Uncertainty Inequalities on Stratified Nilpotent Groups

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Abstract

By a local uncertainty inequality on $\mathbb{R}^n$, we have for any measurable $E \subset \mathbb{R}^n$ and $0 < \theta < \frac{1}{2}$,

$$\int_E \mid \hat{f}(\xi) \mid^2 d\xi \leq C_\theta m(E)^{2\theta} \int_{\mathbb{R}^n} \mid f(x) \mid^2 \mid x \mid^{2n\theta} dx.$$

In this paper, we give analogues of the above inequality for stratified nilpotent Lie groups, connected with the spectral analysis of a given homogeneous sublaplacian. A fully detailed proof is given for step-two groups, and indications are given for the proof in the general case. Further, as in the case of $\mathbb{R}^n$, we shall deduce an analogue of the Heisenberg-Pauli uncertainty inequality.

1 Introduction

It is a well-known fact in classical Fourier analysis that a function f and its Fourier transform cannot both be compactly supported unless $f = 0$ a.e. This is a very simple form of uncertainty principle. One way of stating the uncertainty principle in harmonic analysis is that both function and its Fourier transform cannot be "sharply localized".

Most common quantitative formulation of uncertainty principle is the celebrated Heisenberg-Pauli-Weyl uncertainty inequality. Given $f \in L^2\left(\mathbb{R}^k\right)$, the inequality says

$$\|f\|_2^4 \leq k_n \left(\int_{\mathbb{R}^k} \mid x \mid^2 \mid f(x) \mid^2 dx \right) \left(\int_{\mathbb{R}^k} \mid y \mid^2 \mid \hat{f}(y) \mid^2 dy \right). \qquad (1)$$

For a proof of the above inequality with precise value of C_n, the reader may refer to [7]. It is easy to see that (1) can be rewritten as

$$\|f\|_2^4 \leq k_n \left(\int_{\mathbb{R}^k} \mid x \mid^2 \mid f(x) \mid^2 dx \right) \left(\int_{\mathbb{R}^k} \mid (-\Delta)^{\frac{1}{2}} f(x) \mid^2 dx \right) \qquad (2)$$

where Δ denotes the Laplacian on $\mathbb{R}^k$. A version of the above inequality for the Heisenberg group was established in [16] as follows, with the Laplacian replaced by the $U(n)$-invariant sublaplacian $\mathcal{L}$ on H^k.

Theorem 1.1 *For $f \in L^2(H^k)$, $0 \leq \gamma < \frac{Q}{2}$ one has*

$$\|f\|_2^4 \leq K \left(\int_{H^k} | f(w) |^2 | w |^{2\gamma} \, dw \right) \left(\int_{H^k} | \mathcal{L}^{\frac{\gamma}{2}} f(w) |^2 \, dw \right)$$

where K is a constant, $Q = 2k + 2$ denotes the homogeneous dimension of H^k and, for $w \in H^k$, $| w |$ is a homogeneous norm on H^k.

It is easy to see that once the above inequality is established for small values of γ, it can be extended to any positive γ (see the end of Section 4 below).

That the subLaplacian on H^k is a good substitute for the Laplacian on $\mathbb{R}^k$ in many problems, has been shown by many authors (see [2], [15] and [16] in connection with uncertainty inequalities) . For more details about homogeneous norm and dimension the reader can see [8]. As in the Euclidean case, the above inequality is a consequence of the following local uncertainty inequality, also obtained in [16] (here π_λ, $\lambda \in \mathbb{R}^*$, denotes the Schrödinger representation of H^k on $L^2(\mathbb{R}^k)$ associated to the character $e^{i\lambda t}$ of the center, and the Φ_α^λ, $\alpha \in \mathbb{N}^k$, form the orthonormal basis of scaled Hermite functions; the notation is better explained in Section 3 in greater generality; also consult [15]):

Theorem 1.2 *Let μ denote the Plancherel measure on H^k, ν the counting measure on $\mathbb{N}^k$, and let $\sigma = \mu \times \nu$ on $\mathbb{R}^* \times \mathbb{N}^k$. Then, given $\theta \in [0, \frac{1}{2})$, for each $f \in L^1 \cap L^2(H^k)$ and $E \subset \mathbb{R}^* \times \mathbb{N}^k$ with $\sigma(E) < \infty$ one has*

$$\int_E \|\pi_\lambda(f)\Phi_\alpha^\lambda\|_2^2 d\sigma \leq C_\theta^2 \sigma(E)^{2\theta} \int_{H^k} | f(w) |^2 | w |^{2\theta Q} \, dw$$

where C_θ depends only on θ and k.

The presence of the Hermite functions Φ_α^λ in this formula is closely related to the spectral decomposition of $\pi_\lambda(\mathcal{L})$.

