FOURIER AND SPECTRAL MULTIPLIERS IN R^N AND IN THE HEISENBERG GROUP

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Contents

CHAPTER I
SELF-ADJOINT OPERATORS AND SPECTRAL ANALYSIS

1. Review of notions from spectral theory

Self-adjoint operators.

We sketch some basic facts about the spectral theory of (possibly unbounded) self-adjoint operators on a Hilbert space. For a complete treatment, the reader can consult, e.g. M. Reed, B. Simon *Methods of Modern Mathematical Physics, vol. I, Functional Analysis, vol. II, Fourier Analysis, Self-adjointness.* We shall refer to these books as [RS1, RS2] respectively.

Let H be a Hilbert space, and let T be a linear operator with domain D dense in H and with values in H.

Let D' consist of those elements $u \in H$ for which an element w exists such that[1]

$$\langle Tv|u\rangle = \langle v|w\rangle$$

for all $v \in D$. This w is unique because D is dense, and it is denoted by T^*u. The operator T^* defined on D' is called the adjoint operator of T.

T is called *symmetric* if $D \subseteq D'$ and $T^*{}_{|_D} = T$. It is called *self-adjoint* if it is symmetric and $D' = D$. It is a known fact that a self-adjoint operator is closed (i.e. its graph in $H \times H$ is closed). It follows from the closed graph theorem that a closed densely defined operator T is bounded if and only if $D = H$.

The most notable example of a self-adjoint operator is the following. Take $H = L^2(X, \mu)$, with (X, μ) a measure space. Given an a.e. finite measurable *real-valued* function φ on X, define

$$T_\varphi f(x) = \varphi(x) f(x) \ ,$$

on $D = \{f \in L^2(X,\mu) : \varphi f \in L^2(X,\mu)\}$. That D is dense follows from the fact that, if we denote by X_n the set where $\varphi(x) < n$, then D contains all L^2-functions supported on X_n.

Given $g \in L^2(X,\mu)$, we look for another L^2-function h such that

$$\int_X \varphi(x) f(x) \overline{g(x)}\, d\mu(x) = \int_X f(x) \overline{h(x)}\, d\mu(x) \ ,$$

[1] We adopt the following notation: $\langle \ | \ \rangle$ denotes the Hermitean product on a Hilbert space, whereas $\langle \ , \ \rangle$ denotes a bilinear pairing, e.g. between a distribution and a test function.

for every $f \in D$. This is possible if and only if $h(x) = \varphi(x)g(x)$ a.e. Hence $D' = D$ and then it is pretty obvious that $T_\varphi^* = T_\varphi$.

Observe that T_φ is bounded if and only if $\varphi \in L^\infty(X, \mu)$.

The following statement is easy to prove.

Proposition 1.1. *Let $A : H \to H'$ be a unitary transformation between Hilbert spaces. Given a densely defined operator T on H with domain D, let T' be the operator on H', with domain $D' = AD$, given by $T' = ATA^{-1}$. Then T is self-adjoint if and only if T' is.*

In this case we say that T and T' are *unitarily equivalent.*

As a direct consequence of self-adjointness of T_φ for φ real-valued, we have the following statement for constant coefficient differential operators.

Theorem 1.2. *Let P be a polynomial in n variables with real coefficients, and consider the differential operator $L = P(i^{-1}\partial) = P(i^{-1}\partial_{x_1}, \dots, i^{-1}\partial_{x_n})$ on $\mathbb{R}^n$. With $H = L^2(\mathbb{R}^n)$, take*

$$D = \{f \in L^2(\mathbb{R}^n) : Lf \in L^2(\mathbb{R}^n)\} \ , \tag{1.1}$$

as the domain of L. Then L is self-adjoint.

Proof. Let A be the unitary transformation of $L^2(\mathbb{R}^n)$ onto itself given by the Fourier transform multiplied by $(2\pi)^{-n/2}$. Then L is unitarily equivalent to T_P. Since P has real coefficients, T_P is self-adjoint. The conclusion follows from Proposition 1.1. □

The multiplication operator T_φ described above is more than just an example. The following statement is proved in [RS1].

Theorem 1.3 (Spectral Theorem, version 1). *Let H be a separable Hilbert space and T be a self-adjoint operator on it with domain D. Then T is unitarily equivalent to a multiplication operator T_φ, with φ measurable, a.e. finite, and real-valued on some finite measure space (X, μ).*

The *resolvent set* of a closed operator T on a HIlbert space H is defined as the set of those $\lambda \in \mathbb{C}$ for which $\lambda I - T$ has a bounded inverse. The complement of the resolvent set is the *spectrum* of T, denoted by $\sigma(T)$. The resolvent set is open and the spectrum is closed.

If T_φ is as above, it is simple to show that λ is in the resolvent set if and only if $(\lambda - \varphi)^{-1} \in L^\infty(X, \mu)$, i.e. if and only if there is $\delta > 0$ such that

$$\mu\{x : |\varphi(x) - \lambda| < \delta\} = 0 \ .$$

The complement of this set, called the *essential range* of φ, is the spectrum of T_φ. Since φ is real-valued, clearly $\sigma(T_\varphi) \subseteq \mathbb{R}$.

It is a general fact that the spectrum of a self-adjoint operator is contained in $\mathbb{R}$. For separable H, one can appeal to Theorem 1.3, for general H see [RS1].

Spectral measure.

Definition. *A* regular projection-valued measure, *also called a* regular resolution of the identity, *on $\mathbb{R}$ is a map E assigning to each Borel subset ω of $\mathbb{R}$ an orthogonal projection $E(\omega)$ on some fixed Hilbert space H, satisfying the following properties:*

(1) $E(\emptyset) = 0$ *and* $E(\mathbb{R}) = I$;
(2) $E(\omega \cap \omega') = E(\omega)E(\omega')$;
(3) *if* $\{\omega_j\}$ *is a countable family of pairwise disjoint Borel sets, then*

$$E\Big(\bigcup_j \omega_j\Big) = \sum_j E(\omega_j) \ ,$$

in the strong topology of $\mathcal{L}(H)$;
(4) *for every Borel set* ω,

$$E(\omega) = \sup\big\{E(\omega') : \omega' \subset \omega\,,\, \omega' \text{ compact}\big\} = \inf\big\{E(\omega'') : \omega'' \supset \omega\,,\, \omega'' \text{ open}\big\} \ ,$$

with respect to the partial ordering in the space of bounded self-adjoint operators on H (for which $T \le T'$ if and only if $\langle Tu|u\rangle \le \langle T'u|u\rangle$ for every $u \in H$).

The support *of the measure E is the smallest closed set F such that $E(\mathbb{R}\backslash F) = 0$.*

It follows from (2) that the projections $E(\omega)$ commute with each other.
Observe that, if E is a regular resolution of the identity, then

$$v = \int_{\mathbb{R}} dE(\lambda)v$$

for every $v \in H$. Also, given $v, w \in H$,

$$\nu_{v,w}(\omega) = \langle E(\omega)v|w\rangle$$

is a (scalar-valued) finite Borel measure, and

$$\langle v|w\rangle = \int_{\mathbb{R}} d\nu_{v,w} = \int_{\mathbb{R}} \langle dE(\lambda)v|w\rangle \ .$$

Clearly, $\nu_{v,v}$ is positive for every v, and $\|\nu_{v,v}\|_1 = \|v\|^2$. Hölder's inequality shows that, for general $v, w \in H$, $\|\nu_{v,w}\|_1 \le \|u\|\|v\|$.

Proposition 1.4. *Let m be an E-a.e. finite Borel function on $\mathbb{R}$. Then*

$$D = \Big\{v \in H : \int_{\mathbb{R}} |m(\lambda)|^2 d\nu_{v,v} < \infty\Big\} \ ,$$

is dense in H and the operator

$$S_m v = \int_{\mathbb{R}} m(\lambda)\, dE(\lambda)v$$

is well-defined on D, it commutes with every $E(\omega)$, and

$$S_m E(\omega) = S_{m\chi_\omega} \ .$$

Moreover, S_m is bounded on H if and only if m is E-a.e. bounded, and it is self-adjoint if and only if m is E-a.e. real-valued.

If m_1 and m_2 are E-a.e. bounded, then $S_{m_1 m_2} = S_{m_1} S_{m_2}$.

Proof. Let F_n be the Borel set where $|m(\lambda)| \le n$, and let $E_n = E(F_n)$. If $v \in E_n H$, then $\nu_{v,v}(\mathbb{R} \setminus F_n) = 0$. It follows that $v \in D$.

Since the F_n form an increasing sequence, it follows from condition (3) that the E_n converge in the strong topology to $E\big(\bigcup_n F_n\big)$. This is the identity operator, because m is E-a.e. finite. This implies that

$$\bar{D} \subseteq \overline{\bigcup_n E_n H} = H \ .$$

If $v \in H$ and m is a simple function,

$$m(\lambda) = \sum_j m_j \chi_{\omega_j}(\lambda) \ ,$$

with the ω_j pairwise disjoint Borel sets, then

$$S_m v = \sum_j m_j E(\omega_j) \ ,$$

and

$$\begin{aligned} \|S_m v\|^2 &= \sum_{j,k} m_j \bar{m}_k \langle E(\omega_j)v | E(\omega_k)v\rangle \\ &= \sum_j |m_j|^2 \langle E(\omega_j)v|v\rangle \\ &= \int_R |m(\lambda)|^2 \, d\nu_{v,v}(\lambda) \ . \end{aligned} \tag{1.2}$$

A limiting argument shows that $S_m v$ is well defined for $v \in D$ and that (1.2) holds for every $v \in D$. It also shows that S_m commutes with the $E(\omega)$ and that $S_m E(\omega) = S_{m\chi_\omega}$.

In particular, this implies that each subspace $E_n H$ is mapped into itself by S_m.

That the boundedness of S_m is equivalent to the E-a.e. boundedness of m follows from (1.2). We show now that D is also the domain of the adjoint of S_m.

If $u, v \in D$, then

$$\begin{aligned} \langle S_m v|u\rangle &= \int_{\mathbb{R}} m(\lambda)\, d\nu_{v,u}(\lambda) \\ &= \int_{\mathbb{R}} m(\lambda)\, d\overline{\nu_{u,v}}(\lambda) \\ &= \overline{\langle S_m u|v\rangle} \\ &= \langle v|S_m u\rangle \ . \end{aligned}$$

This shows that S_m^* extends S_m, i.e. S_m is symmetric.

Take now u in the domain of S_m^*. Then there is $w \in H$ such that $\langle S_m v | u \rangle = \langle v | w \rangle$ for every $v \in D$. This is equivalent to saying that

$$\big| \langle S_m v | u \rangle \big| \leq C_u \|v\| \tag{1.3}$$

for some constant C_u and every $v \in D$.

Let $u_n = E_n u$. Then $u_n \in D$ and $u_n \to u$. Also, $S_m u_n$ is also in D. We can then apply (1.3) with $v = S_m u_n$. We find that

$$\begin{aligned} \|S_m u_n\|^2 &= \langle S_m^2 u_n | u_n \rangle \\ &= \langle S_m^2 u_n | E_n u \rangle \\ &= \langle S_m^2 u_n | u \rangle \\ &\leq C_u \|S_m u_n\| \ . \end{aligned}$$

Hence $\|S_m u_n\| \leq C_u$ for every n. From

$$\|S_m u_n\|^2 = \|S_{m\chi_{F_n}} u\|^2 = \int_{F_n} m(\lambda)^2 \, d\nu_{u,u}(\lambda) \leq C_u \ ,$$

we obtain that $u \in D$.

Finally, the identity $S_{m_1 m_2} = S_{m_1} S_{m_2}$ is trivial if m_1, m_2 are simple functions, and the general case follows by approximation. □

The next theorem, the proof of which can be found in [RS1], says that resolutions of the identity are in 1-1 correspondence with self-adjoint operators.

Theorem 1.5 (Spectral Theorem, version 2). *Let T be a self-adjoint operator on a Hilbert space H. Then there is one and only one regular resolution of the identity of H, $\{E(\omega)\}$, on $\mathbb{R}$ such that*

$$T = \int_{\mathbb{R}} \lambda \, dE(\lambda) \ .$$

The measure E is called the spectral measure *of T. Its support coincides with $\sigma(T)$.*

We describe the resolution of the identity of $L^2(X, \mu)$ associated with a multiplication operator T_φ, φ being real-valued and a.e. finite.

Given a Borel set $\omega \subseteq \mathbb{R}$, let

$$E(\omega) f = f \chi_{\varphi^{-1}(\omega)}$$

be the orthogonal projection onto the subspace of L^2-functions on X supported on $\varphi^{-1}(\omega)$. This is clearly a regular resolution of the identity. For $f \in L^2(X, \mu)$, we have

$$\nu_{f,f}(\omega) = \int_{\varphi^{-1}(\omega)} |f(x)|^2 \, d\mu(x) \ .$$

If $g(\lambda) = \sum_j g_j \chi_{\omega_j}(\lambda)$ is a simple function on $\mathbb{R}$, with the ω_j pairwise disjoint, then

$$\int_{\mathbb{R}} g(\lambda) \, d\nu_{f,f}(\lambda) = \sum_j g_j \int_{\varphi^{-1}(\omega_j)} |f(x)|^2 \, d\mu(x) = \int_X g\big(\varphi(x)\big) |f(x)|^2 \, d\mu(x) \ .$$

This identity can be easily extended to more general g.

Consider

$$S = \int_{\mathbb{R}} \lambda \, dE(\lambda) \ .$$

Hence the domain D of S consists of the functions f such that

$$\int_{\mathbb{R}} \lambda^2 \, d\nu_{f,f}(\lambda) = \int_X \varphi(x)^2 |f(x)|^2 \, d\mu(x) < \infty \ ,$$

i.e. those for which $\varphi f \in L^2(X, \mu)$.

This last example gives the spectral resolution for self-adjoint constant coefficient differential operators on $\mathbb{R}^n$.

Proposition 1.6. *Let* $L = P(i^{-1}\partial)$, *where* P *a polynomial in* n *variables with real coefficients, with the domain given in* (1.1). *Then*

$$L = \int_{\mathbb{R}} \lambda \, dE(\lambda) \ ,$$

where, denoting by $\mathcal{F}$ *the Fourier transform,*

$$E(\omega) f = \mathcal{F}^{-1}(\hat{f} \chi_{P^{-1}(\omega)}) \ .$$

The notion of spectral measure associated to a self-adjoint operator T allows to develop a *functional calculus* on T, i.e. to define other operators expressed in terms of the spectral measure, hence depending on T itself.

Let $dE(\lambda)$ be the spectral measure of T. If m is a Borel measurable function on $\mathbb{R}$, E-a.e. finite, define

$$m(T) = \int_{\mathbb{R}} m(\lambda) \, dE(\lambda) \ .$$

The function m is called a *spectral multiplier*.

Extensions of symmetric operators.

The general question if a symmetric operator admits a self-adjoint extension, and if this extension is unique, requires a detailed study, which is out of the scope of these notes. The answer is that self-adjoint extensions do not always exist, and if they exist, they need not be unique. The only fact we want to mention concerns positive operators.

A symmetric operator T with domain D is called *positive* if

$$\langle Tf | f \rangle \geq 0$$

for every $f \in D$.

A well-known fact is that every positive symmetric operator admits at least one self-adjoint extension (see [RS2] for the construcion of a canonical extension, called the *Friedrichs extension*).

For our purposes, it is important to mention the following application. Let $X_1, \ldots, X_m$ be first-order operators on $\mathbb{R}^n$, and denote by X_j' the formal adjoint of X_j, i.e.

$$X_j f(x) = \sum_{k=1}^{n} a_{jk}(x) \partial_{x_k} f(x) + a_0(x) f(x) ,$$

$$X_j' f(x) = -\sum_{k=1}^{n} \partial_{x_k} \big(\bar{a}_{j,k}(x) f(x) \big) + \bar{a}_0(x) f(x) .$$

We shall assume that the coefficients are defined and smooth on all of $\mathbb{R}^n$.

