J_i X, Y) = (Z_i, [XY]) \qquad X, Y \in \mathbf{v}.$$

Explicit calculations give

$$\begin{aligned}
J_1X_1 &= X_2 & \qquad J_2X_1 &= X_4 \\
J_1X_2 &= -X_1 & J_2X_2 &= -X_3 \\
J_1X_3 &= -X_4 & J_2X_3 &= X_2 \\
J_1X_4 &= X_3 & J_2X_4 &= -X_1
\end{aligned}$$

and $J = -J_2J_1 = J_1J_2$.
It is now clear that $\mathbf{n}_\mathbf{R}$ is a Lie algebra of $H-$type: The conditions

$$J_1^2 = J_2^2 = -I$$

are clearly satisfied. The left invariant vectorfields on $N_\mathbf{R}$ are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1} - \frac{1}{2}\left(x_2\frac{\partial}{\partial z_1} + x_4\frac{\partial}{\partial z_2}\right) \\
X_2 &= \frac{\partial}{\partial x_2} + \frac{1}{2}\left(x_1\frac{\partial}{\partial z_1} + x_3\frac{\partial}{\partial z_2}\right) \\
X_3 &= \frac{\partial}{\partial x_3} + \frac{1}{2}\left(x_4\frac{\partial}{\partial z_1} - x_2\frac{\partial}{\partial z_2}\right) \\
X_4 &= \frac{\partial}{\partial x_4} - \frac{1}{2}\left(x_3\frac{\partial}{\partial z_1} - x_1\frac{\partial}{\partial z_2}\right) \\
Z_1 &= \frac{\partial}{\partial z_1} \\
Z_2 &= \frac{\partial}{\partial z_2}
\end{aligned}$$

and the dual left invariant forms are

$$dx_1, dx_2, dx_3, dx_4, \vartheta_1, \vartheta_2$$

with

$$\begin{aligned}
\vartheta_1 &= dz_1 + \frac{1}{2}(x_2\, dx_1 - x_1\, dx_2 - x_4\, dx_3 + x_3\, dx_4) \\
\vartheta_2 &= dz_2 + \frac{1}{2}(x_4\, dx_1 - x_3\, dx_2 + x_2\, dx_3 - x_1\, dx_4).
\end{aligned}$$

4 Contact mappings

On an $H-$type Lie algebra $\mathbf{n} = \mathbf{v} \oplus \mathbf{z}$ we choose an orthogonal system

$$X_1, \ldots, X_n, Z_1, \ldots, Z_m \qquad \begin{array}{lll} X_j \in \mathbf{v} & j = 1, \ldots, n \\ Z_j \in \mathbf{z} & j = 1, \ldots, m \end{array}$$

For the group elements (X, Z) the coordinates $(x_1, \ldots, x_n, z_1, \ldots, z_m)$ are used. If $X = \sum_{j=1}^{n} x_j X_j,\ Z = \sum_{j=1}^{m} z_j Z_j$, then $(X, Z) \in N$ stands for the element

$$\exp(X + Z) = \exp\left(\sum_{j=1}^{n} x_j X_j, + \sum_{j=1}^{m} z_j Z_j\right)$$

In the coordinates, a $1-$parameter group through $0 \in N$ in direction X_j is given by $(tX_j, 0)$. Under left translation by $(X, Z) \in N$ this becomes the curve

$$(X, Z)(tX_j, 0) = (X + tX_j, Z + \tfrac{1}{2}[X, tX_j])$$

through $(X, Z) \in N$. In local coordinates, the tangent vector to this curve at (X, Z) is the vector $(X_j, \frac{1}{2}[X, X_j])$. The left invariant vectorfield $\tilde{X}_j$, which corresponds to $X_j \in \mathbf{v}$ is then

$$\begin{aligned}
\tilde{X}_j(X, Z) &= \tfrac{\partial}{\partial x_j} + \tfrac{1}{2} \sum_{k=1}^{m} \langle Z_k, [X, X_j] \rangle \tfrac{\partial}{\partial z_k} \\
&= \tfrac{\partial}{\partial x_j} - \tfrac{1}{2} \sum_{k=1}^{m} \langle J_{Z_k} X_j, X \rangle \tfrac{\partial}{\partial z_k}
\end{aligned}$$

and the left invariant vectorfields $\tilde{Z}_j$ are $\tilde{Z}_j(X, Z) = \frac{\partial}{\partial z_j}$.