In this paper, we show that the method of proof of [16], using explicit representation theory, can be extended to any connected and simply connected stratified group, in terms of a homogeneous norm and any homogeneous sublaplacian. In Section 3 we give a detailed proof for step-two groups. In Section 5 we sketch the proof in the general case, which requires some further tools from functional calculus and representation theory.

A more general argument, which does not make use of explicit representation theory and allows extensions to non-homogeneous operators and more general groups, will be presented elsewhere [3].

2 Preliminaries

Let $\mathfrak{n}$ be a real two-step nilpotent Lie algebra i.e., $[\mathfrak{n},\mathfrak{n}] \neq \{0\}$ and $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \{0\}$. We write $\mathfrak{n}$ as the sum of two subspaces

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}(\mathfrak{n}),$$

where $\mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ is any subspace of $\mathfrak{n}$ complementary to $\mathfrak{z}(\mathfrak{n})$. We denote the dimension of $\mathfrak{z}(\mathfrak{n})$ by d. We shall choose an inner product on $\mathfrak{n}$ such that $\mathfrak{v}$ and $\mathfrak{z}(\mathfrak{n})$ are orthogonal.

Let N be the connected, simply connected Lie group with Lie algebra $\mathfrak{n}$. Since $\mathfrak{n}$ is nilpotent, the exponential map is surjective. We may therefore parametrize N by $\mathfrak{v} \oplus \mathfrak{z}(\mathfrak{n})$ and write (V, Z) for $\exp(V + Z)$ where $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. By the Campbell-Baker-Hausdorff formula, the product law in N is given by the rule

$$(V,Z)\,(V',Z') = \left(V + V', Z + Z' + \frac{1}{2}[V,V']\right), V, V' \in \mathfrak{v}, Z, Z' \in \mathfrak{z}(\mathfrak{n}).$$

We denote by dV and dZ the Lebesgue measures on $\mathfrak{v}$ and on $\mathfrak{z}(\mathfrak{n})$ respectively. It is easy to check that $dn = dV dZ$ is a Haar measure on N. We write $\mathcal{S}(N)$ for the Schwartz space of N i.e., the set of functions f on N such that $f \circ \exp$ is in the Schwartz space of the Euclidean space $\mathfrak{n}$. We recall some facts from the representation theory of two-step nilpotent Lie groups, following [4] (see also [14]). Fix $\lambda \in \mathfrak{z}(\mathfrak{n})^*$, the dual of $\mathfrak{z}(\mathfrak{n})$, and define the skew-symmetric bilinear form $B(\lambda)$ on $\mathfrak{v}$ by

$$B(\lambda)\,(U,V) = \lambda\left([U,V]\right), U, V \in \mathfrak{v}.$$

Denote the radical of $B(\lambda)$ by $\mathfrak{r}_\lambda$ and let $\mathfrak{w}_\lambda$ be the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{v}$. Since $B(\lambda)$ is skew-symmetric, the dimension of $\mathfrak{w}_\lambda$ is even. Denote by Λ the Zariski-open subset of $\mathfrak{z}(\mathfrak{n})^*$ of vectors λ for which $\dim \mathfrak{w}_\lambda$ is maximum, and define r by requiring that $\dim \mathfrak{w}_\lambda = 2r$ for all λ in Λ. Let $\dim \mathfrak{r}_\lambda = k$.

Fix λ in Λ. Then there exists an orthonormal basis $X_1'(\lambda), \ldots, X_r'(\lambda), Y_1'(\lambda), \ldots, Y_r'(\lambda), Z_1'(\lambda), \ldots, Z_k'(\lambda)$ of $\mathfrak{v}$ such that the matrix of $B(\lambda)$ takes the following form with respect to this basis:

$$\begin{pmatrix} 0 & \cdots & 0 & \eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \eta_r(\lambda) & 0 & \cdots & 0 \\ -\eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_r(\lambda) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0_{k\times k} \end{pmatrix} \tag{1}$$

with each $\eta_j(\lambda) > 0$. We further restrict to the smaller Zariski-open subset $\Lambda_0 \subset \Lambda$ of $\mathfrak{z}(\mathfrak{n})^*$ where the number of distinct $\eta_j(\lambda)$ is maximum. On Λ_0 the functions η_j are real-analytic, hence smooth, and each base vector $X'_j(\lambda), Y'_j(\lambda)$ can be chosen so to depend smoothly on λ. In the same way, we also take the vectors $Z'_j(\lambda)$ smoothly depending on λ.

Now we shall further decompose $\mathfrak{v}$. Denote $\text{span}\{X'_1(\lambda), \ldots, X'_r(\lambda)\}$ by $\mathfrak{p}_\lambda$ and $\text{span}\{Y'_1(\lambda), \ldots, Y'_r(\lambda)\}$ by $\mathfrak{q}_\lambda$. Then we may write $V \in \mathfrak{v}$ as $W + R$ or as $X + Y + R$, with W in $\mathfrak{w}_\lambda$, X in $\mathfrak{p}_\lambda$, Y in $\mathfrak{q}_\lambda$, and R in $\mathfrak{r}_\lambda$. Accordingly, we denote the elements $\exp(W+R+Z)$ and $\exp(X+Y+R+Z)$ of N by (W, R, Z) and (X, Y, R, Z) respectively.