The operator $L = \sum_{j=1}^{m} X_j' X_j$, initially defined on $D_0 = \mathcal{D}(\mathbb{R}^n)$ is symmetric, because its adjoint is an extension of L itself, defined on

$$D = \left\{ f \in L^2(\mathbb{R}^n) : Lf \in L^2(\mathbb{R}^n) \right\} ,$$

(with Lf understood in the sense of distributions). It is also clear that L is positive.

Theorem 1.7. *The operator L with domain D is self-adjoint, and it is the only self-adjoint operator, with domain containing D_0, and equal to L on D_0.*

Proof. The first part of the statement follows from the Friedrichs construction. For the second part[2], let (L, D') be a self-adjoint extension of (L, D_0). If $f \in D'$, by self-adjointness,

$$\langle f | Lg \rangle = \langle Lf | g \rangle$$

for all $g \in D'$. If we restrict to $g \in D_0$,this implies that $h = Lf$ in the sense of distributions, so that $f \in D$.

Hence $(L, D') \subseteq (L, D)$. Passing to the adjoints, the inclusions are reversed; but both operators are self-adjoint, hence $D' = D$. □

Spectral analysis of commuting self-adjoint operators.

If T_1, T_2 are bounded self-adjoint operators on H, and $T_1 T_2 = T_2 T_1$, then any two projections $E_1(\omega)$ and $E_2(\omega')$ in the corresponding resolutions of the identity also commute with each other (see [RS1]).

If we try to extend this statement to unbounded operators, we meet several difficulties. First of all, the composition $T_1 T_2$ is well defined only on those elements v in the domain of T_2 such that $T_2 v$ is in the domain of T_1. These elements can form a very small space, and this space may not coincide with the one constructed by interchanging the rôle of T_1 and T_2. Worse than that, even though the equality $T_1 T_2 = T_2 T_1$ holds on a dense subspace, the projections in the two resolutions of the identity need not commute.

We are so led to impose the following definition.

Definition. *Let T_1, T_2 be self-adjoint operators on H. We say that T_1 and T_2 commute if the operators $\{E_1(\omega)\}$ and $\{E_2(\omega)\}$ forming the corresponding resolutions of the identity all commute with each other.*

We state without proof the following fact (see [RS1] for the proofs).

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[2] Instead of using different symbols, like L' or others for extensions of the "initial" L, we prefer to keep the same symbol, specifying the domain whenever there is ambiguity. We shall also write $(L, D_1) \subseteq (L, D_2)$ to denote that the second operator is an extension of the first from the domain D_1 to the domain D_2.

Theorem 1.8. *Let T_1, T_2 be self-adjoint operators. The following are equivalent:*

(i) *T_1 and T_2 commute;*
(ii) *if λ, μ are non-real numbers, $(\lambda I - T_1)^{-1}$ and $(\mu I - T_2)^{-1}$ commute;*
(iii) *for every $s, t \in \mathbb{R}$, e^{isT_1} and e^{itT_2} commute.*

Observe that the operators in (ii) and (iii) are bounded, so that "commutation" is meant in the ordinary sense.

If T_1 and T_2 commute, we can define a *joint spectral measure* on $\mathbb{R}^2$, by defining

$$E(\omega \times \omega') = E_1(\omega)E_2(\omega') \ ,$$

and extending E to every other Borel set in $\mathbb{R}^2$ so that conditions (3) and (4) are satisfied.

Projecting E onto each component, we find E_1 and E_2 respectively. Hence

$$T_1 = \int_{\mathbb{R}^2} \lambda \, dE(\lambda, \mu) \ , \qquad T_1 = \int_{\mathbb{R}^2} \mu \, dE(\lambda, \mu) \ .$$

A *joint spectral multiplier* of T_1 and T_2 is an E-a.e. finite Borel measurable function $m(\lambda, \mu)$, and one defines

$$m(T_1, T_2) = \int_{\mathbb{R}^2} m(\lambda, \mu) \, dE(\lambda, \mu) \ .$$

The extension of these notions to a larger number of mutually commuting self-adjoint operators is trivial. Proposition 1.4 remains true for multipliers of more than one operator.

The support of E is called the *joint spectrum* of T_1 and T_2, denoted by $\sigma(T_1, T_2)$. In contrast with what happens when taking tensor products of scalar-valued measures, it may happen that the support of E is strictly contained in the product of the support of the E_j; in other words, the joint spectrum $\sigma(T_1, T_2)$ can be strictly smaller than $\sigma(T_1) \times \sigma(T_2)$.

What can happen is that $E_1(\omega)$ and $E_2(\omega')$ are non-zero, but their product is zero.

Consider the case $H = \mathbb{C}^3$, and

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \qquad T_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \ .$$

Then $\sigma(T_1) = \{1, -1\}$ and $\sigma(T_2) = \{2, 3\}$. The spectral projections have the following ranges:

$$\begin{array}{rl} E_1(\{1\}) & \text{span}\,(e_1, e_2) \\ E_1(\{-1\}) & \text{span}\,(e_3) \\ E_2(\{2\}) & \text{span}\,(e_1) \\ E_2(\{3\}) & \text{span}\,(e_2, e_3) \ . \end{array}$$

Hence the only non trivial products are $E(1,2), E(1,3), E(-1,3)$, so that

$$\sigma(T_1,T_2)=\{(1,2),(1,3),(-1,3)\}\ .$$

A more interesting example is the following. On $\mathbb{R}^n$, take $T_1=i^{-1}\partial_{x_1}$ and $T_2=\Delta$, the Laplacian. The domains are those described before, making each of them self-adjoint.

By Proposition 1.6, if ω,ω' are Borel subsets of $\mathbb{R}$,

$$E_1(\omega)f=\mathcal{F}^{-1}(\hat f\chi_{\{\xi:\xi_1\in\omega\}})\ ,\qquad E_2(\omega')f=\mathcal{F}^{-1}(\hat f\chi_{\{\xi:|\xi|^2\in\omega'\}})\ .$$

This implies that $E(\omega\times\omega')=0$ if $\omega\times\omega'$ does not intersect the epi-parabola $\mu\geq\lambda^2$, and one can show easily that the joint spectrum is the full epi-parabola.

Fourier multipliers.

On $\mathbb{R}^n$ take $T_j=i^{-1}\partial_{x_j}$, for $1\leq j\leq n$. One can easily verify that the joint spectrum is all of $\mathbb{R}^n$, and that, if $m(\xi)$ is an a.e. finite Borel function on $\mathbb{R}^n$, then[3]

$$m(i^{-1}\partial)f(x)=\mathcal{F}^{-1}(\hat f m)(x)\ .$$

Hence joint spectral multipliers for the system $i^{-1}\partial$ coincide with the *Fourier multipliers*.

Similarly, if m is an a.e. finite Borel function on $\mathbb{R}$, then

$$m(\Delta)f(x)=\mathcal{F}^{-1}\big(\hat f m(|\cdot|^2)\big)(x)\ ,$$

i.e. the spectral multipliers of Δ coincide with the *radial Fourier multipliers*.

2. The Heisenberg sub-Laplacian

In this section we present what will be for us the main example of an operator of the form described in Theorem 1.7. The group-theoretic notions connected with the operators below are postponed to a future section.

We take $\mathbb{R}^{2n+1}=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ with coordinates (x,y,t), and define $2n$ vector fields $X_1,\dots,X_n,Y_1,\dots,Y_n$ as

$$X_j=\partial_{x_j}-\frac{y_j}{2}\partial_t\ ,\qquad Y_j=\partial_{y_j}+\frac{x_j}{2}\partial_t\ . \tag{2.1}$$

Then $X_j'={}^tX_j=-X_j$, $Y_j'={}^tY_j=-Y_j$, so that the *Heisenberg sub-Laplacian*

$$L=\sum_{j=1}^n(X_j'X_j+Y_j'Y_j)=-\sum_{j=1}^n(X_j^2+Y_j^2) \tag{2.2}$$

is positive on $L^2(\mathbb{R}^{2n+1})$, and self-adjoint on the domain

$$D=\big\{f\in L^2(\mathbb{R}^{2n+1}):Lf\in L^2(\mathbb{R}^{2n+1})\big\}\ .$$

[3] We use the symbol ∂ to denote the system $(\partial_{x_1},\dots,\partial_{x_n})$ in short.

Notice that $\mathcal{S}(\mathbb{R}^{2n+1}) \subset D$. The explicit expression of L,

$$L = \Delta_x + \Delta_y - \frac{|x|^2 + |y|^2}{4}\partial_t^2 + \sum_{j=1}^n (x_j \partial_{y_j} - y_j \partial_{x_j})\partial_t \ ,$$

shows that L is not elliptic (e.g., $L = \Delta_x + \Delta_y$ at the origin). It is however *hypoelliptic*, according to Hörmander's theorem[4]. This follows from the fact that, for every j,

$$[X_j, Y_j] = \partial_t \ ,$$

and the system of vector fields $\{X_1, Y_1, \dots, X_n, Y_n, \partial_t\}$ gives a basis of the tangent space at each point of $\mathbb{R}^{2n+1}$.

The constant vector field ∂_t is usally denoted by T.

In order to work out its spectral decomposition, it is preferable to replace L by another operator, unitarily equivalent to it.

Denote by $\mathcal{F}_t$ the partial Fourier transform in $\mathbb{R}^{2n+1}$ in the variable t. Then $A = (2\pi)^{-1/2}\mathcal{F}_t$ is a unitary operator on $L^2(\mathbb{R}^{2n+1}$. We set

$$\tilde{L} = ALA^{-1} = \mathcal{F}_t L \mathcal{F}_t^{-1} \ .$$

Then $\tilde{L}$ is self-adjoint, with domain

$$\tilde{D} = \left\{ g \in L^2(\mathbb{R}^{2n+1}) : \tilde{L}g \in L^2(\mathbb{R}^{2n+1}) \right\} \ .$$

If we perform the same conjugation by $\mathcal{F}_t$ on the X_j and Y_j, we obtain

$$\tilde{X}_j = \mathcal{F}_t X_j \mathcal{F}_t^{-1} = \partial_{x_j} - i\frac{\lambda}{2}y_j \ , \qquad \tilde{Y}_j = \mathcal{F}_t Y_j \mathcal{F}_t^{-1} = \partial_{y_j} + i\frac{\lambda}{2}x_j \ , \tag{2.3}$$

and

$$\tilde{L} = \sum_{j=1}^n (\tilde{X}_j' \tilde{X}_j + \tilde{Y}_j' \tilde{Y}_j) = -\sum_{j=1}^n (\tilde{X}_j^2 + \tilde{Y}_j^2) \ . \tag{2.4}$$

It must be observed that derivatives are only taken in the variables x_j, y_j, and not in λ. We shall also regard the first-order operators in (2.3) as acting on functions of $(x, y) \in \mathbb{R}^{2n}$, taking λ as a parameter.

When we do so, we call $X_{\lambda,j}, Y_{\lambda,j}$ the operators in (2.3), and L_λ the operator in (2.4). If we set $g^\lambda(x, y) = g(x, y, \lambda)$, this means that

$$\tilde{L}g(x, y, \lambda) = L_\lambda g^\lambda(x, y) \ .$$

By Theorem 1.7, L_λ is self-adjoint on $L^2(\mathbb{R}^{2n})$ for every λ, with domain

$$D_\lambda = \left\{ f \in L^2(\mathbb{R}^{2n}) : L_\lambda f \in L^2(\mathbb{R}^{2n}) \right\} \ .$$

For $\lambda = 0$, $L_\lambda = \Delta$; for $\lambda \neq 0$, L_λ is called the *twisted Laplacian*. The following statement is obvious.

[4]See the notes of the course "Sub-Laplacians on nilpotent Lie groups".

Lemma 2.1. *A function $g(x, y, \lambda) \in L^2(\mathbb{R}^{2n+1})$ belong to $\tilde{D}$ if and only if $g^\lambda \in D_\lambda$ for a.e. λ and*

$$\int_{\mathbb{R}} \|L_\lambda g^\lambda\|_2^2 \, d\lambda < \infty \ .$$

We describe the spectral measure of $\tilde{L}$.

Proposition 2.2. *Denote by $\tilde{E}(\omega)$ the $\tilde{L}$-spectral measure of a Borel subset ω of $\mathbb{R}$, and by $E_\lambda(\omega)$ its L_λ-spectral measure. Then*

$$\tilde{E}(\omega) g(x, y, \lambda) = E_\lambda(\omega) g^\lambda(x, y) \ .$$

Proof. One easily checks that $\tilde{E}$ is a regular projection-valued measure. Then we just need to identify the operator

$$A = \int_{\mathbb{R}} \mu \, d\tilde{E}(\mu)$$

together with its domain.

Setting

$$\nu_{g,h}(\omega) = \langle \tilde{E}(\omega) g | h \rangle \ , \qquad \nu_{g^\lambda, h^\lambda}(\omega) = \langle E_\lambda(\omega) g^\lambda | h^\lambda \rangle \ ,$$

we have

$$\nu_{g,g}(\omega) = \int_{\mathbb{R}} \langle E_\lambda(\omega) g^\lambda | g^\lambda \rangle \, d\lambda = \int_{\mathbb{R}} \int_\omega d\nu_{g^\lambda, g^\lambda}(\mu) \ d\lambda \ .$$

Hence the domain of A consists of the functions g such that

$$\int_{\mathbb{R}} \mu^2 \, d\nu_{g,g}(\mu) = \int_{\mathbb{R}^2} \mu^2 d\nu_{g^\lambda, g^\lambda}(\mu) \ d\lambda < \infty \ .$$

But this means that $g^\lambda \in D_\lambda$ for a.e. λ and that

$$\int_{\mathbb{R}} \|L_\lambda g^\lambda\|_2^2 \, d\lambda < \infty \ ,$$

i.e. g is in the domain of $\tilde{L}$. Moreover,

$$\begin{aligned} \langle Ag | h \rangle &= \int_{\mathbb{R}} \mu \, d\nu_{g,h}(\mu) \\ &= \int_{\mathbb{R}^2} \mu \, d\nu_{g^\lambda, h^\lambda}(\mu) \, d\lambda \\ &= \int_{\mathbb{R}} \langle L_\lambda g^\lambda, h^\lambda \rangle \, d\lambda \\ &= \langle \tilde{L} g | h \rangle \ . \qquad \square \end{aligned}$$

In view of Proposition 2.2, it will be interesting to obtain an explicit description of the spectral measures E_λ. The case $\lambda = 0$ is very simple, because L_0 is the Laplacian, but of course we are more interested in $\lambda \neq 0$.

The following lemma shows that the operators L_λ with $\lambda \neq 0$ can be conjugated among themselves, up to a constant factor, by unitary operators.

Lemma 2.3. *For $s>0$, let A_s be the unitary operator $A_s f(x,y) = s^n f(sx,sy)$ on $L^2(\mathbb{R}^{2n})$. Then*

$$L_{\pm s^2} = s^2 A_s L_{\pm 1} A_s^{-1} \ .$$

If $Bf(x,y) = f(x,-y)$, then

$$L_{-\lambda} = B L_\lambda B^{-1} \ .$$

If F_λ is the spectrum of L_λ, and $\lambda \neq 0$,

$$F_\lambda = |\lambda| F_1 = \big\{ |\lambda|\mu : \mu \in F_1 \big\} \ .$$

Finally, if $m(\mu)$ is a spectral multiplier and $\lambda \neq 0$, and $\varepsilon = 0,1$ depending on whether λ is negative or positive,

$$m(L_\lambda) = B^\varepsilon A_{|\lambda|^{\frac12}} m\big(|\lambda| L_1\big) A^{-1}_{|\lambda|^{\frac12}} B^{-\varepsilon} \ .$$

Proof. Given a Schwartz function f on $\mathbb{R}^{2n}$, it follows from (2.3) that

$$\begin{aligned} X_{\lambda,j}(A_s f)(x,y) &= s^{n+1} \partial_{x_j} f(sx,sy) - i\frac{\lambda}{2} s^n y_j f(sx,sy) \\ &= s^{n+1} X_{\lambda/s^2,j} f(sx,sy) \\ &= s A_s (X_{\lambda/s^2,j} f)(x,y) \ . \end{aligned}$$

It follows that $A_s D_{\lambda/s^2} = D_\lambda$ and

$$L_\lambda A_s = s^2 A_s L_{\lambda/s^2} \ .$$

This gives the first assertion, and the second can be proved in a similar and simpler way.