A dual basis to $\tilde{X}_1, \ldots, \tilde{X}_m, \tilde{Z}_1, \ldots, \tilde{Z}_n$ is

$$dx_1, \ldots, dx_n, \vartheta_1, \ldots, \vartheta_m$$

with

$$\vartheta_k = dz_k + \tfrac{1}{2} \sum_j \langle J_{Z_k} X_j, X \rangle \, dx_j .$$

For fixed k the form $d\vartheta_k$ is calculated to be

$$d\vartheta_k = \tfrac{1}{2} \sum_{j,l} \gamma_{lj}^k \, dx_l \wedge dx_j$$

where $J_{Z_k} X_j = \sum_l \gamma_{lj}^k \, X_l$. This shows that

$$\tilde{Z} \lrcorner \, d\vartheta_k = 0 \qquad \forall \, Z \in \mathbf{z}$$

and

$$-d\vartheta_k(\tilde{X}, \tilde{Y}) = \langle J_{Z_k} X, Y \rangle \qquad \forall \, X, Y \in \mathbf{v}$$

Mappings $f : N \to N$ which preserve the horizontal bundle

$$HN = \text{ span } \{\tilde{X}_1, \ldots, \tilde{X}_n\} \quad \subset TN$$

are called generalized contact mappings. On the Heisenberg group N_0 these mappings are in fact contact mappings as defined classically by the condition $f^* \vartheta = c \, \vartheta$ for a contact form ϑ. On general $H-$type groups we will still call them contact mappings. Formulated in terms of the forms ϑ_i, this means, that $f^* \vartheta_j$ has to be a linear combination of the forms $\vartheta_1, \ldots, \vartheta_m$:

$$f^* \vartheta_j = \sum_{i=1}^m C_{ji} \vartheta_i \qquad j = 1, \ldots, m.$$

In matrix notation

$$f^* \vartheta = C \, \vartheta$$

with $C = (C_{ji})$ a regular matrix at every point of N, and with

$$\vartheta = \begin{pmatrix} \vartheta_1 \\ \vdots \\ \vartheta_n \end{pmatrix} .$$

Grading preserving algebra homomorphisms lift to contact mappings on the groups. An important example is the one parameter group D_t of automorphisms given by

$$D_t \, X = e^t X \qquad X \in \mathbf{v}$$

$$D_t \, Z = e^{2t} Z \qquad Z \in \mathbf{z}$$

which lifts to the dilation group ($s = e^t$)

$$\delta_s(x, z) = (sx, s^2 z) \qquad (x, z) \in N.$$

In general, there will be more contact mappings than just the lifts of grading preserving algebra homomorphisms. But surprisingly, in the case $\dim \mathbf{z} > 2$ it must be expected that the contact diffeomorphisms derive from a finite dimensional Lie group (see [R]). On the Heisenberg group contact diffeomorphisms abound whereas the complexified Heisenberg group (with $\dim \mathbf{z} = 2$) is an intermediate case:

Theorem 4.1 *If f is a C^2−contact mapping on $N_{\mathbf{R}}$ with nonnegative Jacobian determinant* ($\det f_* \geq 0$) *and such that the singular set*

$$S = \{p \in N_{\mathbf{R}} : \det f_*(p) =\}$$

is nowhere dense, then f is holomorphic.

It should be observed, that the contact property is an infinitesimal condition which is formulated in purely real terms. The complex structure on $N_{\mathbf{R}}$ is given by J and the theorem expresses that J commutes with the tangent mapping f_*. Written out in coordinates, f then satisfies the Cauchy-Riemann equations.
The proof of the theorem is based on Pansu's notion of differentiability on Carnot groups and on the fact that the algebra automorphisms of $\mathbf{n}_{\mathbf{R}}$ which preserve the grading necessarily have to be complex linear.