Denote by $\mathfrak{h}_\lambda$ the subalgebra $\mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda \oplus \mathfrak{z}(\mathfrak{n})$ and define $H_\lambda = \exp(\mathfrak{h}_\lambda)$. Note that, by definition, the restriction of λ to the subalgebra $[\mathfrak{h}_\lambda, \mathfrak{h}_\lambda]$ is trivial, hence, for μ in $\mathfrak{r}^*_\lambda$, the dual of $\mathfrak{r}_\lambda$, we may define a character $\sigma_{\lambda,\mu}$ of H_λ by

$$\sigma_{\lambda,\mu}(Y, R, Z) = e^{-\imath(\lambda(Z)+\mu(R))}, (Y, R, Z) \in H_\lambda.$$

Inducing $\sigma(\lambda, \mu)$ from H_α to N, we obtain the irreducible unitary representation $\pi_{\lambda,\mu}$ of N on $L^2(\mathfrak{p}_\lambda)$ given by

$$\left(\pi_{\lambda,\mu}(X, Y, R, Z)\phi\right)(X') = e^{-\imath\lambda(Z+[X'+\frac{1}{2}X,Y])} e^{-\imath\mu(R)} \phi(X + X') \qquad (2)$$

for all functions ϕ in $L^2(\mathfrak{p}_\lambda)$. There are other irreducible unitary representations of N, which do not play any role in the Plancherel formula, and we shall not bother to describe these here. For more details the reader may refer to [4] or [12].

Define the Fourier transform of a function f in $L^1(N)$ by $\hat{f}(\pi_{\lambda,\mu}) = \pi_{\lambda,\mu}(f) = \displaystyle\int_N f(n)\pi_{\lambda,\mu}(n)dn$ for $\lambda \in \Lambda, \mu \in \mathfrak{r}^*_\lambda$. If f is in $L^1 \cap L^2(N)$, then $\pi_{\lambda,\mu}(f)$ is a Hilbert-Schmidt operator with the kernel given by

$$K_{f,\lambda,\mu}(X', X) = \mathcal{F}_{\mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda \oplus \mathfrak{z}(\mathfrak{n})} f\left(X - X', \frac{1}{2}\lambda(X + X'), \mu, \lambda\right).$$

A simple calculation shows that

$$\|\pi_{\lambda,\mu}(f)\|^2_{HS} = (2\pi)^m \mid \text{Pf}\,(\lambda) \mid^{-1} \int_{\mathfrak{w}_\lambda} \mid \mathcal{F}_{\mathfrak{r}_\lambda \oplus \mathfrak{z}(\mathfrak{n})} f(W, \mu, \lambda) \mid^2 dW,$$

where $\text{Pf}\,(\lambda)$ denotes the Pfaffian of the non-degenerate skew-symmetric bilinear form $B(\lambda)|_{\mathfrak{w}_\lambda}$. One has $|\text{Pf}\,(\lambda)| = \prod_{j=1}^r \eta_j(\lambda)$. Therefore, we have the Plancherel formula

$$\int_{\Lambda_0} \int_{\mathfrak{r}^*_\lambda} \|\pi_{\lambda,\mu}(f)\|^2_{HS} \mid \text{Pf}\,(\lambda) \mid d\mu d\lambda = C \int_N \mid f(n) \mid^2 dn,$$

where $d\nu(\lambda, \mu) = \mid \text{Pf}\,(\lambda) \mid d\lambda d\mu$ is the Plancherel measure supported on the subset $\{\pi_{\lambda,\mu}\}_{\lambda \in \Lambda_0, \mu \in \mathfrak{r}^*_\lambda}$ of $\hat{N}$, the unitary dual of N.

In the next section, we shall establish a local uncertainty inequality for the Fourier transform on N.

3 Local Uncertainty Inequality

To state our local uncertainty inequality on N, we modify the smoothly varying family $X_1'(\lambda),\ldots,X_r'(\lambda)$, $Y_1'(\lambda),\ldots,Y_r'(\lambda)$, $Z_1'(\lambda),\ldots,Z_k'(\lambda)$ of orthonormal bases of $\mathfrak{v}$, depending on $\lambda\in\Lambda_0$, chosen so that the matrix of $B(\lambda)$ takes the form (1). Dividing, e.g., each $X_j'(\lambda)$ and $Y_j'(\lambda)$ by the corresponding $\sqrt{\eta_j(\lambda)}$, and leaving the $Z_j'(\lambda)$ unchanged, we obtain new bases $X_1(\lambda),\ldots,X_r(\lambda)$, $Y_1(\lambda),\ldots$, $Y_r(\lambda)$, $Z_1(\lambda),\ldots,Z_k(\lambda)$ of $\mathfrak{v}$ such that the matrix of $B(\lambda)$ takes the normal form

$$\begin{pmatrix} 0 & I_{r\times r} & 0\\ -I_{r\times r} & 0 & 0\\ 0 & 0 & 0_{k\times k}\end{pmatrix}$$

with respect to this basis (i.e. $X_1(\lambda),\ldots,X_r(\lambda)$, $Y_1(\lambda),\ldots,Y_r(\lambda)$ is a symplectic basis of $\mathfrak{w}_\lambda$).