By uniqueness of the spectral measure associated with a self-adjoint operator, setting $C = B^\varepsilon A_{|\lambda|^{\frac12}}$, from the identity

$$\begin{aligned} \int_{\mathbb{R}} \mu \, dE_\lambda(\mu) &= L_\lambda \\ &= |\lambda| C L_1 C^{-1} \\ &= |\lambda| C \Big(\int_{\mathbb{R}} \mu \, dE_1(\mu) \Big) C^{-1} \\ &= C \Big(\int_{\mathbb{R}} \mu \, dE_1\big(|\lambda|^{-1}\mu\big) \Big) C^{-1} \end{aligned}$$

we derive that

$$E_\lambda(\omega) = C E_1\big(|\lambda|^{-1}\omega\big) C^{-1} \ .$$

Hence the support F_λ of E_λ and the support F_1 of E_1 are in the stated relation.

To conclude,

$$\begin{aligned} m(L_\lambda) &= \int_{\mathbb{R}} m(\mu) \, dE_\lambda(\mu) \\ &= C \Big(\int_{\mathbb{R}} m(\mu) \, dE_1\big(|\lambda|^{-1}\mu\big) \Big) C^{-1} \\ &= C \Big(\int_{\mathbb{R}} m(|\lambda|\mu) \, dE_1(\mu) \Big) C^{-1} \\ &= C m\big(|\lambda| L_1\big) C^{-1} \ . \qquad \square \end{aligned}$$

3. The spectral analysis of L_1

In order to complete the analysis, we then have to determine the spectral measure E_1 of L_1. The first remark is that we can reduce our analysis to $n = 1$, i.e. to the operator

$$-\left(\partial_x - \frac{i}{2}y\right)^2 - \left(\partial_y + \frac{i}{2}x\right)^2 \tag{3.1}$$

on $\mathbb{R}^2$.

This reduction is based on the following fact (for notational convenience, we state it for two operators, the extension to n operator being left to the reader).

Lemma 3.1. *Let $\mathcal{L}_1, \mathcal{L}_2$ be self-adjoint differential operators on $L^2(\mathbb{R}^d)$ with domains $D_j = \{f : \mathcal{L}_j f \in L^2(\mathbb{R}^d)\}$ and spectra $F_1, F_2 \subseteq \mathbb{R}$. On $(\mathbb{R}^d)^2 = \mathbb{R}^{2d}$, with coordinates (x_1, x_2) with $x_j \in \mathbb{R}^d$, consider the differential operator $\mathcal{L}$ acting as*

$$\mathcal{L}f = (\mathcal{L}_1)_{x_1} f + (L_2)_{x_2} f \ ,$$

in the sense that each $\mathcal{L}_j$ acts on the corresponding variable $x_j \in \mathbb{R}^d$. Then $\mathcal{L}$ is self-adjoint with domain $D = \{f : \mathcal{L}f \in L^2(\mathbb{R}^{2d})\}$, and with spectrum $F = \overline{F_1 + F_2}$.

Proof. It is easy to verify that the operators $(\mathcal{L}_j)_{x_j}$ commute as self-adjoint operators on $L^2(\mathbb{R}^{2d})$, with domains $\tilde{D}_1 = \{f(x_1, x_2) : f(\cdot, x_2) \in D_1 \text{ for a.e. } x_2\}$ and similarly for $\tilde{D}_2$. It is also easy to verify that their joint spectrum is the cartesian product $F_1 \times F_2$ in $\mathbb{R}^2$. The conclusion follows by applying the spectral multiplier $m(\lambda, \mu) = \lambda + \mu$. $\square$

We prefer to use ad-hoc notations at this stage, and set

$$\mathcal{X} = \partial_x - \frac{i}{2}y \ , \qquad \mathcal{Y} = \partial_y + \frac{i}{2}x \ ,$$

calling $\mathcal{L}$ the operator (3.1).

We shall see that $\mathcal{L}$ has a discrete spectrum, and we can explicitely construct a complete system of eigenfunctions. For this construction it is crucial to introduce the complex operators

$$\begin{aligned} \mathcal{Z}f &= \frac{1}{2}(\mathcal{X} - i\mathcal{Y})f = \partial_z f + \frac{\bar{z}}{4}f \\ \bar{\mathcal{Z}}f &= \frac{1}{2}(\mathcal{X} + i\mathcal{Y})f = \partial_{\bar{z}} f - \frac{z}{4}f \ . \end{aligned} \tag{3.2}$$

For reasons that will be immediately clear, we call $\mathcal{Z}$ the *annihilation operator* and $\bar{\mathcal{Z}}$ the *creation operator*.

Since

$$\mathcal{Z}f = e^{-\frac{|z|^2}{4}} \partial_z (e^{\frac{|z|^2}{4}} f) \ , \qquad \bar{\mathcal{Z}}f = e^{\frac{|z|^2}{4}} \partial_{\bar{z}} (e^{-\frac{|z|^2}{4}} f) \ , \tag{3.3}$$

it follows that

$$\mathcal{Z}f = 0 \iff f = e^{-\frac{|z|^2}{4}} \times \text{an antiholomorphic function} \ ,$$

$$\bar{\mathcal{Z}}f = 0 \iff f = e^{\frac{|z|^2}{4}} \times \text{a holomorphic function} \ .$$

This implies that $\bar{\mathcal{Z}}$ is injective on $L^2(\mathbb{C})$, whereas $\mathcal{Z}$ has a big null-space in $L^2(\mathbb{C})$. For $j,k \in \mathbb{N}$, define

$$\begin{aligned} h_{j,0}(z) &= \bar{z}^j e^{-\frac{|z|^2}{4}} \ , \\ h_{j,k}(z) &= \bar{\mathcal{Z}}^k h_{j,0}(z) \ . \end{aligned} \tag{3.4}$$

Proposition 3.2. *The functions $h_{j,k}$, with $j,k \in \mathbb{N}$, form a complete orthogonal system in $L^2(\mathbb{C})$, and*

$$L_1 h_{j,k} = (2k+1) h_{j,k} \ .$$

Proof. Since

$$\bar{\mathcal{Z}}\mathcal{Z} f = \frac{1}{4}(\mathcal{X}+i\mathcal{Y})(\mathcal{X}-i\mathcal{Y}) = \frac{1}{4}\big(\mathcal{X}^2+\mathcal{Y}^2 - i[\mathcal{X},\mathcal{Y}]\big) \ ,$$

and $[\mathcal{X},\mathcal{Y}] = iI$ by (2.3), we have that

$$\mathcal{L} = -4\bar{\mathcal{Z}}\mathcal{Z} + I \ . \tag{3.5}$$

But $\mathcal{Z} h_{j,0} = 0$, hence $\mathcal{L} h_{j,0} = h_{j,0}$. A similar computation shows that

$$\mathcal{L} = -4\mathcal{Z}\bar{\mathcal{Z}} - I \ , \tag{3.6}$$

so that

$$\begin{aligned} \mathcal{L} h_{j,k} &= (-4\bar{\mathcal{Z}}\mathcal{Z} + I)\bar{\mathcal{Z}} h_{j,k-1} \\ &= \bar{\mathcal{Z}}(-4\mathcal{Z}\bar{\mathcal{Z}} + I) h_{j,k-1} \\ &= \bar{\mathcal{Z}}(\mathcal{L} + 2I) h_{j,k-1} \ . \end{aligned}$$

By induction, $\mathcal{L} h_{j,k} = (2k+1) h_{j,k}$.

It also follows by induction that $h_{j,k}$ equals a polynomial in $z, \bar{z}$ times $e^{-\frac{|z|^2}{4}}$. In particular, $h_{j,k} \in \mathcal{S}(\mathbb{C})$, hence in the domain of L_1 in $L^2(\mathbb{C})$. We then have

$$\begin{aligned} (2k+1)\langle h_{j,k} | h_{j',k'} \rangle &= \langle L_1 h_{j,k} | h_{j',k'} \rangle \\ &= \langle h_{j,k} | L_1 h_{j',k'} \rangle \\ &= (2k'+1)\langle h_{j,k} | h_{j',k'} \rangle \ , \end{aligned}$$

so that the two functions are orthogonal if $k \neq k'$.

If $k = k' = 0$, then

$$\begin{aligned} \langle h_{j,0} | h_{j',0} \rangle &= \int_{\mathbb{C}} \bar{z}^j z^{j'} e^{-\frac{|z|^2}{2}} \, dz \\ &= \int_0^{+\infty} r^{j+j'+1} e^{-\frac{r^2}{2}} \, dr \int_0^{2\pi} e^{i(j'-j)\theta} \, d\theta \ , \end{aligned} \tag{3.7}$$

which is zero if $j \neq j'$. The same conclusion follows by induction for $k = k' > 0$, since

$$\begin{aligned}\langle h_{j,k} | h_{j',k}\rangle &= \langle \bar{\mathcal{Z}} h_{j,k-1} | \bar{\mathcal{Z}} h_{j',k-1}\rangle \\ &= -\langle h_{j,k-1} | \mathcal{Z}\bar{\mathcal{Z}} h_{j',k-1}\rangle \\ &= \frac{1}{4}\langle h_{j,k-1} | (\mathcal{L}+I) h_{j',k-1}\rangle \\ &= \frac{k}{2}\langle h_{j,k-1} | h_{j',k-1}\rangle \ .\end{aligned} \tag{3.8}$$

From (3.3) we obtain that

$$h_{j,k}(z) = e^{\frac{|z|^2}{4}} \partial_{\bar z}^k \big(\bar z^j e^{-\frac{|z|^2}{2}}\big) \ , \tag{3.9}$$

and, by Leibniz's formula,

$$\begin{aligned} h_{j,k}(z) &= e^{-\frac{|z|^2}{4}} \sum_{\ell=0}^{\min\{j,k\}} (-1)^{k-\ell} \binom{k}{\ell} \frac{j(j-1)\cdots(j-\ell+1)}{2^{k-\ell}} \bar z^{j-\ell} z^{k-\ell} \\ &= \begin{cases} \bar z^{j-k} P_{j,k}\big(|z|^2\big) e^{-\frac{|z|^2}{4}} & \text{if } j \geq k \ , \\ z^{k-j} P_{j,k}\big(|z|^2\big) e^{-\frac{|z|^2}{4}} & \text{if } j < k \ , \end{cases}\end{aligned} \tag{3.10}$$

where $P_{j,k}$ is a polynomial of degree equal to $\min\{j,k\}$.

This implies that the linear span of the $h_{j,k}$ contains all functions of the form $Q(z,\bar z)e^{-\frac{|z|^2}{4}}$, where Q is an arbitrary polynomial in two variables. Switching back to real coordinates, we find that for every m,n the function $x^m y^n e^{-\frac{x^2+y^2}{4}}$ is in the linear span of the $h_{j,k}$.

Assume that $f \in L^2(\mathbb{R}^2)$ is orthogonal to all the $h_{j,k}$. Then

$$\int_{\mathbb{R}^2} f(x,y) x^m y^n e^{-\frac{x^2+y^2}{4}}\, dx\, dy = 0 \ , \tag{3.11}$$

for all m,n. Consider the functions $g(x,y) = f(x,y)e^{-\frac{x^2+y^2}{4}}$ and

$$G(\zeta_1,\zeta_2) = \int_{\mathbb{R}^2} g(x,y) e^{-i(x\zeta_1+y\zeta_2)}\, dx\, dy \ .$$

Because of the rapid decay of g at infinity due to the Gaussian factor, G is defined on all of $\mathbb{C}^2$, and holomorphic. Hence $\hat g = G_{|_{\mathbb{R}^2}}$, i.e. the Fourier transform of g, is real-analytic on all $\mathbb{R}^2$. Condition (3.11) says that all derivatives of $\hat g$ at the origin are zero. Hence $\hat g = 0$, i.e. $g = 0$ and finally $f = 0$. □

This shows that $\mathcal{L}$ has a discrete spectrum, the eigenvalues being the odd positive integers. Combining this with Lemma 3.1 and with Lemma 2.3, we obtain the following description of the spectral measure E_λ of L_λ on $\mathbb{R}^{2n}$.

Corollary 3.3. *Let $n = 1$. If $\lambda \neq 0$, the spectrum of L_λ, as an operator on $L^2(\mathbb{R}^2)$, is $\{|\lambda|(2k+1) : k \in \mathbb{N}\}$.*

If $\lambda > 0$, $E_\lambda\big(\{\lambda(2k+1)\}\big)$ is the orthogonal projection onto span $\{h_{j,k}\big(\sqrt{\lambda}\, z\big) : j \in \mathbb{N}\}$.

If $\lambda < 0$, $E_\lambda\big(\{|\lambda|(2k+1)\}\big)$ is the orthogonal projection onto span $\{h_{j,k}\big(\sqrt{|\lambda|}\, \bar{z}\big) : j \in \mathbb{N}\}$.

For general n, the spectrum of L_λ is $\{|\lambda|(2k+n) : k \in \mathbb{N}\}$. If $\lambda > 0$, a complete orthogonal system of eigenfunctions for the eigenvalue $\lambda(2k+n)$ is given by the products

$$\prod_{i=1}^{n} h_{j_i,k_i}\big(\sqrt{\lambda}\, z_i\big) ,$$

with $j_i, k_i \in \mathbb{N}$ and $k_1 + \cdots + k_n = k$ (and similarly for $\lambda < 0$, replacing λ by $|\lambda|$ and z_i by $\bar{z}_i$).

One may observe that the coefficient in (3.10)

$$(-1)^{k-\ell}\binom{k}{\ell} j(j-1)\cdots(j-\ell+1) = \frac{(-1)^{k-\ell}k!j!}{\ell!(k-\ell)!(j-\ell)!}$$

exhibits a symmetry in j and k, which give the identity

$$h_{j,k}(z) = (-2)^{j-k} h_{k,j}(\bar{z}) = (-2)^{j-k}\overline{h_{k,j}(z)} . \tag{3.12}$$

Modulo normalizations, the polynomials $P_{j,k}$ appearing in (3.10) are the so-called *Laguerre polynomials* $L_m^{(\alpha)}$, with $\alpha = |j-k|$ and $m = \min\{j,k\}$. The Laguerre polynomial belong to the class of confluent hypergeometric functions, and they are described, e.g., in the book *Higher transcendental functions*, vol. II, by A. Erdelyi, W. Magnus, F. Oberhettinger, F. Tricomi.

4. The joint spectrum of L and $i^{-1}T$

We have already observed in Section 2 that

$$T = \partial_t = [X_j, Y_j] ,$$

this last equality holding for every j.

Since T has constant coefficients, $TLf = LTf$ for every $f \in \mathcal{D}(\mathbb{R}^{2n+1})$. It takes a little thought to realize that, as self-adjoint operators, L and $i^{-1}T$ commute.