5 Quasiconformality and Differentiability

The Carnot-Caratheodory metric D on N is defined as follows [CDKR, p. 11]: Set

$$\begin{aligned} B(X, Z) &= \left(\tfrac{|X|^4}{16} + |Z|^2\right)^{\frac{1}{4}} && (X, Z) \in N \\ d(n, n') &= B(n^{-1} n') && n, n' \in N \end{aligned}$$

and for piecewise smooth curves $\gamma : [0, 1] \to N$ set

$$\lambda(\gamma) = \limsup \left\{ \sum_{j=1}^{n} d(\gamma(t_j), \gamma(t_{j-1}) : 0 = t_0 < t_1, \ldots < t_n = 1 \right\}$$

the lim sup being taken as $\max_j |t_j - t_{j-1}| \to 0$. Then, the Carnot-Caratheodory distance between two points n and n' is defined as the infimum of the arc length $\lambda(\gamma)$ of all piecewise smooth curves γ connecting n to n'

$$D(n, n') = \inf_{\gamma} \lambda(\gamma).$$

The rectifiable curves are the curves with $\lambda(\gamma) < \infty$. Tangent vectors to rectifiable curves (when they exist) are horizontal: they are left translates of vectors in $\mathbf{v}$.

A homeomorphism $f : U \to U'$ between open subsets of an $H-$type group is a $K-$quasiconformal mapping, if

$$H(X) = \limsup_{r \to o} \frac{\max\limits_{D(x,y)=r} D(f(x), f(y))}{\min\limits_{D(x,y)=r} D(f(x), f(y))}$$

is uniformly bounded on n with $||H||_\infty \leq K$.

There is a notion of differentiability for $H-$type groups (introduced by Pansu in 1987): A continuous mapping $f : U \to U'$ defined in a neighbourhood of $0 \in N$ and such that $f(0) = 0$ is differentiable at 0 if

$$\delta_t \circ f \circ \delta_t^{-1}$$

converges uniformly on compact sets when t tends to ∞. The limit is called the $P-$derivative of f at 0. Using left translation on N, the derivative of $f, Df(n)$, at an arbitrary point n is defined. Pansu [P, 1987] proved the following Theorem.

Theorem 5.1 *(Pansu). Quasiconformal mappings $f : U \to U'$ between open subsets of an $H-$type group are a.e. $P-$differentiable and the $P-$derivative is a group homomorphism which commutes with δ.*

In fact, Pansu states this theorem for more general nilpotent groups, the so called Carnot groups. Furthermore, the mapping f can be taken to be a quasiconformal mapping between two different groups.
The derivative Df at a point $n \in N$ is a group homomorphism and it induces an algebra homomorphism $df : \mathbf{n}$ to $\mathbf{n}$ which leaves the subspaces $\mathbf{v}$ and $\mathbf{z}$ invariant (since Df commutes with δ_t).

We specialize now to the complexified Heisenberg group. The mappings will be written in coordinates

$$f(x_1, x_2, x_3, x_4, z_1, z_2) = f(x, z) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \psi_1, \psi_2)(x, z) = (\varphi, \psi)(x, z).$$

The tangent mappings of f_* will then be represented by the matrix

$$I = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = (dx_i(f_*X_j)) \qquad B = (dx_i(f_*Z_\ell))$$
$$C = (\vartheta_k(f_*X_j)) \qquad D = (\vartheta_k(f_*Z_\ell))$$

If f is a contact mapping, then $C = 0$. Furthermore

$$dx_i(f_*X_j) = X_j\varphi_i \quad \text{and} \quad dx_i(f_*Z_\ell) = Z_\ell\varphi_i$$

whereas $\vartheta_k(f_*Z_\ell)$ is expressed through $\vartheta_k = dz_k + \sum_{i=1}^4 q_{ki}dx_i$ in a more complicated way as

$$\vartheta_k(f_*Z_\ell) = Z_\ell\psi_k + \sum_{i=1}^4 q_{ki}Z_\ell\varphi_i.$$

(The q_{ki} have to be taken at the image point. They are linear combinations of the $\varphi_1, \varphi_2, \varphi_3, \varphi_4$.)
The $P-$derivative at a point p is the linear mapping determined by the matrices A and D of the tangent mapping f_* at p:

$$Df(x,z) = (Ax, Dz).$$

6 The grading preserving algebra homomorphisms

According to Pansu's result, C^1-contact diffeomorphisms are $P-$differentiable. This brings us to the problem of classifying the Lie algebra homomorphisms which preserve the grading.
The groups $\mathrm{Aut}_{\mathbf{v}}(\mathbf{n})$ of grading preserving automorphisms of the $H-$type Lie algebras have been classified by Riehm [R] and L. Saal [S]. For the complexified Heisenberg group their considerations simplify. This allows us to include a complete argument.
Let us write (g,h) with $g \in \mathrm{End}(\mathbf{v})$, $h \in \mathrm{End}(\mathbf{z})$ for a grading preserving homomorphism of the algebra of the complexified Heisenberg group. This algebra is considered as a real algebra with additional complex structure J. The condition is then, that