For each multi-index $\alpha\in\mathbb{N}^r$, let $\Phi_\alpha(x)$ stand for the normalized Hermite function on $\mathbb{R}^r$. For $\lambda\in\Lambda_0$ and $\alpha\in\mathbb{N}^r$, we let $\Phi_\alpha^\lambda(x)=\mid \mathrm{Pf}(\lambda)\mid^{\frac14} \prod_{i=1}^{r}\Phi_{\alpha_i}\left(\mid\eta_i(\lambda)\mid^{\frac12} x_i\right)$ where $\alpha=(\alpha_1,\ldots,\alpha_r)$ in $\mathbb{N}^r$ and $x=(x_1,\ldots,x_r)$ in $\mathbb{R}^r$. For definition and properties of Hermite functions we refer to [17]. For each $(\lambda,\mu,\alpha)\in\Lambda\times\mathbb{R}^k\times\mathbb{N}^r$, we define the linear functional $F(\lambda,\mu,\alpha)$ by

$$F(\lambda,\mu,\alpha)\phi=\left(\phi,\hat f(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\right),\phi\in L^2(\mathfrak{p}_\lambda)\ ,$$

where $(.,.)$ denotes the inner product on $L^2(\mathfrak{p}_\lambda)$. The smooth dependence of the bases on λ makes it sure that $F(\lambda,\mu,\alpha)\phi$ depends at least continuously on λ and μ.

From this definition it is easy to see that

$$\begin{aligned}\mathrm{tr}\left(\hat f(\pi_{\lambda,\mu})\hat f(\pi_{\lambda,\mu})^*\right) &= \sum_\alpha \|\hat f(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2\\ &= \sum_\alpha \|F(\lambda,\mu,\alpha)\|^2 \qquad (1)\end{aligned}$$

where $\|F(\lambda,\mu,\alpha)\|$ denotes the norm of the linear functional $F(\lambda,\mu,\alpha)$.

Let δ denote the counting measure on $\mathbb{N}^r$ and let $\sigma=\nu\times\delta$ on $\Lambda_0\times\mathbb{R}^k\times\mathbb{N}^r$. Let $\mid(V,Z)\mid$ denote the homogeneous norm of (V,Z) in N.

We can now state and prove the local uncertainty principle:

Theorem 3.1 *Given $\theta\in[0,\frac12)$, for each $f\in L^1\cap L^2(N)$ and $E\subset\Lambda_0\times\mathbb{R}^k\times\mathbb{N}^r$ with $\sigma(E)<\infty$ one has*

$$\int_E\|F(\lambda,\mu,\alpha)\|^2d\sigma\le C_\theta^2\sigma(E)^{2\theta}\int_N\mid f(V,Z)\mid^2\mid(V,Z)\mid^{2\theta Q}\,dVdZ \qquad (2)$$

where C_θ is a constant depending only on θ and Q, the homogeneous dimension of N.

Proof: Let $s > 0$ be a positive number to be chosen later. Let B_s denote the ball centered at the identity of N with 'homogeneous radius' s. We shall denote the characteristic function of a set B by χ_B. Write $f = f_1 + f_2 = f \cdot \chi_{B_s} + f \cdot \chi_{B_s^c}$ where '$\cdot$' denotes the pointwise multiplication of the functions and B_s^c is the complement of B_s. We have then

$$\int_E \|F(\lambda,\mu,\alpha)\|^2 d\sigma \leq 2\{\int_E \|\hat{f}_1(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma + \int_E \|\hat{f}_2(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma\}.$$

But since $\hat{f}_1(\pi_{\lambda,\mu})$ is a bounded operator on $L^2(\mathbb{R}^r)$, we have $\|\hat{f}_1(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2 \leq \|\hat{f}_1(\pi_{\lambda,\mu})\|_{op}\|\Phi_\alpha^\lambda\|_2 = \|\hat{f}_1(\pi_{\lambda,\mu})\|_{op} \leq \|f_1\|_1$, where $\|\hat{f}_1(\pi_{\lambda,\mu})\|_{op}$ denotes the operator norm of $\hat{f}_1(\pi_{\lambda,\mu})$ on $L^2(\mathbb{R}^r)$.