Proposition 4.1. *L and $i^{-1}T$ commute with each other. The joint spectrum of L and $i^{-1}T$ consist of the pairs $\big(\lambda, |\lambda|(2k+n)\big)$ with $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$, and of the pairs $(0,\mu)$ with $\mu \geq 0$.*

Proof. It is convenient to replace the two operators by the unitarily equivalent ones obtained by conjugating both of them with $(2\pi)^{-\frac{1}{2}}\mathcal{F}_t$. Instead of L, we then consider $\tilde{L}$, introduced in Section 2, and instead of $i^{-1}T$ the multiplication operator

$$Sf(x,y,\lambda) = \lambda f(x,y,\lambda) .$$

We call $\tilde{E}$ and $\tilde{E}'$ the spectral measures of $\tilde{L}$ and S respectively. The example given in Section 1 shows that

$$\tilde{E}'(\omega')f(x,y,\lambda) = \chi_{\omega'}(\lambda)f(x,y,\lambda) ,$$

and, by Proposition 2.2, calling $f^\lambda(x,y) = f(x,y,\lambda)$,

$$\tilde{E}(\omega)f(x,y,\lambda) = \big(E_\lambda(\omega)f^\lambda\big)(x,y) .$$

Then

$$\tilde{E}(\omega)\tilde{E}'(\omega')f(x,y,\lambda) = \tilde{E}'(\omega')\tilde{E}(\omega)f(x,y,\lambda) = \chi_{\omega'}(\lambda)\big(E_\lambda(\omega)f^\lambda\big)(x,y) , \tag{4.1}$$

hence L and $i^{-1}T$ commute.

In order to determine the joint spectrum of $\tilde{L}$ and S, we discuss the possibility that

$$\tilde{E}(\omega)\tilde{E}'(\omega') = 0 . \tag{4.2}$$

We can take ω, ω' open intervals; for the moment we assume that $0 \notin \omega'$, say $\omega' \subset \mathbb{R}^+$, to fix the notation.

By (4.1), (4.2) happens if $E_\lambda(\omega) = 0$ for every $\lambda \in \omega'$. We show that the converse is also true.

Let $\lambda_0 \in \omega'$ be such that $E_{\lambda_0}(\omega) \neq 0$. Then ω contains a point $\lambda_0(2k+n)$ for some $k \in \mathbb{N}$. Then there is $\delta > 0$ such that $\lambda(2k+n) \in \omega$ for $|\lambda - \lambda_0| < \delta$. Let $\omega'' = (\lambda_0 - \delta, \lambda_0 + \delta) \subset \omega'$.

Take $h(x,y)$ a non-trivial function in the $(2k+n)$-eigenspace of L_1. By Corollary 3.3, $h^\lambda(x,y) = h\big(\sqrt{\lambda}\,x, \sqrt{\lambda}\,y\big)$ is in the $\lambda(2k+n)$-eigenspace of L_λ for $\lambda > 0$, hence $E_\lambda(\omega)h^\lambda = h^\lambda$ for $\lambda \in \omega''$.

Let $f(x,y,\lambda) = \chi_{\omega''}(\lambda)h^\lambda(x,y)$. By (4.1),

$$\tilde{E}(\omega)\tilde{E}'(\omega')f = f ,$$

contradicting (4.2).

This proves that a point (λ,μ) with $\lambda \neq 0$ is in the joint spectrum if and only if $\mu = |\lambda|(2k+n)$ for some $k \in \mathbb{N}$. Then the joint spectrum must also contain the points in the closure of this set, i.e. the points $(0,\mu)$ with $\mu \geq 0$.

On the other hand $\tilde{E}$ is zero on the negative half-line, and this concludes the proof. □

The joint spectrum of L and $i^{-1}T$ is called the *Heisenberg fan.*

CHAPTER II
MIHLIN-HÖRMANDER MULTIPLIERS FOR CONSTANT COEFFICIENT OPERATORS

1. Spectral and Fourier L^p-multipliers

Definition. *Let $\{T_1, \dots, T_n\}$ be a commuting system self-adjoint operators, acting on $L^2(X, \mu)$. If $1 \le p \le \infty$, we say that a measurable, a.e. finite, function m on $\mathbb{R}^n$ is a* spectral L^p-multiplier *if the operator $m(T_1, \dots, T_n)$ extends to a bounded operator from $L^p(X, \mu)$ to itself.*

In particular, By Proposition 1.4 in Chapter I, m is a spectral L^2-multiplier if and only if it is bounded a.e. with respect to the joint spectral measure of the T_j.

In this chapter we shall take $X = \mathbb{R}^n$, with μ the Lebesgue measure, and we shall discuss spectral multipliers for

(1) the system $i^{-1}\partial_{x_j}$, $1 \le j \le n$, i.e. the Fourier multipliers on $\mathbb{R}^n$;
(2) the Laplacian, i.e. the radial Fourier multipliers;
(3) other constant coefficient operators, satisfying homogeneity conditions that will be defined below.

We shall mainly restrict ourselves to $p \ne 1, \infty$. In this section we make some preliminary considerations, starting with case (1).

Lemma 1.1. *If m is a Fourier L^p-multiplier on $\mathbb{R}^n$ for some $p \in (1, \infty)$, then $m \in L^\infty(\mathbb{R}^n)$. The set of points $1/p \in (0,1)$ such that m is a Fourier L^p-multiplier is an interval, symmetric w.r. to $1/2$.*

Proof. Assume that m is real and that $S_m = m(i^{-1}\partial)$ is bounded on $L^p(\mathbb{R}^n)$. By Proposition 1.4 in Chapter I, S_m is self-adjoint. For $f, g \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle S_m f | g \rangle = \langle f | S_m g \rangle \ ,$$

which implies that S_m is also bounded on the dual space $L^{p'}(\mathbb{R}^n)$. By the Riesz-Thorin interpolation theorem, S_m is bounded on $L^q(\mathbb{R}^n)$ for every q between p and p'. This proves the second part of the statement. In particular S_m is L^2-bounded, hence $m \in L^\infty(\mathbb{R}^n)$.

If $m = m_1 + im_2$ is not real and S_m is L^p-bounded, take $f, g \in \mathcal{D}(\mathbb{R}^n)$ real. Then

$$\big|\langle S_m f | g \rangle\big| \le \big|\langle S_{m_1} f | g \rangle\big| + \big|\langle S_{m_2} f | g \rangle\big| \le C \|f\|_p \|g\|_{p'} \ ,$$

and this easily implies that both S_{m_1} and S_{m_2} are L^p-bounded. □

One next remarks concern homogeneity.

For $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}^+)^n$, define

$$\mathbf{r} \cdot x = (r_1 x_1, \dots, r_n x_n)$$

on $\mathbb{R}^n$, and the dilation operator

$$\delta_{\mathbf{r}} f(x) = f_{\mathbf{r}}(x) = f(r_1 x_1, \dots, r_n x_n) \ .$$

It is easy to verify that, if f is in the domain of $i^{-1}\partial_{x_j}$, the same is true for $f_{\mathbf{r}}$ and that

$$(i^{-1}\partial_{x_j}) \circ \delta_{\mathbf{r}} = r_j \delta_{\mathbf{r}} \circ (i^{-1}\partial_{x_j}) \ .$$

Hence

$$\begin{aligned} i^{-1}\partial_{x_j} &= \int_{\mathbb{R}^n} \xi_j \, dE(\xi) \\ &= \delta_{\mathbf{r}}^{-1} \circ \left(r_j^{-1} \int_{\mathbb{R}^n} \xi_j \, dE(\xi) \right) \circ \delta_{\mathbf{r}} \\ &= \delta_{\mathbf{r}}^{-1} \circ \left(\int_{\mathbb{R}^n} \xi_j \, dE(\mathbf{r} \cdot \xi) \right) \circ \delta_{\mathbf{r}} \ , \end{aligned}$$

showing that

$$E(\omega) = \delta_{\mathbf{r}}^{-1} \circ E(\mathbf{r} \cdot \omega) \circ \delta_{\mathbf{r}} \ , \tag{1.1}$$

for every Borel set ω.

Hence, for every multiplier m,

$$\begin{aligned} S_{m_{\mathbf{r}}} &= \int_{\mathbb{R}^n} m(\mathbf{r} \cdot \xi) \, dE(\xi) \\ &= \int_{\mathbb{R}^n} m(\xi) \, dE(\mathbf{r}^{-1} \cdot \xi) \\ &= \delta_{\mathbf{r}}^{-1} \circ \left(\int_{\mathbb{R}^n} m(\xi) \, dE(\xi) \right) \circ \delta_{\mathbf{r}} \\ &= \delta_{\mathbf{r}}^{-1} \circ S_m \circ \delta_{\mathbf{r}} \ . \end{aligned} \tag{1.2}$$

For every p, $\|f_{\mathbf{r}}\|_p = (r_1 r_2 \cdots r_n)^{-1/p} \|f\|_p$, so that

$$\|S_{m_{\mathbf{r}}} f\|_p = \|S_m f\|_p \ .$$

This proves the following statement.

Proposition 1.2. *If m is a Fourier L^p-multiplier, and $\mathbf{r} \in (\mathbb{R}^+)^n$, then $m_{\mathbf{r}}$ is also a Fourier L^p-multiplier, and the operators S_m and $S_{m_{\mathbf{r}}}$ have the same norm.*

Spectral multipliers of the Laplacians correspond to a special subclass of the Fourier multipliers. In fact, if $m(\lambda)$ is Borel measurable on the positive half-line, then

$$m(\Delta) = \tilde{m}(i^{-1}\partial) \ , \tag{1.3}$$

with $\tilde{m}(\xi) = m\big(|\xi|^2\big)$. Observe that if E denotes now the spectral measure of Δ on the positive half-line, then (1.1) and (1.2) hold for $\mathbf{r} = (r, \dots, r)$, with $\mathbf{r} \cdot \omega$ replaced by $r\omega$. As a consequence, we have the following analogue of Proposition 1.2.

Proposition 1.3. *If m is a spectral L^p-multiplier of Δ, and $r > 0$, then $m_r(\lambda) = m(r^{-1}\lambda)$ is also a spectral L^p-multiplier, and the operators S_m and S_{m_r} have the same norm.*

This kind of homogeneity argument is quite general. Staying in the context of $\mathbb{R}^n$, let $L = P(i^{-1}\partial)$ be any constant coefficient, self-adjoint differential operator, and assume that there are positive exponents $\lambda_1, \dots, \lambda_n$ and k such that

$$P(r^{\lambda_1}\xi_1, \dots, r^{\lambda_n}x_n) = r^k P(x_1, \dots, x_n) \ .$$

The expressions *non-isotropic dilations* and *non-isotropic homogeneity* refer to this more general situation[5].

One example is

$$L = -\partial_{x_1}^2 + \Big(\sum_{j=2}^n \partial_{x_j}^2\Big)^2 ,$$

corresponding to $P(\xi) = \xi_1^2 + \Big(\sum_{j=2}^n \xi_j^2\Big)^2$, with $\lambda_1 = 2$, $\lambda_2 = \cdots = \lambda_n = 1$, and $k = 4$.

Again, spectral multipliers m of L correspond to special Fourier multipliers, because (1.3) holds with $\tilde m(\xi) = m\big(P(\xi)\big)$. In this more general case, (1.1) and (1.2) hold for $\mathbf{r} = (r^{d_1}, \dots, r^{d_n})$, with $r > 0$. Proposition 1.3 holds unchanged.

The rest of the chapter is devoted to the presentation of conditions on the multipliers which are invariant under dilations and which assure that they define bounded operators on L^p for all $p \in (1, \infty)$. This requires the introduction of the Calderón-Zygmund theory of singular integrals.

2. The Hardy-Littlewood maximal function

Il punto di partenza della teoria di Calderón-Zygmund è l'analisi dell'operatore massimale di Hardy-Littlewood. In questo paragrafo ne presentiamo gli aspetti che ci consentiranno più avanti di inglobare in un unico contesto lo studio dei moltiplicatori spettrali di tutti gli operatori differenziali presentati nel Capitolo I.

Sia X un insieme. Una *quasi-distanza* su X è una funzione d da $X \times X$ in $\mathbb{R}$ tale che

(1) $d(x, y) \geq 0$ per ogni $x, y \in X$;
(2) $d(x, y) = 0$ se e solo se $x = y$;
(3) $d(x, y) = d(y, x)$ per ogni $x, y \in X$;
(4) esiste una costante $c \geq 1$ tale che

$$d(x, z) \leq c\big(d(x, y) + d(y, z)\big) \tag{2.1}$$

per ogni $x, y, z \in X$.

[5] Despite the name, the *isotropic* case, $\lambda_1 = \cdots = \lambda_n$, is also included.

Una quasi-distanza induce in modo naturale una topologia su X, una cui base è costituita dalle palle $B(x,r)=\{y: d(x,y)<r\}$. Sia ora m una misura di Borel positiva su X. Si dice che m è *doubling* se esiste una costante c' tale che

$$m\big(B(x,2r)\big)\le c' m\big(B(x,r)\big) \tag{2.2}$$

per ogni $x\in X$ e $r>0$.

Definizione. *Una terna (X,d,m), dove d è una una quasi-distanza su X e m è una misura doubling, si dice uno* spazio di natura omogenea*.*

Esempi.

(1) Ovviamente $\mathbb{R}^n$, con la distanza Euclidea e la misura di Lebesgue, è di natura omogenea.
(2) Anche $\mathbb{Z}$, con la distanza $d(n,m)=|n-m|$ e la misura del conteggio $m(E)=\mathrm{card}E$, è di natura omogenea.
(3) Sia $\alpha>-n$. Allora $\mathbb{R}$, con la distanza Euclidea e la misura $dm(x)=|x|^\alpha dx$, è di natura omogenea.
(4) Si prenda $X=\mathbb{R}^n$, e siano $d_1,\dots,d_n>0$. Si ponga

$$d(x,y)=\max\left\{|x_1-y_1|^{1/d_1},\dots,|x_n-y_n|^{1/d_n}\right\}\ .$$

Allora d è una quasi-distanza e, con la misura di Lebesgue, $\mathbb{R}^n$ è uno spazio di natura omogenea.
(5) La sfera unitaria S^{n-1}, dotata della distanza indotta da $\mathbb{R}^n$ e della misura di Hausdorff σ, è uno spazio di natura omogenea.

Sia X uno spazio di natura omogenea.

Definizione. *Sia f localmente integrabile rispetto alla misura m. La funzione*

$$Mf(x)=\sup_{x\in B}\frac{1}{m(B)}\int_B |f(y)|\,dy\ , \tag{2.3}$$

si chiama funzione massimale di Hardy-Littlewood *di f e l'operatore $M: f\longmapsto Mf$* operatore massimale di Hardy-Littlewood*.*

Chiaramente M non è lineare (si osservi che $Mf(x)\ge 0$ per ogni f), ma solo *sub-lineare*, nel senso che

$$M(f+g)\le Mf+Mg\ ,\qquad M(\lambda f)=|\lambda|Mf\ . \tag{2.4}$$

Lemma 2.1. *La funzione Mf è semicontinua inferiormente, e dunque misurabile.*

Proof. Sia $M(x_0)>\alpha$. Esiste allora una palla B contenente x tale che

$$\frac{1}{m(B)}\int_B |f(y)|\,dy>\alpha\ .$$

Ma allora per ogni $x\in B$ $Mf(x)>\alpha$. $\square$

Osservazione. La definizione classica di funzione massimale di Hardy-Littlewood (nel contesto $X = \mathbb{R}^n$, con distanza euclidea e misura di Lebesgue) è la seguente:

$$M'f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \ , \tag{2.5}$$

limitandosi quindi a considerare le medie di $|f|$ sulle palle centrare in x. Chiaramente $M'f(x) \leq Mf(x)$; tuttavia $M'f$ non è necessariamente semicontinua inferiormente. La misurabilità di $M'f$ segue dal fatto che la funzione

$$F(x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

è continua in r, per cui l'estremo superiore in (2.5) può essere ristretto a $r \in \mathbb{Q}$.

In un generico spazio di natura omogenea, nulla assicura che $F(x,r)$ sia continua in r, per cui è preferibile ricorrere alla (2.3).

Ci interessa ora discutere la limitatezza di M sugli spazi $L^p(X)$, ossia la validità di disuguaglianze del tipo

$$\|Mf\|_p \leq C\|f\|_p \ .$$

Si noti che dalle (2.4) segue che $|Mf - Mg| \leq M(f-g)$, e dunque la limitatezza su L^p equivale alla continuità, come per gli operatori lineari.