$$[gx, gy] = h[x,y] \qquad \text{for all } x,y \in \mathbf{v}.$$

Furthermore, the mappings

$$(J_z, -\rho_z) \qquad |z| = 1$$

with ρ_z the reflection in the line orthogonal to z generate a subgroup called Pin(2) of the automorphism group. Note that $\{\rho_z : |z| = 1\}$ generates all of $0(2)$.
A grading preserving homomorphism (g,h) can therefore be represented by decomposing h as

$$h = k_1\, d\, k_2 \quad d = \mathrm{diag}(d_1, d_2) \quad d_1 \geq 0, d_2 \geq 0,\ k_1, k_2 \in 0(2)$$

and it takes the form

$$(g,h)=(g_1,k_1)(g_0,d)(g_2,k_2)$$

with $(g_i,k_i)\in \mathrm{Pin}(2)$ for $i=1,2$. For the grading preserving homomorphism (g_0,d) one finds for $i=1,2$

$$(J_i g_0 X, g_0 Y)=(Z_i,[g_0X,g_0Y])=(Z_i,d[X,Y])=d_i(J_iX,Y).$$

This then shows that

$$\begin{aligned} g_0^t J_i g_0 &= d_i J_i \\ \det g_0^2 &= d_1^2=d_2^2. \end{aligned}$$

We distinguish now two cases. Either $d_1=d_2\neq 0$ wich leads to

$$\begin{aligned} (g_0,h_0) &= \delta_s(g',I) \qquad s^2=d_1=d_2, \\ (g,h) &= \delta_3(g_3k_3)(g'',I) \end{aligned}$$

with $(g_3,k_3)\in \mathrm{Pin}(2)$, $g''\in GL(4)$ with $\det g''=\pm 1$, or $d_1=d_2=0$ and the homomorphism is of the form

$$(g,0) \qquad \det g=0.$$

Let us first look at $(g,I)\in \mathrm{Aut}_{\mathbf{v}}(\mathbf{n_R})$. From

$$(J_i gx,gy)=(J_ix,y) \qquad i=1,2$$

it follows that $g^tJ_ig=J_i \quad i=1,2$ and therefore $Jg=J_1J_2g=J_1(g^t)^{-1}J_2=-(g^tJ_1)^{-1}J_2=-gJ_1^{-1}J_2=gJ_1J_2=gJ.$
This equality expresses, that g is complex linear (hence $\det g=1$). The elements $(J_z,-\rho_z)$ which generate $\mathrm{Pin}(2)$ are all complex antilinear, since

$$\begin{aligned} JJ_1=J_1J_2J_1=-J_1J \\ JJ_2=J_1J_2J_2=-J_2J. \end{aligned}$$

Consequently, (g_3,k_3) is complex linear or complex antilinear according to whether $\det k_3$ is $+1$ or -1. Altogether it follows that an automorphism (g,k) is either complex linear or complex antilinear. If it is complex linear, (g,k) can be expressed with respect to the basis $\{W_1,W_2,Z\}$ as a complex matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}=\begin{pmatrix} \alpha_{11} & \alpha_{12} & \\ \alpha_{21} & \alpha_{22} & \\ & & 0 \end{pmatrix}$$

and the commutator relation $[gW_1, gW_2] = kZ$ gives

$$\delta = \det \alpha.$$

For complex antilinear mappings $\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$ describes the mapping $T^{1,0}\mathbf{C}^3 \to T^{0,1}\mathbf{C}^3$ with respect to the bases $\{W_1, W_2, Z\}$ and $\{\overline{W}_1, \overline{W}_2, \overline{Z}\}$.