Therefore by Cauchy-Schwartz inequality,

$$\begin{aligned}\int_E \|\hat{f}_1(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma &\leq \sigma(E)\left(\int_N \mid f_1(V,Z) \mid dVdZ\right)^2 \\ &\leq \sigma(E)\left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} dVdZ\right) \\ &\quad \left(\int_{\mid(V,Z)\mid \leq s} \mid (V,Z) \mid^{-2\theta Q} dVdZ\right). \end{aligned} \tag{3}$$

By definition of the homogeneous norm, we get,

$$\int_E \|\hat{f}_1(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \leq C\sigma(E)s^{-(2\theta-1)Q}\left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} dVdZ\right) \tag{4}$$

for some constant C. On the other hand, we have by Plancherel theorem,

$$\begin{aligned}\int_E \|\hat{f}_2(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma &\leq \int_{\Lambda_0\times\mathbb{R}^k\times\mathbb{N}^r} \|\hat{f}_2(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \\ &= \int_{\Lambda_0\times\mathbb{R}^k} \|\hat{f}_2(\pi_{\lambda,\mu})\|_{HS}^2 d\nu(\lambda,\mu) \\ &= C\int_N \mid f_2(V,Z) \mid^2 dVdZ \\ &= C\int_N \mid f_2(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} \mid (V,Z) \mid^{-2\theta Q} dVdZ \\ &\leq Cs^{-2\theta Q}\left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} dVdZ\right) \end{aligned} \tag{5}$$

where C is a constant depending on Q. Therefore we have proved the inequality,

$$\begin{aligned}\int_E \|F(\lambda,\mu,\alpha)\|^2 d\sigma \leq &\left(2C\sigma(E)s^{(1-2\theta)Q} + 2Cs^{-2\theta Q}\right) \\ &\left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} dVdZ\right). \end{aligned} \tag{6}$$

By an appropriate choice of s, we minimize the right hand side to get

$$\int_E \|F(\lambda,\mu,\alpha)\|^2 d\sigma \le C_\theta \sigma(E)^{2\theta} \left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\theta Q} dV dZ \right). \tag{7}$$

This completes the proof of the inequality (2).

In the next section we shall use the local uncertainty inequality (2) to prove an analogue of Heisenberg-Pauli-Weyl uncertainty inequality, as in the Euclidean case.

4 Heisenberg-Pauli-Weyl uncertainty inequality

To state the global uncertainty inequality, we need to introduce some more notation. Suppose $\{V_1, V_2, \ldots, V_{2r+k}\}$ be a basis of $\mathfrak{v}$. We can assume that the inner product on $\mathfrak{n}$ initially chosen is such that the V_j form an orthonormal basis of $\mathfrak{v}$. The subLaplacian $\mathfrak{L}$ on N, defined as $\mathfrak{L} = -\sum_{j=1}^{2r+k} V_j^2$ is hypoelliptic, and for fixed $\lambda \in \Lambda_0$,

$$\mathfrak{L} = -\sum_{i=1}^{r} \eta_i(\lambda) \left(X_i(\lambda)^2 + Y_i(\lambda)^2 \right) - \sum_{j=1}^{k} Z_j(\lambda)^2 .$$

Let $H(\lambda)$ be the Hermite operator whose spectral decomposition is given by $H(\lambda) = \sum_{\alpha \in \mathbb{N}^r} (2\alpha_1+1)\eta_1(\lambda) + \cdots + (2\alpha_r+1)\eta_r(\lambda)$. The relation between $\mathfrak{L}$ and $H(\lambda)$ is given by

$$\widehat{(\mathfrak{L}f)}(\pi_{\lambda,\mu}) = \hat{f}(\pi_{\lambda,\mu}) \left(H(\lambda) + \mid \mu \mid^2 \right),$$

for any reasonably nice function f on N. Here $\mid \mu \mid$ denotes the Euclidean norm of the vector $\mu \in \mathbb{R}^k$.

For $s > 0$, define

$$E_s = \{(\lambda,\mu,\alpha) \in \Lambda_0 \times \mathbb{R}^k \times \mathbb{N}^r : \sum_{i=1}^{r} (2\alpha_i+1)\eta_i(\lambda) + \mid \mu \mid^2 \le s^2\}.$$

The smoothness of the functions η_i implies that E_s is measurable. We shall now establish an estimate that is crucial in the proof of Heisenberg inequality:

Lemma 4.1 *There is a constant C depending only on θ such that $\sigma(E_s) \le C s^Q$.*

Proof: Since N is a two step nilpotent Lie group and the homogeneous dimension of N is greater than 2, there is a unique global fundamental solution K of

$\mathfrak{L}$ (see [6]), which is smooth away from the origin and homogeneous of degree $2-Q$. Hence we have,

$$\mathfrak{L}K = \delta_e,$$

where δ_e denotes the Dirac delta function supported only at the identity element of N. Applying group Fourier transform on both sides, we get,

$$\widehat{K}(\lambda,\mu)\left(H(\lambda)+\mid \mu\mid^2\right) = Id,$$

where Id denotes the identity operator on $\mathcal{H}_{\lambda,\mu}$. Hence

$$\left(\widehat{K}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right) = \frac{1}{\displaystyle\sum_{i=1}^{r}(2\alpha_i+1)\eta_i(\lambda)+\mid\mu\mid^2}.$$