Ovviamente M è limitato su $L^\infty(X)$. All'altro estremo, per $p = 1$, non si ha limitatezza in generale. Per esempio, nella situazione classica ($X = \mathbb{R}^n$ ecc.), se f è la funzione caratteristica della palla unitaria, si vede facilmente che

$$Mf(x) \geq \frac{C}{1+|x|^n} \ ,$$

per cui $Mf \notin L^1$.

Tuttavia, M risulta limitato in un senso più debole, fatto che costituisce il punto fondamentale della teoria di Calderón-Zygmund.

Definizione. *Sia T un operatore sub-lineare definito da $L^p(X)$ a $L^q(X)$ (con $1 \leq p,q < \infty$) a valori nelle funzioni misurabili su X. Si dice che T è* di tipo debole *(p,q), se per ogni $\alpha > 0$*

$$|\{x : Mf(x) > \alpha\}| \leq C\left(\frac{\|f\|_p}{\alpha}\right)^q \ . \tag{15.3}$$

Si noti che, se T è limitato da $L^p(X)$ a $L^q(X)$, allora T è di tipo debole (p,q) per la disuguaglianza di Chebishev.

Dimostreremo per prima cosa che M è di tipo debole $(1,1)$, basandoci sul seguente *lemma di ricoprimento di Vitali.*

Lemma 2.2. *In uno spazio di natura omogenea X, sia $\{B_j\}_{j\in J}$ un ricoprimento finito di un insieme misurabile E mediante palle. Esiste allora una sottofamiglia $\{B_j\}_{j\in J'}$ tale che $B_j \cap B_k = \emptyset$ per $j,k \in J'$, $j \neq k$, e inoltre*

$$\Big|\bigcup_{j\in J'} B_j\Big| \geq \kappa|E| \ ,$$

dove κ dipende solo dalle costanti c, c' in (2.1), (2.2).

Proof. Sia B_{j_1} una palla che abbia misura massima. Induttivamente si prenda $B_{j_{k+1}}$ in modo che abbia misura massima tra le palle disgiunte da $B_{j_1} \cup \cdots \cup B_{j_k}$. Ovviamente il procedimento si arresta dopo un numero finito di passi, precisamente quando non ci sono più palle disgiunte da $B_{j_1} \cup \cdots \cup B_{j_k}$. Poniamo $J' = \{j_1, \ldots, j_k\}$.

Data una palla B di raggio r, sia B^* la palla con lo stesso centro e raggio $2cr$, dove c è la costante che appare nella (2.1). Si noti che, se due palle B, B' hanno intersezione non vuota e il raggio di B' è minore o uguale al raggio di B, allora $B' \subseteq B^*$.

Sia B' una delle palle avanzate. Necessariamente essa interseca almeno una di quelle selezionate. Sia $\bar{\ell}$ il primo intero ℓ tale che $B' \cap B_{j_\ell} \neq \emptyset$. Allora il raggio di $B_{j_{\bar{\ell}}}$ è maggiore o uguale al raggio di B', e dunque

$$B' \subseteq B^*_{j_{\bar{\ell}}} \ .$$

Di conseguenza

$$E \subseteq \bigcup_{j \in J} B_j \subseteq \bigcup_{j \in J'} B^*_j \ ,$$

Se $2^\nu \geq 2c$, si ha allora

$$m(B^*_j) \leq m\big(B(x, 2^k r)\big) \leq {c'}^\nu m(B) \ .$$

per cui, con $\kappa = {c'}^{-\nu}$,

$$|E| \leq \sum_{j \in J'} |B^*_j| = \kappa^{-1} \sum_{j \in J'} |B_j| = \kappa^{-1} \left| \bigcup_{j \in J'} B_j \right| \ . \qquad \square$$

Teorema 2.3. *L'operatore M è di tipo debole* $(1,1)$.

Proof. Data $f \in L^1(X)$ e dato $\alpha > 0$, sia $E_\alpha = \{x : Mf(x) > \alpha\}$. Per il Lemma 2.1, E_α è aperto e la sua misura è l'estremo superiore delle misure dei suoi sottoinsiemi compatti.

Sia E un sottoinsieme compatto di E_α. Preso $x \in E$, essendo $Mf(x) > \alpha$, esiste una palla B_x contenente x tale che

$$\frac{1}{|B_x|} \int_{B_x} |f(y)| \, dy > \alpha \ ,$$

ossia

$$|B_x| \leq \frac{1}{\alpha} \int_{B_x} |f(y)| \, dy \ .$$

Per la compattezza di E, si può estrarre da $\{B_x\}$ un sottoricoprimento finito, e quindi, applicando il Lemma 2.2, un sottoinsieme finito $\{B_{x_j}\}$ di palle disgiunte tali che $\sum_j |B_{x_j}| \geq \kappa |E|$. Si ha allora

$$|E| \leq \kappa^{-1} \sum_j |B_{x_j}| \leq \frac{\kappa^{-1}}{\alpha} \sum_j \int_{B_{x_j}} |f(y)| \, dy \leq \kappa^{-1} \frac{\|f\|_1}{\alpha} \ .$$

Di conseguenza $|E_\alpha| \leq \kappa^{-1} \|f\|_1 / \alpha$. $\square$

Avendo a disposizione la limitatezza su L^∞ e il tipo debole (1,1), possiamo applicare il seguente *Teorema di interpolazione di Marcinkiewicz*, la cui dimostrazione si trova su E.M. Stein, G. Weiss, *An introduction to Fourier analysis on Euclidean spaces.*

Teorema 2.4. *Sia T un operatore sub-lineare che sia di tipo debole (p_0, q_0) e di tipo debole (p_1, q_1), con $1 \le p_0 \le q_0$, $1 \le p_1 \le q_1$, $p_0 \ne p_1$, $q_0 \ne q_1$*

Allora, dato $t \in [0,1]$ e posto

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \ ,$$

T è di tipo forte (p_t, q_t).

La stessa tesi vale se uno o entrambi dei q_j è infinito, e T è limitato da L^{p_j} in L^{q_j}.

Applicando questo teorema con $p_0 = q_0 = 1$ e $p_1 = q_1 = \infty$, si ottiene immediatamente il seguente risultato.

Corollario 2.5. *Se $1 < p \le \infty$, M è limitato su $L^p(X)$.*

3. Calderón-Zygmund operators on spaces of homogeneous type

Gli operatori di Calderón-Zygmund (o a integrali singolari) costituiscono una classe più ampia di quella degli operatori integrali. In questo paragrafo essi vengono presentati su generici spazi di natura omogenea; in questa generalità la loro definizione risulta alquanto implicita, mentre nei contesti più comuni, come $\mathbb{R}^n$ o le varietà differenziabili, la teoria delle distribuzioni ne consente una descrizione più precisa.

Cominciamo con il considerare, dato uno spazio di natura omogenea X, un operatore lineare T con le seguenti proprietà:

(1) è definito sullo spazio $C_c(X)$ delle funzioni continue a supporto compatto, o su un suo sottospazio denso V;
(2) è a valori in $L^1_{\text{loc}}(X)$;
(3) esiste una funzione $K(x,y)$, localmente integrabile su $(X \times X) \setminus \{\text{diagonale}\}$, tale che, se $f, g \in V$ e $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, allora

$$\int_X (Tf)(x) g(x)\, dx = \iint_{X\times X} K(x,y) f(y) g(x)\, dy\, dx \ , \tag{3.1}$$

o, equivalentemente,

$$Tf(x) = \int_X K(x,y) f(y)\, dy \tag{3.2}$$

per quasi ogni $x \notin \operatorname{supp} f$.

Si noti che K non determina univocamente T: se, per esempio, $Tf = \varphi f$, le tre proprietà sono soddifatte con $K = 0$, indipendentemente da φ. Infatti, il nucleo K non descrive completamente l'operatore, perché non dice quanto valga $\int_X (Tf)(x) g(x)\, dx$ se i supporti di f e g hanno intersezione non vuota.

Definizione. *Si chiama* operatore di Calderón-Zygmund *un operatore lineare T che*

(i) *sia limitato su $L^q(X)$ per qualche $q \in (1, \infty)$;*

(ii) *sia soddisfatta la condizione* (3)*, ed esista una costante* $C > 0$ *tale che per ogni coppia di punti distinti* y, y' *di* X

$$\int_{\{x:d(x,y)>4cd(y,y')\}} \big|K(x,y) - K(x,y')\big|\, dx < C \ , \tag{3.3}$$

dove c *indica la costante nella disuguaglianza triangolare* (2.1)*.*

La condizione (3.3) *si chiama* condizione di Calderón-Zygmund[6].

La (3.3) va interpretata come una forma integrale di regolarità. Una condizione di carattere puntuale che implica la (3.3) è la seguente: indichiamo con $v(x,y)$ il volume della palla di centro x e raggio $r = d(x,y)$, e supponiamo che per qualche $\alpha > 0$ e per ogni terna di punti x, y, y' con $d(x,y) > 4cd(y,y')$ valga la disuguaglianza

$$\big|K(x,y) - K(x,y')\big| \le C \frac{d(y,y')^\alpha}{v(x,y)d(x,y)^\alpha} \ , \tag{3.4}$$

(che appare come una condizione di Lipschitz[7] in y, con una costante che dipende dalla distanza da x).

Allora, posto $E_j = \{x : 4c2^j d(y,y') \le d(x,y) < 4c2^{j+1} d(y,y')\}$,

$$\begin{aligned}
\int_{\{x:d(x,y)>4cd(y,y')\}} & \big|K(x,y) - K(x,y')\big|\, dx \le \\
&\le Cd(y,y')^\alpha \int_{\{x:d(x,y)>4cd(y,y')\}} \frac{1}{v(x,y)d(x,y)^\alpha}\, dx \\
&\le \sum_{j=0}^\infty Cd(y,y')^\alpha \int_{E_j} \frac{1}{v(x,y)d(x,y)^\alpha}\, dx \\
&\le \sum_{j=0}^\infty C' 2^{-\alpha j} \int_{E_j} \frac{1}{v(x,y)}\, dx \ .
\end{aligned}$$

Se $x \in E_j$, posto $d = d(y,y')$ e $r = d(x,y)$, allora $B(x,r) \supseteq B(x, 4c2^j d)$, per cui $v(x,y) \ge m\big(B(x,4c2^j d)\big)$. D'altra parte,

$$m(E_j) = m\big(B(x,4c2^{j+1}d)\big) - m\big(B(x,4c2^j d)\big) \le (c'-1) m\big(B(x,4c2^j d)\big) \ ,$$

per la (2.2). In conclusione,

$$\int_{\{x:d(x,y)>4cd(y,y')\}} \big|K(x,y) - K(x,y')\big|\, dx \le \sum_{j=0}^\infty C'' 2^{-\alpha j} \ ,$$

il che fornisce la condizione di Calderón-Zygmund.

Dimostreremo ora che gli operatori di Calderón-Zygmund sono di tipo debole (1,1). La dimostrazione richiede due strumenti. Il primo è il *lemma di ricoprimento di Whitney* per spazi di natura omogenea.

[6]Il coefficiente 4 nella (3.3) può essere sostituito da un qualunque numero maggiore di 1.

[7]Qui e nel seguito, diremo "lipschitziana di ordine α" in luogo della dizione più comune "hölderiana di ordine α".

Lemma 3.1. *Sia F un chiuso non vuoto di X, e sia A il suo complementare. Esistono costanti $1 < k < k'$, indipendenti da F, e una famiglia numerabile di palle $B_j = B(x_j, r_j) \subset A$ tali che*

(i) *le palle B_j sono a due a due disgiunte;*
(ii) *l'unione delle palle $B_j^* = B(x_j, kr_j)$ è uguale a A;*
(iii) *ogni palla $B_j^{**} = B(x_j, k'r_j)$ ha intersezione non vuota con F.*

Proof. Sia c la costante nella (2.1). Per ogni $x \in A$, sia $d_x = d(x, F)$ e si prenda $B_x = B(x, \delta d_x)$, con $\delta < 1$ da determinarsi. Si scelga quindi una famiglia $\{B_j = B_{x_j}\}_{j \in J}$ di tali palle che sia massimale rispetto alla proprietà di essere a due a due disgiunte.

La famiglia $\{B_j\}$ è numerabile. Ciò segue dal fatto che, fissati $x_0 \in X$ e un intero n, le palle B_j di misura maggiore di $1/n$ e contenute in $B(x_0, n)$ possono essere solo in numero finito. Si noti anche che se una palla avesse misura nulla, tutto X avrebbe misura nulla per la (2.2).

La (i) è dunque verificata. Per la (iii), basta prendere $B_j^{**} = B(x_j, 2d_{x_j})$. Passiamo quindi alla (ii).

Si considerino le palle $B_j^* = B(x_j, d_{x_j}/2)$. Esse sono chiaramente contenute in A. Sia ora $x \in A$. Per la massimalità della famiglia $\{B_j\}$, la palla B_x interseca una delle B_j. Vogliamo mostrare che $x \in B_j^*$, se δ è stato scelto opportunamente. Sia y un punto in $B_x \cap B_j$ e sia $z \in F$ tale che $d(x_j, z) < 2d_{x_j}$. Allora

$$\begin{aligned} d_x &\le d(x,z) \\ &\le c\big(d(x,y) + d(y,z)\big) \le c^2\big(d(x,y) + d(y,x_j) + d(x_j,z)\big) \\ &< c^2\left(\delta d_x + \delta d_{x_j} + 2d_{x_j}\right) \ . \end{aligned}$$

Quindi, se $\delta c^2 < 1$,

$$d_x < \frac{(\delta+2)c^2}{1-\delta c^2} d_{x_j} = \sigma d_{x_j} \ .$$

Ora

$$\begin{aligned} d(x, x_j) &\le c\big(d(x,y) + d(y,x_j)\big) \\ &< c\delta(d_x + d_{x_j}) \\ &< c\delta(1+\sigma) d_{x_j} \ . \end{aligned}$$

Si tratta ora di prendere δ tale che

$$\begin{cases} \delta < \frac{1}{c^2} \\ \delta\left(1 + \frac{(\delta+2)c^2}{1-\delta c^2}\right) < \frac{1}{2} \end{cases} ,$$

cioè $\delta < 1/(2+5c^2)$. $\square$

Il secondo risultato è la *decomposizione di Calderón-Zygmund*. Per semplicità supponiamo che $m(X) = \infty$.

Lemma 3.2. *Sia $f \in L^1(X,m)$ e sia $\alpha > 0$. È possibile decomporre f come*

$$f(x) = g(x) + \sum_{j=0}^{\infty} b_j(x)$$

in modo che

(i) $|g(x)| \le \alpha$;

(ii) *le funzioni b_j hanno supporto in palle B'_j e sono tali che*

$$\frac{1}{m(B'_j)} \int |b_j(x)|\, dm(x) \le \alpha \ , \qquad \int b_j(x)\, dx = 0 \ ;$$

(iii) $\sum_j m(B'_j) \le \frac{C}{\alpha} \|f\|_1$.

Proof. Sia $A = \{x : Mf(x) > \kappa\alpha\}$, con κ da determinarsi, e si costruiscano le palle B_j come nel Lemma 2.1. Si ponga quindi

$$Q_0 = B_0^* \setminus \left(\bigcup_{\ell \ge 1} B_\ell \right)$$

$$Q_1 = B_1^* \setminus \left(Q_0 \cup \bigcup_{\ell \ge 2} B_\ell \right)$$

$$\dots$$

$$Q_j = B_j^* \setminus \left(\bigcup_{\ell < j} Q_\ell \cup \bigcup_{\ell > j} B_\ell \right) \ .$$

I Q_j danno una partizione di A e $B_j \subset Q_j \subset B_j^*$. Quindi

$$\begin{aligned} \frac{1}{m(Q_j)} \int_{Q_j} |f(x)|\, dm(x) &\le \frac{1}{m(B_j)} \int_{B_j^*} |f(x)|\, dm(x) \\ &\le \frac{c'}{m(B_j^*)} \int_{B_j^*} |f(x)|\, dm(x) \\ &\le c' M_1 f(x_j) \\ &\le c' \kappa\alpha \ , \end{aligned}$$

se c' è la costante nella (2.2) e $x_j \in B_j^* \subset A$.