At last we have to look at the homomorphisms

$$(g, 0) \qquad \det g = 0.$$

They satisfy

$$(J_i gx, gy) = 0 \qquad \forall x, y \in \mathbf{v}.$$

The range $R = \{gx \in \mathbf{v} : x \in \mathbf{v}\}$ and its image J_iR under J_i are therefore orthogonal. This implies $\dim_{\mathbf{R}} R \leq 2$. Furthermore, if $\dim_{\mathbf{R}} R = 2$, then $J_1R = J_2R$ by orthogonality. Consequently, the range is complex invariant: $JR = R$. In general, it cannot be concluded that g is complex linear and in fact, the kernel of g can be an arbitrary real subspace of dim ≥ 2. The classification is now complete (cf. [R], [S]). It is summarized in the

Proposition 6.1 *The grading preserving automorphisms of $\mathbf{n}_{\mathbf{R}} = \mathbf{v} + \mathbf{z}$ are complex linear if the orientation is preserved and complex antilinear, if it is reversed. These automorphisms can be represented by complex matrices*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \\ \alpha_{21} & \alpha_{22} & \\ & & \lambda \end{pmatrix} = \begin{pmatrix} \alpha & \\ & \det \alpha \end{pmatrix}.$$

The grading preserving homomorphisms, which are not invertible are of the form $(g, 0)$ with $\mathrm{rank}_{\mathbf{R}}\, g \leq 2$. In the case $\mathrm{rank}_{\mathbf{R}} g = 2$ the range of the homomorphism is complex invariant.

7 Proof of the theorem

The C^2-contact transformations f are $P-$differentiable. if the derivative f_* is given by the matrix

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

then the $P-$derivative is the group homomorphism (A, D). This can be seen directly from the definition of $P-$differentiability. The induced Lie algebra homomorphism *df* is also described by (A, D). Let us now assume that $\det f_* > 0$ at $p \in N_{\mathbf{R}}$. Then (A, D) is an automorphism and by the proposition it is complex linear:

$$JA = AJ \quad \text{and} \quad JD = DJ$$

where J is the complex structure (restricted to $\mathbf{v}$ and $\mathbf{z}$ respectively). It remains to show that B commutes with J. For this purpose let us first reformulate the $J-$invariance of A: The tangent mapping f_* maps the left invariant horizontal vector fields $X = \sum c_j X_j,\ c_j \in \mathbf{R}$, into the horizontal vector fields

$$\sum_{i,j} a_{ij} c_j X_i = \sum_{i,j} (X_j \varphi_i) c_j X_i = \sum_i (X\varphi_i) X_i.$$

(Here and in the following we omit the ˜ in the notation for the left invariant vector fields).

$J-$invariance of A then means

$$\sum_i (X\varphi_i) JX_i = \sum_i (JX\varphi_i) X_i.$$

We now claim that

$$\sum_i (Z\varphi_i) JX_i = \sum_i (JZ\varphi_i) X_i$$

for all $Z \in \mathbf{z}$, and this is the same as saying that B commutes with J.

Since f_* is continuous and since the singular set S is assumed to be nowhere dense, we can then conclude that f_* commutes with J at all points, i.e. f is holomorphic.

For the proof of our claim we start with the basic equation

$$\sum_i (X\varphi_i) JX_i = \sum_i (JX\varphi_i) X_i.$$

If $s = \sum s_j X_j$ is an arbitrary horizontal vector field, then

$$\Big(\sum (X\varphi_i) JX_i, s\Big) = \sum_{ij} X\varphi_i s_j (JX_i, X_j) = \Big(\sum JX\varphi_i X_i, s\Big) = \sum_i (JX\varphi_i) s_i.$$

If $\det f_*(p) > 0$, then f is injective in a neighbourhood U of p and the horizontal C^2-vector fields on fU can be represented in the form

$$s = \sum_j s_j(p) X_j(f(p)) \qquad p \in U$$

with X_j the basic left invariant vector fields ($j = 1, \dots, 4$).

We express the previous equation in form of an integral equation and use partial integration. For all horizontal C^2-vector fields s with compact support in fU

$$0 = \int_U \sum_{i,j} \varphi_i X s_j (JX_i, X_j) dp - \int_U \sum_i \varphi_i (JXs_i) dp$$

(dp denotes the invariant measure on $N_{\mathbf{R}}$). Conversely, if we establish that for all $Z \in \mathbf{z}$ and for all such vector fields s

$$0 = \int_U \sum_{i,j} \varphi_i Z s_j (JX_i, X_j) dp - \int_U \sum_i \varphi_i (JZ s_i) dp$$

then our claim is proved.
Start then by differentiating the integral equation with respect to the left invariant horizontal vector field Y:

$$\begin{aligned} &\int \textstyle\sum_{i,j} Y\varphi_i X s_j (JX_i, X_j) dp - \int \sum_i Y\varphi_i JX s_i dp \\ &+ \int \textstyle\sum_{i,j} \varphi_i Y X s_j (JX_i, X_j) dp - \int \sum_i \varphi_i Y JX s_i \, dp = 0. \end{aligned}$$