Using this relation, we can rewrite the set E_s as

$$E_s = \{(\lambda,\mu,\alpha)\in\Lambda_0\times\mathbb{R}^k\times\mathbb{N}^r : \left(\widehat{K}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\geq \frac{1}{s^2}\}.$$

Define

$$\begin{aligned} K_l(x) &= K(x) \text{ if } |x|\leq l \\ &= 0 \qquad \text{if } |x|>l, \end{aligned}$$

and $K^l(x) = K(x) - K_l(x)$. Then, we have

$$\begin{aligned} \sigma(E_s) &= \sigma\left(\{(\lambda,\mu,\alpha)\in\Lambda_0\times\mathbb{R}^k\times\mathbb{N}^r :\mid \left((\widehat{K_l+K^l})(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\mid\geq\frac{1}{s^2}\}\right) \\ &\leq \sigma\left(\{(\lambda,\mu,\alpha) :\mid \left(\widehat{K_l}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\mid\geq\frac{1}{2s^2}\}\right) \\ &\quad +\sigma\left(\{(\lambda,\mu,\alpha) :\mid \left(\widehat{K^l}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\mid\geq\frac{1}{2s^2}\}\right) \end{aligned} \tag{1}$$

Consider that

$$\sigma\left(\{(\lambda,\mu,\alpha) :\mid \left(\hat{K}_l(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\mid\geq\frac{1}{2s^2}\}\right)\leq (2s^2)^{p'}\int\|\pi_{\lambda,\mu}(K_l)\|_{l_{p'}}^{p'}d\nu, \tag{2}$$

by Chebyshev's inequality for each fixed $\lambda\in\Lambda_0$, $\mu\in\mathfrak{r}_\lambda^*$ and $p'\geq 1$. By the non-commutative Hausdorff-Young inequality (see [?]) for $1\leq p\leq 2$ and $p' = \dfrac{p}{p-1}$, we have

$$\int\|\pi_{\lambda,\mu}(K_l)\|_{l_{p'}}^{p'}d\nu\leq\|K_l\|_{L^p(N)}^{p'}. \tag{3}$$

Therefore by (2) and (3), we get,

$$\sigma\left(\{(\lambda,\mu,\alpha) :\mid \left(\widehat{K_l}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\mid\geq\frac{1}{2s^2}\}\right)\leq (2s^2)^{p'}\|K_l\|_{L^p(N)}^{p'}. \tag{4}$$

A similar argument shows that K^l also satisfies the estimate:

$$\sigma\left(\{(\lambda,\mu,\alpha):\left|\left(\widehat{K^l}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\right|\geq\frac{1}{2s^2}\}\right)\leq(2s^2)^{q\prime}\|K^l\|_{L^q(N)}^{q\prime}. \qquad (5)$$

where $1\leq p,q\leq 2$. Putting together (4) and (5), we get

$$\begin{aligned}&\sigma\left(\{(\lambda,\mu,\alpha):\left|\left(\widehat{K}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\right|\geq\frac{1}{s^2}\}\right)\\ \leq\;& (2s^2)^{p\prime}\left(\int_N|K_l(x)|^p\,dx\right)^{\frac{p\prime}{p}}+(2s^2)^{q\prime}\left(\int_N|K^l(x)|^q\,dx\right)^{\frac{q\prime}{q}}. \qquad (6)\end{aligned}$$

Since K is homogeneous of degree $2-Q$ and smooth for $x\neq e$, if $C=\max\{|K(x)|:|x|=1\}$, we have,

$$|K(x)|\leq C|x|^{2-Q},$$

for $x\neq e$. Therefore,

$$\begin{aligned}\sigma\Bigg(\Big\{(\lambda,\mu,\alpha)\;&:\;|\hat{K}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda|\geq\frac{1}{s^2}\Big\}\Bigg)\\ &\leq\; Cs^{2p\prime}\left(\int_{\{x\in N:C|x|\leq l\}}|x|^{(2-Q)p}\,dx\right)^{\frac{p\prime}{p}}\\ &+\; Cs^{2q\prime}\left(\int_{\{x\in N:C|x|>l\}}|x|^{(2-Q)q}\,dx\right)^{\frac{q\prime}{q}}. \qquad (7)\end{aligned}$$