In particolare, se

$$\beta_j = \frac{1}{m(Q_j)} \int_{Q_j} f(x)\, dm(x) \ ,$$

risulta

$$|\beta_j| \le c' \kappa\alpha \ .$$

Si ponga allora

$$g = f\chi_{X \setminus A} + \sum_j \beta_j \chi_{Q_j} \ .$$

Poiché $|f(x)| \leq Mf(x)$ quasi ovunque, risulta

$$|g(x)| \leq c'\kappa\alpha$$

quasi ovunque.

Si ponga ora

$$b_j = (f - \beta_j)\chi_{Q_j} \ .$$

Allora b_j ha supporto nella palla B_j^*, ha integrale nullo e

$$\begin{aligned}\frac{1}{m(B_j^*)}\int |b_j(x)|\,dm(x) &\leq \frac{1}{m(B_j^*)}\int_{Q_j} |f(x)|\,dm(x) + \frac{m(Q_j)}{m(B_j^*)}|\beta_j| \\ &\leq (1+c')\kappa\alpha \ .\end{aligned}$$

Quindi se $\kappa = 1/(1+c')$ e $B_j' = B_j^*$, le condizioni (i) e (ii) sono verificate. Quanto alla (iii), abbiamo

$$\sum_j m(B_j^*) \leq c' \sum_j m(B_j) \leq c' m(A) \leq C\frac{\|f\|_1}{\alpha}$$

perché M è di tipo debole (1,1). □

Teorema 3.3. *Un operatore T di Calderón-Zygmund è di tipo debole* $(1,1)$.

Proof. Si supponga T limitato su $L^q(X)$, e sia $f \in L^1$. Dato $\alpha > 0$, si consideri la decomposizione di Calderón-Zygmund

$$f(x) = g(x) + \sum_{j=0}^{\infty} b_j(x) \ ,$$

come dal Lemma 3.2, corrispondente al valore di α fissato. Se $b(x) = \sum_j b_j(x)$, si ha

$$m\big(\{x : |Tf(x)| > 2\alpha\}\big) \leq m\big(\{x : |Tg(x)| > \alpha\}\big) + m\big(\{x : |Tb(x)| > \alpha\}\big) \ .$$

Osserviamo ora che $g \in L^q$; più precisamente, segue da (ii) e (iii) che

$$\sum_j \|b_j\|_1 \leq C\|f\|_1 \ ,$$

per cui anche $\|g\|_1 \leq C\|f\|_1$. Usando quindi anche (i), si ha

$$\|g\|_q^q \leq C\alpha^{q-1}\|f\|_1$$

Poiché T è limitato su $L^q(X)$ e per la disuguaglianza di Chebishev,

$$\begin{aligned}m\big(\{x : |Tg(x)| > \alpha\}\big) &\leq \frac{\|Tg\|_q^q}{\alpha^q} \\ &\leq C\frac{\|g\|_q^q}{\alpha^q} \\ &\leq C\frac{\|f\|_1}{\alpha} \ .\end{aligned}$$

Passiamo ora a Tb. Sia $B'_j = B(x_j, r_j)$. Poniamo $B''_j = B(x_j, 4cr_j)$. Se $x \notin B'_j$, si ha

$$Tb_j(x) = \int_{B'_j} K(x,y) b_j(y)\, dy = \int_{B'_j} \big(K(x,y) - K(x,x_j)\big) b_j(y)\, dy \ ,$$

in quanto b_j ha integrale nullo. Allora

$$\begin{aligned}
\int_{X \setminus B''_j} |Tb_j(x)|\, dx &\leq \int_{X\setminus B''_j} \int_{B'_j} \big|K(x,y) - K(x,x_j)\big| |b_j(y)|\, dy\, dx \\
&= \int_{B'_j} |b_j(y)| \int_{X\setminus B''_j} \big|K(x,y) - K(x,x_j)\big|\, dx\, dy \\
&\leq \int_{B'_j} |b_j(y)| \int_{\{x: d(x,x_j) > 4cd(y,x_j)\}} \big|K(x,y) - K(x,x_j)\big|\, dx\, dy \\
&\leq C \int_{B'_j} |b_j(y)|\, dy \\
&\leq C\alpha |B'_j| \ .
\end{aligned}$$

Di conseguenza

$$\int_{X \setminus \bigcup_j B''_j} |Tb(x)|\, dx \leq C\alpha \sum_j |B'_j| \leq C \|f\|_1 \ .$$

Per la disuguaglianza di Chebishev,

$$\Big|\{x \notin \bigcup_j B''_j : |Tb(x)| > \alpha\}\Big| \leq C \frac{\|f\|_1}{\alpha} \ .$$

Rimane da considerare la misura dell'insieme $\{x \in \bigcup_j B''_j : |Tb(x)| > \alpha\}$. Ma questa è sicuramente minore o uguale a

$$m\Big(\bigcup_j B''_j\Big) \leq \sum_j m(B''_j) \leq C \sum_j m(B'_j) \leq C \frac{\|f\|_1}{\alpha} \ . \qquad \square$$

Corollario 3.4. *Un operatore di Calderón-Zygmund limitato su $L^q(X)$ è anche limitato su $L^p(X)$ se $1 < p \leq q$. Se anche $k^*(x,y) = \overline{k(y,x)}$ soddisfa la condizione di Calderón-Zygmund* (3.3)*, allora T è limitato su $L^p(X)$ per ogni $p \in (1,\infty)$.*

Proof. La prima parte dell'enunciato segue direttamente dal Teorema di interpolazione di Marcinkiewicz. Se $q = \infty$ non c'è altro da dimostrare.

Se $q < \infty$ e k^* soddisfa la (3.3), allora T^*, che è limitato su $L^{q'}(X)$, è pure un operatore di Calderón-Zygmund, e dunque è limitato su $L^r(X)$ per $1 < r \leq q'$. Ma allora T è limitato su $L^p(X)$ per $q \leq p < \infty$. $\square$

4. Integral Lipschitz conditions

Lasciamo per questo paragrafo gli spazi di natura omogenea, e introduciamo alcune nozioni preliminari al seguito del capitolo. Indicheremo con $\|x\|$ la norma euclidea di $x \in \mathbb{R}^n$.

Si dice che una funzione $f \in L^p(\mathbb{R}^n)$ soddisfa una condizione L^p-Lipschitz di ordine $\alpha \in (0,1)$, o che $f \in \Lambda_p^\alpha(\mathbb{R}^n)$, se esiste una costante c tale che

$$\left(\int |f(x-h) - f(x)|^p \, dx \right)^{1/p} \leq c\|h\|^\alpha \tag{4.3}$$

per ogni $h \in \mathbb{R}^n$. Si pone in tal caso

$$\|f\|_{\Lambda_p^\alpha} = \|f\|_p + \sup_{h \neq 0} \|h\|^{-\alpha} \left(\int |f(x-h) - f(x)|^p \, dx \right)^{1/p} . \tag{4.4}$$

La norma (4.4) può essere sostituita da ciascuna delle norme equivalenti

$$\|f\|'_{\Lambda_p^\alpha} = \|f\|_p + \sup_{0 < \|h\| < a} \|h\|^{-\alpha} \left(\int |f(x-h) - f(x)|^p \, dx \right)^{1/p} . \tag{4.5}$$

Infatti basta maggiorare, per $\|h\| \geq a$,

$$\|h\|^{-\alpha} \left(\int |f(x-h) - f(x)|^p \, dx \right)^{1/p} \leq 2a^{-\alpha} \|f\|_p \, .$$

Le condizioni di Lipschitz integrali sono localmente più deboli delle ordinarie condizioni di Lipschitz. Per esempio, sia $f(x) = \|x\|^{-\frac{n}{p}+\alpha}\varphi(x)$, dove $0 < \alpha < 1$ e φ è una funzione C^∞ a supporto compatto uguale a 1 in un intorno di 0. Allora

$$\begin{aligned}
&\left(\int \left| \|x-h\|^{-\frac{n}{p}+\alpha}\varphi(x-h) - \|x\|^{-\frac{n}{p}+\alpha}\varphi(x) \right|^p dx \right)^{1/p} \\
&\quad \leq \left(\int_{\|x\| > 2\|h\|} \left| \|x-h\|^{-\frac{n}{p}+\alpha}\varphi(x-h) - \|x\|^{-\frac{n}{p}+\alpha}\varphi(x) \right|^p dx \right)^{1/p} \\
&\qquad\quad + 2\left(\int_{\|x\| < 3\|h\|} \|x\|^{-n+\alpha p} \, dx \right)^{1/p} \\
&\quad \leq C\|h\| \left(\int_{\|x\| > 2\|h\|} \|x\|^{-n+\alpha p - p} \, dx \right)^{1/p} + C\left(\int_0^{3\|h\|} r^{\alpha p - 1} \, dr \right)^{1/p} \\
&\quad = C|h| \left(\int_{2\|h\|}^\infty r^{\alpha p - p - 1} \, dr \right)^{1/p} + C\left(\int_0^{3\|h\|} r^{\alpha p - 1} \, dr \right)^{1/p} \\
&\quad = C|h|^\alpha \, .
\end{aligned}$$

Dunque $f \in \Lambda_p^\alpha(\mathbb{R}^n)$.

Più in generale, lo *spazio di Besov* $B^\alpha_{p,q}(\mathbb{R}^n)$ è definito, per $\alpha > 0$ e $1 \le p, q \le \infty$, come lo spazio delle funzioni $f \in L^p$ tali che

$$\|f\|_{B^\alpha_{p,q}} = \begin{cases} \|f\|_p + \left(\int_{\mathbb{R}^n} (\|h\|^{-\alpha}\|\tau_h f - f\|_p)^q \, \frac{dh}{\|h\|^n}\right)^{1/q} & \text{se } q < \infty \\ \|f\|_p + \sup_{h\neq 0} \|h\|^{-\alpha}\|\tau_h f - f\|_p & \text{se } q = \infty \ , \end{cases}$$

dove $\tau_h f(x) = f(x-h)$. Ovviamente $\Lambda^\alpha_p(\mathbb{R}^n) = B^\alpha_{p,\infty}(\mathbb{R}^n)$.

Diamo ora due risultati riguardanti $\Lambda^\alpha_1(\mathbb{R}^n)$ e che useremo nel prossimo paragrafo.

Lemma 4.1. *Se $f \in \Lambda^\alpha_1(\mathbb{R}^n)$, allora*

$$|\hat f(\xi)| \le C\|f\|_{\Lambda^\alpha_1}(1+\|\xi\|)^{-\alpha} \ .$$

Proof. Si osservi che

$$\int f\left(x - \pi\frac{\xi}{\|\xi\|^2}\right) e^{-ix\cdot\xi}\, dx = -\int f(x) e^{-ix\cdot\xi}\, dx = -\hat f(\xi) \ .$$

Quindi

$$\begin{aligned} |\hat f(\xi)| &= \frac{1}{2}\left|\int f\left(x - \pi\frac{\xi}{\|\xi\|^2}\right) e^{-ix\cdot\xi}\, dx - \int f(x) e^{-ix\cdot\xi}\, dx\right| \\ &\le \frac{1}{2}\int \left| f\left(x - \pi\frac{\xi}{\|\xi\|^2}\right) - f(x)\right| dx \\ &\le C\|\xi\|^{-\alpha}\|f\|_{\Lambda^\alpha_1} \ . \end{aligned}$$

D'altra parte,

$$|\hat f(\xi)| \le \|f\|_1 \le \|f\|_{\Lambda^\alpha_1} \ .$$

Quindi

$$(1+\|\xi\|)^\alpha |\hat f(\xi)| \le C(1+\|\xi\|^\alpha)|\hat f(\xi)| \le C\|f\|_{\Lambda^\alpha_1} \ . \qquad \square$$

Lemma 4.2. *Se $f \in \Lambda^\alpha_1$, allora $f \in L^p$ per ogni $p < \frac{n}{n-\alpha}$ e $\|f\|_p \le C_p\|f\|_{\Lambda^\alpha_1}$.*

Proof. Sia $\varphi \in \mathcal{D}(\mathbb{R}^n)$ tale che $\int_{\mathbb{R}^n}\varphi = 1$ e $\operatorname{supp}\varphi$ sia contenuto nella palla unitaria. Poniamo

$$\psi_0(x) = \varphi(x) \ , \psi_j(x) = 2^{nj}\varphi(2^j x) - 2^{n(j-1)}\varphi(2^{j-1}x) \ ,$$

così che

$$f = \lim_{j\to+\infty} f * \left(2^{nj}\varphi(2^j\cdot)\right) = \sum_{j=0}^\infty f * \psi_j \ .$$

Si ha $\|\psi_j\|_1 \le 2$ e $\|\psi_j\|_\infty \le C2^{nj}$. Inoltre $\operatorname{supp}\psi_j \subset B(0, 2^{-(j-1)})$ e $\int_{\mathbb{R}^n}\psi_j = 0$ per $j \ge 1$. Quindi, se $j \ge 1$,

$$\begin{aligned} \|f * \psi_j\|_1 &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\big(f(x-y) - f(x)\big)\psi_j(y)\, dy\right| dx \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big|f(x-y) - f(x)\big||\psi_j(y)|\, dx\, dy \\ &\le \|f\|_{\Lambda^\alpha_1}\int_{\mathbb{R}^n}|y|^\alpha|\psi_j(y)|\, dy \\ &\le C2^{-\alpha j}\|f\|_{\Lambda^\alpha_1} \ . \end{aligned}$$

Inoltre $\|f * \psi_j\|_\infty \le C2^{nj}\|f\|_1 \le C2^{nj}\|f\|_{\Lambda_1^\alpha}$. Quindi

$$\|f * \psi_j\|_p^p \le C\|f * \psi_j\|_\infty^{p-1} \int_{\mathbb{R}^n} |f * \psi_j(x)|\, dx \le C2^{nj(p-1)-\alpha j}\|f\|_{\Lambda_1^\alpha} \ .$$

Sommando su j, la serie delle norme di ordine p converge se $p < n/(n-\alpha)$. □

5. Non-isotropic dilations in $\mathbb{R}^n$ and Calderón-Zygmund kernels

Per la parte restante di questo capitolo l'insieme ambiente X sarà $\mathbb{R}^n$, dotato della misura di Lebesgue. A completare la struttura di spazio di natura omogenea, prenderemo in esame diverse quasi-distanze (tra cui quella euclidea), associate a famiglie di dilatazioni.

Dati numeri positivi (non necessariamente interi) $\lambda_1, \dots, \lambda_n$, si chiamano *dilatazioni non isotropiche* di $\mathbb{R}^n$ relative agli esponenti λ_j le trasformazioni lineari

$$r \cdot x = (r^{\lambda_1}x_1, \dots, r^{\lambda_n}x_n) \ .$$

Se $Q = \lambda_1 + \cdots + \lambda_n$, si ha chiaramente $d(r \cdot x) = r^Q\, dx$. Il numero Q si chiama la *dimensione omogenea* di $\mathbb{R}^n$ rispetto alle date dilatazioni.

Si chiama *norma omogenea* associata alle date dilatazioni una funzione continua $x \longmapsto |x|$ da $\mathbb{R}^n$ a $[0, +\infty)$ tale che

(1) $|x| = 0$ se e solo se $x = 0$;
(2) $|-x| = |x|$ per ogni x;
(3) $|r \cdot x| = r|x|$

Un esempio è dato da

$$|x| = |x_1|^{1/\lambda_1} + \cdots + |x_n|^{1/\lambda_n} \ .$$

La norma euclidea sarà indicata con $\| \ \|$.

Lemma 5.1. *Sia $| \ |$ una norma omogenea.*

(i) *Gli insiemi $B_r = \{x : |x| \le r\}$ sono compatti.*
(ii) *Esiste una costante $c \ge 1$ tale che*

$$|x + y| \le c\big(|x| + |y|\big)$$

per ogni $x, y \in \mathbb{R}^n$.