Interchange X with Y and subtract to get

$$\begin{aligned} &\int \{ \sum_{i,j} Y\varphi_i X s_j (JX_i, X_j) - \sum_{i,j} X\varphi_i Y s_j (JX_i, X_j) \} dp \\ &- \int \{ \sum_i Y\varphi_i JX s_i - \sum_i X\varphi_i JY s_i \} dp \\ &- \int \sum_i \varphi_i [X, Y] s_i (JX_i, X_j) dp \\ &- \int \{ \sum_i \varphi_i Y JX s_i - \sum \varphi_i X JY s_i \} dp = 0. \end{aligned}$$

The same equation with X, Y replaced by JX, JY is

$$\begin{aligned} &\int \{ \sum JY\varphi_i JX s_j (JX_i, X_j) - \sum JX\varphi_i JY s_j (JX_i, X_j) \} dp \\ &+ \int \{ \sum JY\varphi_i X s_i - \sum JX\varphi_i Y s_i \} dp \\ &- \int \sum \varphi_i [JX, JY] s_j (JX_i, X_j) dp \\ &+ \int \{ \sum \varphi_i JY X s_i - \sum \varphi_i JX Y s_i \} dp = 0. \end{aligned}$$

Subtract the 2 equations to get

$$\begin{aligned} D = &\int \sum Y\varphi_i X s_j (JX_i, X_j) dp - \int \sum JY\varphi_i X s_i dp \\ &- \int \sum X\varphi_i Y s_j (JX_i, X_j) dp + \int \sum JX\varphi_i Y s_i dp \\ &- \int \sum JY\varphi_i JX s_j (JX_i, X_j) dp - \int \sum Y\varphi_i JX s_i dp \end{aligned}$$

$$+\int\sum JX\varphi_i JY s_j(JX_i,X_j)dp+\int\sum X\varphi_i JY s_i dp$$
$$-\int\sum \varphi_i[X,Y]s_j(JX_i,X_j)dp+\int\sum\varphi_i[JX,JY]s_j(JX_i,X_j)dp$$
$$-\int\sum\varphi_i YJX s_i dp+\int\sum\varphi_i JXY s_i dp$$
$$+\int\sum\varphi_i XJY s_i dp-\int\sum\varphi_i JYX s_i dp.$$

By hypothesis, the first four pairs each add up to zero. Therefore

$$0=-2\int\sum\varphi_i[X,Y]s_j(JX_i,X_j)dp+\int\sum\varphi_i[JX,Y]s_i dp+\int\sum\varphi_i[X,JY]s_i dp$$
$$0=\int\sum\varphi_i[X,Y]s_j(JX_i,X_j)dp-\int\sum\varphi_i J[X,Y]s_i dp.$$

8 Local one-parameter families of contact mappings

Let us come back to the general case of $H-$type groups. The contact mappings f on N preserve the horizontal bundle $HN=\text{span}\{X_1,\ldots,X_n\}\subset TN$. This can be expressed with the dual forms as

$$f^*\vartheta=C\vartheta$$

with the matrix notation of section 4.

Assume now u is a vector field on N with associated flow f_t satisfying

$$f_t^*\vartheta=C_t\vartheta.$$

Then differentiation in t gives

(*) $$u\lrcorner\, d\vartheta+d(u\lrcorner\,\vartheta)=A_t\vartheta$$

with

$$A_t=(C_t^\bullet C_t^{-1})\circ f_t=(a_{ij}).$$

Decompose

$$u=h+p$$

with $h\in HN$ and $p\in ZN=\text{span}\{Z_1,\ldots,Z_n\},\ p=\sum p_jZ_j$.
Observing that $p\lrcorner\, d\vartheta=0$ and $h\lrcorner\,\vartheta=0$, equation (*) can be expressed as

(1) $$h\lrcorner\, d\vartheta+d(p\lrcorner\,\vartheta)=A_t\vartheta$$

with

$$p\lrcorner\,\vartheta \;=\; \begin{pmatrix}\vartheta_1(p)\\ \vdots\\ \vartheta_m(p)\end{pmatrix}=\begin{pmatrix}p_1\\ \vdots\\ p_m\end{pmatrix}$$

$$d(p \lrcorner \vartheta) \quad = \quad \begin{pmatrix} dp_1 \\ \vdots \\ dp_m \end{pmatrix}$$