The first of the above integrals is convergent if $(2-Q)p>-Q$, while the second one is convergent if $(2-Q)q<-Q$, so we should have

$$p>\frac{Q}{Q-2}\quad\text{and}\quad q<\frac{Q}{Q-2}.$$

But to apply Hausdorff-Young inequality, we must have $p,q\leq 2$. Hence, from the above Q has to be bigger than 4. If we assume that N is non-abelian and $N\neq H^1$ (the case $N=H^1$ case has already been proved by Thangavelu, see [15]), this condition is satisfied. So the integrals in the last inequality (7) are finite for appropriate choices of $p,q\leq 2$, and

$$\sigma\left\{(\lambda,\mu,\alpha):\left|\left(\widehat{K}(\lambda,\mu)\Phi_\alpha^\lambda,\Phi_\alpha^\lambda\right)\right|\geq\frac{1}{s^2}\right\}\leq C\big(s^{2p'}l^{2p\prime-Q}+s^{2q'}l^{2q\prime-Q}\big).$$

To minimize the right hand side, we impose that

$$s^{2p'}l^{2p\prime-Q}=s^{2q'}l^{2q\prime-Q},$$

that is $l=s^{-1}$. Hence, we have $s^{2p'}l^{2p\prime-Q}=s^Q$, from which the lemma follows.

Theorem 4.1 *For $f \in L^2(N)$ and $\gamma > 0$ one has*

$$\|f\|_2^4 \leq C \left(\int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\gamma} \, dV dZ \right) \left(\int_N \mid \mathfrak{L}^{\frac{\gamma}{2}} f(V,Z) \mid^2 \, dV dZ \right) \quad (8)$$

where C is a constant.

Proof: Assume first that $\gamma < \frac{Q}{2}$. Let $s > 0$ be a positive number to be chosen later. Let E_s be the subset of $\Lambda_0 \times \mathbb{R}^k \times \mathbb{N}^r$ defined as defined earlier and E_s^c denotes the complement of E_s in $\Lambda_0 \times \mathbb{R}^k \times \mathbb{N}^r$. Then, by Plancherel theorem, we have,

$$\begin{aligned} \|f\|_2^2 &= C_Q \int \|\hat{f}(\pi_{\lambda,\mu})\|_{HS}^2 d\nu \\ &= C_Q \left(\int\limits_{E_s} \|\hat{f}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma + \int\limits_{E_s^c} \|\hat{f}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \right). \end{aligned} \quad (9)$$

Consider

$$\begin{aligned} & \int\limits_{E_s^c} \|\hat{f}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \\ \leq \; & s^{-2\gamma} \int\limits_{E_s^c} \{\sum_{i=1}^r (2\alpha_i+1)|\eta_i(\lambda)| + \mid \mu \mid^2\}^\gamma \|\hat{f}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \\ = \; & s^{-2\gamma} \int\limits_{E_s^c} \|\hat{f}(\pi_{\lambda,\mu}) \left(H(\lambda) + \mid \mu \mid^2\right)^{\frac{\gamma}{2}} \Phi_\alpha^\lambda\|_2^2 d\sigma \\ \leq \; & s^{-2\gamma} \int\limits_{\widehat{N}} \|\widehat{(\mathfrak{L}^{\frac{\gamma}{2}} f)}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \\ = \; & C_1 s^{-2\gamma} \int\limits_N \mid \mathfrak{L}^{\frac{\gamma}{2}} f(V,Z) \mid^2 \, dV dZ, \end{aligned} \quad (10)$$

where C_1 is a constant depending only on the homogeneous dimension Q. By the local uncertainty inequality (2), with $\theta = \frac{\gamma}{Q} < \frac{1}{2}$, we get,

$$\int\limits_{E_s} \|\hat{f}(\pi_{\lambda,\mu})\Phi_\alpha^\lambda\|_2^2 d\sigma \leq C s^{2\gamma} \int\limits_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\gamma} \, dV dZ. \quad (11)$$

Putting together, (10) and (11), we have the inequality

$$\begin{aligned} \|f\|_2^2 \leq C\{s^{2\gamma} & \int_N \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\gamma} \, dV dZ + \\ s^{-2\gamma} & \int_N \mid \mathfrak{L}^{\frac{\gamma}{2}} f(V,Z) \mid^2 \, dV dZ\}, \end{aligned} \quad (12)$$

and by minimizing the right hand side, we have the desired inequality

$$\|f\|_2^4 \leq C \left(\int \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\gamma} dV dZ \right) \left(\int \mid \mathfrak{L}^{\frac{\gamma}{2}} f(V,Z) \mid^2 dV dZ \right). \quad (13)$$

Using the two estimates

$$\int \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{2\gamma} dV dZ \leq \|f\|_2 \left(\int \mid f(V,Z) \mid^2 \mid (V,Z) \mid^{4\gamma} dV dZ \right)^{\frac{1}{2}}$$

(by Hölder's inequality), and

$$\int \mid \mathfrak{L}^{\frac{\gamma}{2}} f(V,Z) \mid^2 dV dZ \leq \|f\|_2 \left(\int \mid \mathfrak{L}^{\gamma} f(V,Z) \mid^2 dV dZ \right)^{\frac{1}{2}}$$

(by self-adjointness of $L^{\frac{\gamma}{2}}$ and Hölder), the value of γ in (13) can be can be replaced by 2γ, and doing it repeatedly, (13) is extended to any positive γ.