(iii) *Se $\lambda' = \min_i \lambda_i$ e $\lambda'' = \max_i \lambda_i$, allora esistono due costanti $A, B > 0$ tali che, per ogni x con $|x| > 1$,*

$$A|x|^{\lambda'} \le \|x\| \le B|x|^{\lambda''} \ .$$

(iv) *Esistono due costanti $A', B' > 0$ tali che, per ogni x con $|x| < 1$,*

$$A'|x|^{\lambda''} \le \|x\| \le B'|x|^{\lambda'} \ .$$

(v) *Se $| \ |'$ è un'altra norma omogenea, allora $| \ |$ e $| \ |'$ sono equivalenti, nel senso che esistono costanti $a, b > 0$ tali che*

$$a|x| \le |x|' \le b|x|$$

per ogni $x \in \mathbb{R}^n$.

Proof. Sia S la sfera unitaria chiusa nella norma euclidea, e sia $m > 0$ il minimo della funzione $|\ |$ su S. Dimostriamo che B_m è contenuto nella palla unitaria chiusa euclidea $\tilde{B}_1$. Sia x tale che $\|x\| > 1$. Poiché $\lim_{r\to 0} r \cdot x = 0$ e per la continuità della norma omogenea, esiste $\delta < 1$ tale che $\delta \cdot x \in \tilde{B}_1$ e $|\delta \cdot x| = \delta|x| < m$. Poiché l'applicazione $r \longmapsto r \cdot x$ è continua, esiste $r \in [\delta, 1)$ tale che $r \cdot x \in S$. Allora $|r \cdot x| \geq m$, da cui $|x| = m/r > m$.

Quindi B_m è limitato e dunque compatto. Ma allora $B_r = (r/m) \cdot B_m$ è pure compatto per ogni $r > 0$.

Sia $c = \max\{|x + y| : x, y \in B_1\}$. Dati $x, y \neq 0$, sia $t^{-1} = |x| + |y| > 0$. Allora $t \cdot x$ e $t \cdot x$ sono in B_1, per cui

$$t|x + y| = |t \cdot (x + y)| = |t \cdot x + t \cdot y| \leq c \ ,$$

e questo dimostra la (ii).

Per dimostrare la (iii), osserviamo che esistono $s, \sigma > 0$ tali che

$$\tilde{B}_s \subseteq B_1 \subseteq B_\sigma \ .$$

Quindi

$$r \cdot \tilde{B}_s \subseteq B_r \subseteq r \cdot \tilde{B}_\sigma \ ,$$

per ogni $r > 0$. Se $r > 1$,

$$r \cdot \tilde{B}_s \supseteq \tilde{B}_{r^{\lambda'} s} \ , \qquad r \cdot B_\sigma \subseteq \tilde{B}_{r^{\lambda''} \sigma} \ ,$$

e da questo segue facilmente la tesi. La (iv) si dimostra in modo analogo.

Per la (v), siano a, b rispettivamente il minimo e il massimo di $|x|'$ sulla sfera $S' = \{x : |x| = 1\}$. Se $x = r \cdot y$ con $y \in S'$, allora

$$a|x| = ar \leq r|y|' = |x|' \leq rb = b|x| \ . \qquad \square$$

In particolare ogni norma omogenea è equivalente a

$$|x| = \sum_{j=1}^{n} |x_j|^{1/\lambda_j} \ .$$

Proposizione 5.2. *Sia $|\ |$ una norma omogenea associata a una famiglia di dilatazioni non isotropiche. Allora $\mathbb{R}^n$, dotato della misura di Lebesgue e della quasi-distanza $d(x, y) = |x - y|$, è uno spazio di natura omogenea.*

Proof. Poiché la quasi-distanza è invariante per traslazioni, basta confrontare le misure di B_r e B_{2r}. Ma $m(B_r) = cr^Q$, dove $c = m(B_1)$, per cui la (2.2) vale con $c' = 2^Q$. $\square$

Supporremo ora fissata una famiglia di dilatazioni non isotropiche e una corrispondente norma omogenea. Gli operatori che discuteremo saranno operatori di convoluzione $Tf = f * k$, con $k \in \mathcal{S}'(\mathbb{R}^n)$.

La (3.1) equivale allora alla condizione che k coincida, su $\mathbb{R}^n \setminus \{0\}$, con una funzione $k(x) \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. In tal caso[8]

$$K(x,y) = k(x-y) \; .$$

La condizione di Calderón-Zygmund (3.3) equivale a richiedere che per ogni $h \in \mathbb{R}^n$, $h \neq 0$,

$$\int_{|x|>4c|h|} \big|k(x+h) - k(x)\big| \, dx \leq C \; . \tag{5.1}$$

Allo stesso modo, la (3.4) equivale a richiedere che $k(x)$ sia lipschitziana[9] di ordine $\alpha > 0$ fuori dall'origine e che esista $C > 0$ tale che per $|x| > 4c|h|$ sia

$$\big|k(x+h) - k(x)\big| \leq C \frac{|h|^\alpha}{|x|^{Q+\alpha}} \; . \tag{5.2}$$

Si noti che una distribuzione che soddisfi la condizione (ii) nella definizione di operatore di Calderón-Zygmund non soddisfa necessariamente la condizione (i): si prenda, per es., come k la funzione identicamente uguale a 1. La (ii) è banalmente verificata, ma $Tf(x) = f * 1(x) = \int f$ non è limitato su nessun L^q. Per operatori di convoluzione, l'ipotesi più naturale da imporre è la limitatezza su $L^2(\mathbb{R}^n)$, che equivale a richiedere che $\hat{k} \in L^\infty(\mathbb{R}^n)$.

Chiameremo quindi *nucleo di Calderón-Zygmund* una distribuzione $k \in \mathcal{S}'(\mathbb{R}^n)$ per cui valga la (5.1) e con $\hat{k} \in L^\infty(\mathbb{R}^n)$.

Vedremo ora un procedimento abbastanza generale di costruzione di nuclei di Calderón-Zygmund. La distribuzione k si ottiene come "somma diadica", a partire da una successione di funzioni integrabili φ_j che, al variare di $j \in \mathbb{Z}$, hanno norme uniformemente limitate in qualche $\Lambda^\alpha_1(\mathbb{R}^n)$, soddisfano una condizione di decadimento all'infinito, e hanno tutte media nulla. Ogni φ_j viene poi dilatata di un fattore 2^j.

Dati una funzione f e $j \in \mathbb{Z}$, poniamo $f^{(j)}(x) = 2^{-Qj} f(2^{-j} \cdot x)$.

Teorema 5.3. *Sia* $\{\varphi_j\}_{j\in\mathbb{Z}} \subset L^1(\mathbb{R}^n)$ *una famiglia di funzioni tali che esistano costanti* $\varepsilon > 0, \alpha \in (0,1), C > 0$ *per cui valgano le seguenti proprietà:*

(a) $\int |\varphi_j(x)|(1 + \|x\|)^\varepsilon \, dx \leq C$;
(b) $\int \varphi_j(x) \, dx = 0$;
(c) $\|\varphi_j\|_{\Lambda^\alpha_1} \leq C$.

Allora la serie $\sum_{j\in\mathbb{Z}} \varphi_j^{(j)}$ *converge in* $\mathcal{S}'$ *a un nucleo di Calderón-Zygmund.*

Proof. Consideriamo la serie delle trasformate di Fourier

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi_j^{(j)}}(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\varphi_j}(2^j \cdot \xi) \; , \tag{5.3}$$

[8] Questa notazione crea un'ambiguità tra la distribuzione k e la funzione $k(x)$. Occorre non confondere le due entità. Per esempio, quando si parla di trasformata di Fourier di k, questa va intesa come distribuzione.

[9] Nel senso della quasi-distanza d.

e dimostriamone la convergenza assoluta.

Se $|2^j \cdot \xi| \geq 1$, usiamo il Lemma 4.1 e il Lemma 5.1 (iii) per ricavare che

$$\left|\widehat{\varphi_j}(2^j \cdot \xi)\right| \leq C\left(1 + \|2^j \cdot \xi\|\right)^{-\alpha} \leq C'\left(1 + |2^j \cdot \xi|\right)^{-\alpha\lambda'} .$$

Se invece $|2^j \cdot \xi| < 1$, usando la (b), la disuguaglianza $|e^{it} - 1| \leq C|t|^\varepsilon$, e il Lemma 5.1 (iv), si ha

$$\begin{aligned}\left|\widehat{\varphi_j}(2^j \cdot \xi)\right| &= \left|\int_{\mathbb{R}^n} \varphi_j(x)(e^{-i(2^j \cdot \xi)\cdot x} - 1)\, dx\right| \\ &\leq C\|2^j \cdot \xi\|^\varepsilon \int_{\mathbb{R}^n} |\varphi_j(x)|\|x\|^\varepsilon\, dx \\ &\leq C'|2^j \cdot \xi|^{\varepsilon\lambda'} .\end{aligned}$$

Allora

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\varphi_j^{(j)}}(\xi)\right| \leq C|\xi|^{-\alpha\lambda'} \sum_{j:2^j|\xi|\geq 1} 2^{-\alpha\lambda' j} + C|\xi|^{\varepsilon\lambda'} \sum_{j:2^j|\xi|<1} 2^{\varepsilon\lambda' j} \leq C' .$$

Poniamo dunque

$$u(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\varphi_j^{(j)}}(\xi) \in L^\infty(\mathbb{R}^n) .$$

Per convergenza dominata, data $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} uf = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \widehat{\varphi_j^{(j)}} f ,$$

cioè $u = \sum_{j\in\mathbb{Z}} \widehat{\varphi_j^{(j)}}$ nel senso delle distribuzioni.

Se $k = \mathcal{F}^{-1}u$, si ha dunque

$$k = \sum_{j\in\mathbb{Z}} \varphi_j^{(j)}$$

nel senso delle distribuzioni.

Rimane da dimostrare che, fuori dall'origine, k coincide con una funzione che soddisfa la (5.1).

Mostriamo che la serie

$$\sum_{j\in\mathbb{Z}} \varphi_j^{(j)}(x)$$

converge in $L^1(K)$ per ogni compatto $K \subset \mathbb{R}^n \setminus \{0\}$.

Possiamo supporre che $K = \{x : 2^m \leq |x| \leq 2^{m+1}\}$, con $m \in \mathbb{Z}$. Cambiando variabile,

$$\int_{2^m\leq|x|\leq 2^{m+1}} \left|\varphi_j^{(j)}(x)\right| dx = \int_{2^{m-j}\leq|x|\leq 2^{m+1-j}} \left|\varphi_j(x)\right| dx .$$

Per $j \le m$, usiamo la (a) e il Lemma 5.1 (iii) per ottenere che

$$\begin{aligned}\int_{2^{m-j}\le|x|\le 2^{m+1-j}} \big|\varphi_j(x)\big|\,dx &\le 2^{(j-m)\varepsilon\lambda'} \int_{2^{m-j}\le|x|\le 2^{m+1-j}} \big|\varphi_j(x)\big||x|^{\varepsilon\lambda'}\,dx \\ &\le 2^{(j-m)\varepsilon\lambda'} \int_{2^{m-j}\le|x|\le 2^{m+1-j}} \big|\varphi_j(x)\big|\|x\|^{\varepsilon}\,dx \\ &\le C2^{(j-m)\varepsilon\lambda'}\ .\end{aligned}$$

Per $j > m$, usiamo invece la (c) e il Lemma 4.2 per ottenere che

$$\begin{aligned}\int_{2^{m-j}\le|x|\le 2^{m+1-j}} \big|\varphi_j(x)\big|\,dx &\le \bigg(\int_{|x|\le 2^{m+1-j}} \big|\varphi_j(x)\big|^p\,dx\bigg)^{1/p} m(B_{2^{m+1-j}})^{1/p'} \\ &\le C2^{(m-j)Q/p'}\ ,\end{aligned}$$

per un opportuno $p > 1$. Quindi

$$\sum_{j\in\mathbb{Z}} \|\varphi_j^{(j)}\|_{L^1(K)} \le C\sum_{j\le m} 2^{(j-m)\varepsilon\lambda'} + C\sum_{j>m} 2^{(m-j)Q/p'}\ ,$$

dove entrambe le somme convergono.

Verifichiamo ora la condizione di Calderón-Zygmund. Sia $h \in \mathbb{R}^n \setminus \{0\}$, e si supponga $2^m \le 4c|h| < 2^{m+1}$. Allora

$$\begin{aligned}\int_{|x|>4c|h|}\big|k(x+h)-k(x)\big|\,dx &\le \sum_{j\in\mathbb{Z}}\int_{|x|>2^m} \big|\varphi_j^{(j)}(x+h)-\varphi_j^{(j)}(x)\big|\,dx \\ &= \sum_{j\in\mathbb{Z}}\int_{|y|>2^{m-j}} \big|\varphi_j(y+2^{-j}\cdot h)-\varphi_j(y)\big|\,dy \\ &\le 2\sum_{j<m}\int_{|y|>\frac{2^{m-j}}{2c}} |\varphi_j(y)|\,dy + \sum_{j\ge m}\int \big|\varphi_j(y+2^{-j}\cdot h)-\varphi_j(y)\big|\,dy \\ &\le C\sum_{j<m} 2^{(j-m)\varepsilon\lambda'}\int_{|y|>\frac{2^{m-j}}{2c}} |\varphi_j(y)||y|^{\varepsilon\lambda'}\,dy + C\sum_{j\ge m}\|2^{-j}\cdot h\|^{\alpha} \\ &\le C\sum_{j<m} 2^{(j-m)\varepsilon\lambda'} + C\sum_{j\ge m} 2^{(m-j)\alpha\lambda'}\ .\end{aligned}$$

Nel passaggio alla terza riga siè utilizzato il fatto che

$$2^{m-j} < |y| \le c\big(|y+2^{-j}\cdot h| + 2^{-j}|h|\big) < c\Big(|y+2^{-j}\cdot h| + \frac{2^{m-j}}{2c}\Big)\ ,$$

da cui segue che

$$|y+2^{-j}\cdot h| > \frac{2^{m-j}}{2c}\ . \qquad \square$$

6. Mihlin-Hörmander conditions on Fourier multipliers

Si consideri una famiglia di dilatazioni non isotropiche in $\mathbb{R}^n$. La condizione di Mihlin-Hörmander su un moltiplicatore di Fourier $m(\xi)$ ha le seguenti caratteristiche:

(1) è invariante rispetto alle dilatazioni fissate, nel senso che se $m(\xi)$ la soddisfa, anche $m(r\cdot\xi)$ la soddisfa per ogni $r>0$;
(2) implica che la distribuzione $k=\mathcal{F}^{-1}m$ è un nucleo di Calderón-Zygmund adattato alle dilatazioni fissate.

Di conseguenza, l'operatore $S_m f = m(i^{-1}\partial)f = f*k$ è limitato su $L^p(\mathbb{R}^n)$ per ogni $p\in(1,\infty)$.

Premettiamo la definizione e alcune proprietà degli spazi di Sobolev con esponente frazionario.

Definizione. *Lo spazio di Sobolev $H^s(R^n)$, con $s\in\mathbb{R}$, consiste delle distribuzioni $f\in\mathcal{S}'(\mathbb{R}^n)$ tali che $\hat f$ è una funzione localmente integrabile e*

$$\|f\|^2_{H^s} = \int_{\mathbb{R}^n} |\hat f(\tau)|^2\big(1+\|\tau\|^2\big)^s\,d\tau < \infty\ .$$

Vale l'inclusione $H^s(R^n)\subset H^t(R^n)$ per $s>t$. In particolare, $H^s(R^n)\subset H^0(R^n)=L^2(R^n)$ per $s>0$.

Lemma 6.1. *Se $s\in\mathbb{N}$, $H^s(\mathbb{R}^n)$ coincide con lo spazio delle funzioni $f\in L^2(\mathbb{R}^n)$ tali che $\partial^\alpha f\in L^2(\mathbb{R}^n)$ per ogni multiindice α con $|\alpha|\le s$.*

Tralasciamo la dimostrazione, che segue facilmente dalla formula di Plancherel.