Applied to the vectorfield Z_k equation (1) gives

$$\left(\sum_{j=1}^{m} a_{ij}\vartheta_j - dp_i \right) (Z_k) = 0 \qquad \text{for all } i, k$$

$$Z_k p_i = a_{ik}.$$

Equation (1) becomes

$$(2) \qquad \begin{aligned} h \lrcorner d\vartheta_i &= \sum_{k=1}^{m} (Z_k p_i)\vartheta_k - dp_i \\ &= -\sum_{j=1}^{n} (X_j p_i) dx_j. \end{aligned}$$

The problem then is to find conditions on the $p_1, \ldots, p_m$ such that the equations (2) have a common solution h.
We set $h = \sum_{j=1}^{n} h_j X_j$ where h_j are functions of $(X, Z) \in N$

$$(h \lrcorner \mathrm{d}\vartheta_i)(X) = \mathrm{d}\vartheta_i(h, X) = -Xp_i \qquad i = 1, \ldots, m.$$

If the mappings J_{Z_i} of $\mathbf{v}$ onto itself are considered as mappings on the left invariant vectorfields, then $-d\vartheta_i(X, Y) = \langle J_{Z_i} X, Y \rangle$. Therefore the equations have the solution

$$(3) \qquad h = -J_{Z_i} \mathrm{grad}_0 p_i \qquad i = 1, \ldots, m$$

where $\mathrm{grad}_0 p_i = \sum_{j=1}^{n} (X_j p_i) X_j$ is the "horizontal gradient". The equation (1) can then be solved for h if and only if $J_{Z_i} \mathrm{grad}_0 p_i$ is independent of i. The solution is given by (3) and the vector fields in (*) must be of the form

$$u = -J_{Z_i} \mathrm{grad}_0 p_i + \sum_{i=1}^{m} p_i Z_i$$

with $J_{Z_i} \mathrm{grad}_0 p_i$ independent of i. Conversely, if $p_1, \ldots, p_m$ can be found such that $J_{Z_i} \mathrm{grad}_0 p_i$ is independent of i, then the flow generated by the vector field

$$u = -J_{Z_i} \mathrm{grad}_0 p_i + \sum_{i=1}^{m} p_i Z_i$$

is a local flow of contact transforms. We thus have a method at hand to construct contact mappings.

We summarize our findings in the following

Theorem 8.1 *On $H-$type groups the vector fields u which generate local one-parameter flows of contact mappings are of the form*

$$u = -J_{Z_i}\text{grad}_0 p_i + \sum p_j Z_j$$

with $p_1, \ldots, p_m$ such that $J_{Z_i}\text{grad}_0 p_i$ is independent of i.

9 Contact flows on the complexified Heisenberg group

We specialize now to the complexified Heisenberg group. $\{X_1, \ldots, X_4, Z_1, Z_2\}$ is the basis of the Lie algebra (as a real vector space) introduced in section 3. The mappings J_1, J_2 and $J = J_1 J_2$ are given by

$$\begin{array}{lllllllll} J_1X_1 & = & X_2 & J_2X_1 & = & X_4 & JX_1 & = & X_3 \\ J_1X_2 & = & -X_1 & J_2X_2 & = & -X_3 & JX_2 & = & X_4 \\ J_1X_3 & = & -X_4 & J_2X_3 & = & X_2 & JX_3 & = & -X_1 \\ J_1X_4 & = & X_3 & J_2X_4 & = & -X_1 & JX_4 & = & -X_2. \end{array}$$

They reflect multiplication by i, j, k in the space of quaternions. The requirement

$$J_1\text{grad}_0 p_1 = J_2\text{grad}_0 p_2$$

is equivalent to the "Cauchy-Riemann" equation

$$\begin{array}{ll} X_1p_1 = X_3p_2 & X_2p_1 = X_4p_2 \\ X_3p_1 = -X_1p_2 & X_4p_1 = -X_2p_2. \end{array}$$