5 The case of a general stratified group

Let G be a connected, simply connected, stratified nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}$ admits a decomposition into subspaces

$$\mathfrak{g} \ = \ W_1 \oplus W_2 \oplus \ldots \oplus W_r \ ,$$

where $[W_1, W_j] = W_{j+1}$, for every j. In particular, W_1 generates $\mathfrak{g}$. Let $Q = \sum_{i=1}^{r} i \, \mathrm{dim} W_i$ be the homogeneous dimension of G.

For $r > 0$, we define the dilation δ_r as the automorphism of $\mathfrak{g}$ such that $\delta_r X = r^j X$ for $X \in W_j$. Since G is simply connected, the exponential map allow to transfer the dilations δ_r to automorphisms of G. We fix a corersponding homogeneous norm function $| \cdot |$ on G which is smooth away from the origin.

Let $\mathcal{S}(G)$ denote the Schwartz class on G.

By Kirillov's theory, to any coadjoint orbit in $\mathfrak{g}^*$ there corresponds a unitary irreducible representation of G. Moreover, there is a linear subspace Λ of $\mathfrak{g}^*$ whose points belong to different coadjoint orbits, and such that these orbits carry the full Plancherel measure μ of G. We can then regard μ as a measure on Λ. The representation associated to the orbit through $\lambda \in \Lambda$ will be denoted by π_λ. We refer the reader to [4] or [9] for more details.

Let $\{X_j\}_{j=1}^{s}$ be a basis of W_1, then the sub-Laplacian

$$L = -\sum_{j=1}^{s} X_j^2$$

on $\mathcal{S}(G)$ is essentially self-adjoint, positive, homogeneous of order 2, and hypoelleptic. The main step in the proof is the following lemma.

Lemma 5.1 *There is a subset Λ_0 of Λ, carrying the full Plancherel measure, and such that*

(i) $\pi_\lambda(L)$ is self-adjoint, positive, and with compact resolvent; we denote by $0 < l_0(\lambda) \leq l_1(\lambda) \leq \dots$ the diverging sequence of eigenvalues of $\pi_\lambda(L)$, arranged in non-decreasing order, each repeated according to its multiplicity;

(ii) each function $l_j(\lambda)$ is measurable on Λ_0;

(iii) there is a constant C such that for every $r > 0$,

$$\mu \times \nu \left\{(\lambda, n) \in \Lambda_0 \times \mathbb{N} : l_n(\lambda) < r^2\right\} \leq C\, r^Q \ ,$$

where ν is the counting measure on $\mathbb{N}$.

We sketch the proof of the lemma. By [6], L has a fundamental solution K which is smooth away from the origin and homogeneous of degree $-Q+2$. Unless $G = H_1$ or is abelian (two situations that we can exclude), $Q > 4$.

Fix $s > 0$ and set $K_s = K\chi_{\{|x|<s\}}$, $K^s = K - K_s$. Then $K_s \in L^p(G)$ for $p < p_0 \frac{Q}{Q-2} < 2$, and $K^s \in L^q(G)$ for $q > p_0$, with

$$\|K_s\|_p \leq C_p s^{2-\frac{Q}{p'}} \ , \qquad \|K^s\|_q \leq C_q s^{2-\frac{Q}{q'}} \ .$$

By the non-commutative version of the Hausdorff-Young inequality [13] involving Schatten class norms, if $p < p_0$,

$$\int_\Lambda \|\pi_\lambda(K_s)\|^{p'}_{\mathcal{S}^{p'}}\, d\mu(\lambda) \leq C_p s^{2p'-Q} \ , \qquad \int_\Lambda \|\pi_\lambda(K^s)\|^{q'}_{\mathcal{S}^{q'}}\, d\mu(\lambda) \leq C_q s^{2q'-Q} \ , \quad (14)$$

provided $p < p_0 < q \leq 2$. In particular, for a.e. $\lambda \in \Lambda$, $\pi_\lambda(K)$ is compact and self-adjoint. Since $LK = \delta_0$, $d\pi_\lambda(L) = \big(\pi_\lambda(K)\big)^{-1}$, and (i) follows.

If $h_1 \in \mathcal{S}(G)$ is the heat kernel $e^{-L}\delta_0$ at time $t = 1$, and $g_r = \chi_{[0,r]}(L)h_1 \in L^2(G)$, then

$$N(\lambda, r) := \#\{n : l_n(\lambda) < r\} = \operatorname{rank} \pi_\lambda(g_r) \ ,$$

which is measurable in λ for any fixed r [5]. This implies (ii).

Then (iii) follows from (14) and Chebishev's inequality.

The results of Sections 3 and 4 can now be extended to the higher step case.

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