In complex notation, with

$$\begin{array}{rclrcl} w_1 & = & x_1 + ix_3 & & & \\ w_2 & = & x_2 + ix_4 & & & \\ z & = & z_1 + iz_2 & & & \\ W_1 = \frac{1}{2}(X_1 - iX_3) & = & \frac{\partial}{\partial w_1} - \frac{1}{2}w_2\frac{\partial}{\partial z} & \overline{W}_1 & = & \frac{\partial}{\partial \overline{w}_1} - \frac{1}{2}\overline{w}_2\frac{\partial}{\partial \overline{z}} \\ W_2 = \frac{1}{2}(X_2 - iX_4) & = & \frac{\partial}{\partial w_2} + \frac{1}{2}w_1\frac{\partial}{\partial z} & \overline{W}_2 & = & \frac{\partial}{\partial \overline{w}_2} + \frac{1}{2}\overline{w}_1\frac{\partial}{\partial \overline{z}} \\ Z = \frac{1}{2}(Z_1 - iZ_2) & = & \frac{\partial}{\partial z} & \overline{Z} & = & \frac{\partial}{\partial \overline{z}} \end{array}$$

the "potential" $p = p_1 + ip_2$ then has to satisfy the equations

$$\begin{aligned} \overline{W}_1 p &= 0 \\ \overline{W}_2 p &= 0 \end{aligned}$$

and the bracket relations $[W_1, W_2] = Z$, $[\overline{W}_1, \overline{W}_2] = \overline{Z}$ imply

$$\overline{Z} p = 0.$$

As a consequence, $\frac{\partial}{\partial \bar{z}} p = \frac{\partial}{\partial \overline{w}_1} p = \frac{\partial}{\partial \overline{w}_2} p = 0$, which means that p has to be holomorphic in its Lie algebra coordinates w_1, w_2, z. Let us summarize our findings about contact flows on the complexified Heisenberg group:

Theorem 9.1 *On the complexified Heisenberg group the vector field*

$$u = p_1 Z_1 + p_2 Z_2$$

with

$$h = -J_1 \mathrm{grad}_0 p_1$$

generates a local flow of contact transformations if and only if $p_1 + ip_2$ *is holomorphic. In this situation,* $J_1 \mathrm{grad}_0 p_1 = J_2 \mathrm{grad}_0 p_2$.

We conclude by writing the flow in complex notation. Using the standard isomorphism $T\mathbf{C}^3 \to T^{1,0}\mathbf{C}^3$ of the real tangent space to $\mathbf{C}^3$ onto the $(1,0)-$vectors in the complexified tangent space $(T\mathbf{C}^3) \otimes \mathbf{C}$, the vector field u is given by

$$u = pZ + W_2 p W_1 - W_1 p W_2.$$

The tangent mapping f_{t_*} of the holomorphic flow f_t can be described by a $3 \times 3-$complex matrix

$$f_{t_*} = \begin{pmatrix} dw_1(f_* W_1) & dw_1(f_* W_2) & dw_1(f_* Z) \\ dw_2(f_* W_1) & dw_2(f_* W_2) & dw_2(f_* Z) \\ 0 & 0 & \vartheta(f_* Z) \end{pmatrix}$$

with

$$\begin{aligned} \vartheta &= \vartheta_1 + i\vartheta_2 \\ &= dz + \frac{1}{2} w_2 dw_1 - \frac{1}{2} w_2 dw_1. \end{aligned}$$

Furthermore, if

$$f_{t_*} = \begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & \lambda \end{pmatrix}$$

then the equality $\lambda = ad - bc$ holds. This follows from

$$f_t^* \vartheta = \lambda \vartheta$$
$$f_t^* d\vartheta|_{HN} = \lambda d\vartheta|_{HN}$$

and the fact that $d\vartheta = -dw_1 \wedge dw_2$.

References

[CDKR] M. Cowling, A. Dooley, A. Korányi, F. Ricci: *H−type groups and Iwasawa decompositions.* Adv. in Math. 87 (1991), 1 - 41.

[K] A. Korányi: *Geometric properties of Heisenberg-type groups.* Adv. in Math. 56 (1985), 28 - 38.

[P] P. Pansu: *Quasiisométries des variétés à courbure négative.* Thèse, Paris 1987.

[R] C. Riehm: *The automorphism group of a composition of quadratic forms.* Trans AMS 269 (1982), 403 - 414.

[S] L. Saal: *The automorphism group of a Lie algebra of Heisenberg type.* Rend. Sem. Mat. Univ. Torino 54

(1996), 101-113.