Lemma 6.2. *Sia φ tale che $\hat\varphi\in H^s(R^n)$ con $s>\frac{n}{2}$, allora $\varphi\in L^1(R^n)$. Inoltre per ogni $\varepsilon < s-\frac{n}{2}$*

$$\int_{\mathbb{R}^n} |\varphi(x)|\big(1+\|x\|\big)^\varepsilon\,dx \le C_\varepsilon\|\hat\varphi\|_{H^s}\ .$$

Proof. Per la disuguaglianza di Hölder,

$$\begin{aligned}\int_{\mathbb{R}^n} |\varphi(x)|(1+\|x\|)^\varepsilon\,dx &\le \int_{\mathbb{R}^n} |\varphi(x)|(1+\|x\|^2)^{\varepsilon/2}\,dx\\ &\le C\left(\int_{\mathbb{R}^n}|\varphi(x)|^2(1+\|x\|^2)^s\,dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\frac{1}{(1+\|x\|^2)^{s-\varepsilon}}\,dx\right)^{1/2}\\ &\le C_\varepsilon\|\hat\varphi\|_{H^s}\ .\end{aligned}$$

La conclusione segue dal fatto che $\int(1+\|x\|^2)^{-s+\varepsilon}\,dx$ converge. □

Si osservi che il Lemma 6.2 e il Teorema di Riemann-Lebesgue implicano la *immersione di Sobolev* $H^s(R^n)\subseteq C_0(R^n)$ per $s>\frac{n}{2}$ e la relativa disuguaglianza

$$\|f\|_\infty \le C\|f\|_{H^s}\ . \tag{6.1}$$

Lemma 6.3. *Siano $f \in H^s(\mathbb{R}^n)$ e $g \in \mathcal{S}(\mathbb{R}^n)$. Allora $fg \in H^s(\mathbb{R}^n)$ e $\|fg\|_{H^s} \leq C_g \|f\|_{H_s}$.*

Proof. Poiché $\widehat{fg} = (2\pi)^{-n} \hat{f} * \hat{g}$, si trova che

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{fg}(\tau)|^2 \big(1 + \|\tau\|^2\big)^s \, d\tau &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{f}(\tau - \tau')\hat{g}(\tau') \, d\tau' \right|^2 \big(1 + \|\tau\|^2\big)^s \, d\tau \\ &\leq C\|\hat{g}\|_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{f}(\tau - \tau')|^2 |\hat{g}(\tau')| \, d\tau' \big(1 + \|\tau\|^2\big)^s \, d\tau \\ &\leq C\|\hat{g}\|_1 \int_{\mathbb{R}^n} |\hat{g}(\tau')| \int_{\mathbb{R}^n} |\hat{f}(\tau)|^2 \big(1 + \|\tau + \tau'\|^2\big)^s \, d\tau \, d\tau' \ .\end{aligned}$$

Ma

$$1 + \|\tau + \tau'\|^2 \leq 1 + 2\big(\|\tau\|^2 + \|\tau'\|^2\big) \leq 2\big(1 + \|\tau\|^2\big)\big(1 + \|\tau'\|^2\big) \ ,$$

per cui

$$\|fg\|_{H^s}^2 \leq C\|\hat{g}\|_1 \int_{\mathbb{R}^n} |\hat{g}(\tau')|\big(1 + \|\tau'\|^2\big) \, d\tau' \int_{\mathbb{R}^n} |\hat{f}(\tau)|^2 \big(1 + \|\tau\|^2\big)^s \, d\tau \ .$$

I due termini contenenti $\hat{g}$ sono finiti, in quanto $\hat{g} \in \mathcal{S}(\mathbb{R}^n)$, e questo dimostra la tesi. □

Siano ora $a_0 < a_1 < b_1 < b_0$ numeri positivi. Indichiamo con η una funzione in $\mathcal{D}(R^n)$ tale che[10]

(i) $\operatorname{supp} \eta \subseteq \{\xi : a_0 \leq |\xi| \leq b_0\}$,
(ii) $\eta(\xi) \geq 0$ per ogni ξ e $\eta(\xi) > 0$ per $a_1 \leq |\xi| \leq b_1$.

Poniamo inoltre $m_r(\xi) = m(r \cdot \xi)$.

Definizione. *Si chiama* moltiplicatore di Mihlin-Hörmander*, adattato alle dilatazioni fissate, una funzione $m(\xi)$ tale che*

$$\sup_{r>0} \|m_r \eta\|_{H^s} = \|m\|_{MH_s} < \infty$$

per qualche $s > \frac{n}{2}$.

Indicheremo con $MH_s(\mathbb{R}^n)$ la classe dei moltiplicatori su $\mathbb{R}^n$ con $\|m\|_{MH_s} < \infty$.
Si noti che la definizione stessa implica che m e m_r hanno la stessa norma MH_s per ogni $r > 0$. Inoltre, per la (6.1), le norme $\|m_r \eta\|_\infty$ sono uniformemente limitate, da cui segue che $m \in L^\infty(\mathbb{R}^n)$.

Per s intero, la condizione puntuale

$$\big|\partial^\alpha m(\xi)\big| \leq C|\xi|^{-\sum \lambda_i \alpha_i} \tag{6.2}$$

per $\xi \neq 0$ e $|\alpha| \leq s$ implica che $m \in MH_s(\mathbb{R}^n)$. Questa è la condizione inizialmente enunciata da Mihlin (per dilatazioni isotropiche).

[10] Si noti che nelle condizioni (i) e (ii) si può sostituire la norma euclidea alla norma omogenea.

Si può dimostrare, usando il Lemma 6.3, che scelte diverse di η inducono norme MH_s equivalenti, per cui la condizione di Mihlin-Hörmander non dipende dalla scelta di η. Per i nostri scopi è utile scegliere η in modo che

$$\sum_{j\in\mathbb{Z}} \eta(2^j \cdot \xi) = 1 \tag{6.2}$$

per ogni $\xi \neq 0$. Per ottenere una tale η, si parta da una $\eta_0 \in \mathcal{D}(\mathbb{R}^n)$ soddisfacente (i) e (ii) con $a_1 \le 1$ e $b_1 \ge 2$. Posto

$$\tilde\eta(\xi) = \sum_{j\in\mathbb{Z}} \eta_0(2^j \cdot \xi) \ ,$$

la funzione $\eta = \eta_0/\tilde\eta$ soddisfa (i), (ii) e la (6.2).

Teorema 6.4. *Sia m un moltiplicatore di Mihlin-Hörmander, adattato a una fissata famiglia di dilatazioni. Allora l'operatore $S_m = m(i^{-1}\partial)$ è limitato su $L^p(\mathbb{R}^n)$ per $1 < p < \infty$.*

Proof. Sia $m_j(\xi) = m(2^{-j} \cdot \xi)\eta(\xi)$. Allora

$$\sum_{j\in\mathbb{Z}} m_j(2^j \cdot \xi) = m(\xi)$$

quasi ovunque e nel senso delle distribuzioni. Ne segue che, se poniamo

$$k = \mathcal{F}^{-1}m \ , \varphi_j = \mathcal{F}^{-1}m_j \ ,$$

allora

$$k = \sum_{j\in\mathbb{Z}} \varphi_j^{(j)}$$

nel senso delle distribuzioni. Mostriamo ora che le φ_j soddisfano le ipotesi (a), (b), (c) del Teorema 5.3.

Poiché $\|m_j\|_{H^s} \le \|m\|_{MH_s}$, la (a) segue dal Lemma 6.2. La (b) segue dal fatto che $m_j(0) = 0$.

Dimostriamo ora la (c) con $\alpha = 1$, osservando preliminarmente che per ogni $i = 1, \dots, n$ e per ogni $j \in \mathbb{Z}$, $\xi_i m_j(\xi) \in H^s(\mathbb{R}^n)$ e che $\|\xi_i m_j\|_{H^s} \le C\|m\|_{MH_s}$. Infatti, sia $\omega(\xi) \in \mathcal{D}(\mathbb{R}^n)$, con $\omega = 1$ sul supporto di η. Allora $\xi_i m_j = (\xi_i\omega)m_j$, e possiamo dunque applicare il Lemma 6.3.

Usando la disuguaglianza di Hölder come nella dimostrazione del Lemma 6.2, si ottiene che

$$\int_{\mathbb{R}^n} \left|\varphi_j(x-h) - \varphi_j(x)\right| dx \le C\bigg(\int_{\mathbb{R}^n} \left|\varphi_j(x-h) - \varphi_j(x)\right|^2 \left(1 + \|x\|^2\right)^s dx\bigg)^{\frac12} .$$

Essendo m_j a supporto compatto, φ_j è C^∞, per cui

$$\left|\varphi_j(x-h) - \varphi_j(x)\right|^2 = \left|\int_0^1 h \cdot \nabla\varphi_j(x-th)\, dt\right|^2 \le \|h\|^2 \int_0^1 \|\nabla\varphi_j(x-th)\|^2\, dt \ .$$

Quindi, poiché $\widehat{\partial_{x_i}\varphi_j} = i\xi_i m_j$,

$$\begin{aligned}
\sup_{0<\|h\|<1} \|h\|^{-1} \int_{\mathbb{R}^n} \big|\varphi_j(x-h) - \varphi_j(x)\big|\, dx \leq \\
&\leq \sup_{0<\|h\|<1} \|h\|^{-1} \bigg(\int_{\mathbb{R}^n} \big|\varphi_j(x-h) - \varphi_j(x)\big|^2 \big(1+\|x\|^2\big)^s\, dx \bigg)^{\frac{1}{2}} \\
&\leq \sup_{0<\|h\|<1} \bigg(\int_{\mathbb{R}^n} \int_0^1 \|\nabla\varphi_j(x-th)\|^2\, dt \big(1+\|x\|^2\big)^s\, dx \bigg)^{\frac{1}{2}} \\
&\leq \sup_{0<\|h\|<1} \bigg(\int_0^1 \int_{\mathbb{R}^n} \|\nabla\varphi_j(x)\|^2 \big(1+\|x+th\|^2\big)^s\, dx\, dt \bigg)^{\frac{1}{2}} \\
&\leq 2 \bigg(\int_{\mathbb{R}^n} \|\nabla\varphi_j(x)\|^2 \big(1+\|x\|^2\big)^s\, dx \bigg)^{\frac{1}{2}} \\
&= 2 \bigg(\sum_{i=1}^n \|\xi_i m_j\|^2_{H^s} \bigg)^{\frac{1}{2}} \\
&\leq C\|m\|_{MH_s}\ .
\end{aligned}$$

Usando la norma (4.5) e il fatto che $\|\varphi_j\|_1 \leq C\|m\|_{MH_s}$ per il Lemma 6.2, si ha che $\|\varphi_j\|_{\Lambda_1^1} \leq C\|m\|_{MH_s}$.

Applicando allora il Teorema 5.3 si ha la conclusione. □

7. Applications

Results about L^p-boundedness for Fourier multipliers have important consequences for differential operator with constant coefficients.

Our first application concerns spectral multipliers. We shall make the following two assumptions on the symbol $P(\xi)$ of the operator $L = P(i^{-1}\partial)$:

(1) there are dilations $r \cdot \xi = (r^{\lambda_1}\xi_1, \ldots, r^{\lambda_n}\xi_n)$ such that P is homogeneous of degree $k > 0$, i.e. $P(r\cdot\xi) = r^k P(\xi)$;
(2) $P(\xi) > 0$ for $\xi \neq 0$.

Condition (1) is equivalent to saying that each α such that the monomial ξ^α has a non-zero coefficient in P has a non-isotropic degree

$$d(\alpha) = \sum_{j=1}^n \lambda_j \alpha_j$$

equal to k.

Condition (2) implies that L is self-adjoint and positive with domain $D = \{f \in L^2 : Lf \in L^2\}$.

Theorem 7.1. *Let $L = P(i^{-1}\partial)$ with P satisfying* (1) *and* (2)*, and let $m(\lambda) \in MH_s(\mathbb{R})$ with $s > \frac{n}{2}$. Then $m(L)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Observe that the Mihlin-Hörmander condition on the real line makes sense also for multipliers defined only on $\mathbb{R}^+$ or on $\mathbb{R}^-$. Moreover, a multiplier m on the whole line satisfies the Mihlin-Hörmander condition if and only if both $m^\pm = m\chi_{\mathbb{R}^\pm}$ do. Since the spectrum of L is the positive half line, it is sufficient to assume that m is defined for $\lambda > 0$.

The proof of Theorem 7.1 requires some further remarks on Sobolev spaces and non-isotropic norms.

Lemma 7.2. *Let T be a linear operator, bounded from $H^{s_0}(\mathbb{R}^n)$ to $H^{s_0}(\mathbb{R}^m)$ and from $H^{s_1}(\mathbb{R}^n)$ to $H^{s_1}(\mathbb{R}^m)$, with $0 \le s_0 < s_1$, and let C_0, C_1 be the corresponding operator norms. Then, for $s_0 < s < s_1$, T is bounded from $H^s(\mathbb{R}^n)$ to $H^s(\mathbb{R}^m)$. If $s = (1-\theta)s_0 + \theta s_1$, then the operator norm of T acting between the H^s spaces is not larger than $C_0^{1-\theta} C_1^\theta$.*

Proof. Consider $T' = \mathcal{F} T \mathcal{F}^{-1}$. By assumption, for $j = 0, 1$, T' is bounded from the weighted L^2 spaces $L^2_{s_j} = L^2\big(\mathbb{R}^n, (1 + \|\tau\|^2)^{s_j}\, d\tau\big)$ to the same space on $\mathbb{R}^m$, and we want to prove that it is bounded between the L^2_s spaces, with the stated bound on the norm.

Like in the proof of the Riesz-Thorin theorem, we use the three-lines theorem. It is sufficient to prove to prove that, if g, h are continuous functions with compact support in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and $\|g\|_{L^2_s} = \|h\|_{L^2_s} = 1$, then

$$\left| \int_{\mathbb{R}^m} (T'g)(x) h(x) \big(1 + \|x\|^2\big)^s \, dx \right| \le C_0^{1-\theta} C_1^\theta \ . \tag{7.1}$$

For $z \in \mathbb{C}$, define

$$g_z(x) = g(x)\big(1 + \|x\|^2\big)^z \ , \qquad h_z(x) = h(x)\big(1 + \|x\|^2\big)^z \ ,$$

and let

$$F(z) = \int_{\mathbb{R}^m} (T'g_z)(x) h_z(x) \big(1 + \|x\|^2\big)^{s+2z} \, dx \ .$$

Since g_z, h_z have compact support, F is defined and holomorphic in the whole plane. We restrict F to the vertical strip S where

$$\frac{s_0 - s}{2} \le \Re\mathrm{e} z \le \frac{s_1 - s}{2} \ .$$

For $z \in S$,

$$\begin{aligned} |F(z)| &\le \int_{\mathbb{R}^m} \big|(T'g_z)(x)\big| \big|h_z(x)\big| \big(1 + \|x\|^2\big)^{s_1} \, dx \\ &\le C_1 \|g_z\|_{L^2_{s_1}} \|h_z\|_{L^2_{s_1}} \ . \end{aligned}$$

If g and h are supported on the ball of radius r, using the normalization of g and h in L^2_s,

$$\begin{aligned} \|g_z\|^2_{L^2_{s_1}} &= \int_{\|x\|<r} |g(x)|^2 \big(1 + \|x\|^2\big)^{2s_1 - s} \, dx \\ &\le (1 + r^2)^{2(s_1 - s)} \ , \end{aligned}$$

and similarly for h_z. Hence F is bounded on S.

For $\Re e z = \sigma_j = \frac{s_j - s}{2}$,

$$\begin{aligned} |F(z)| &\le C_j \|g_z\|_{L^2_{\sigma_j}} \|h_z\|_{L^2_{\sigma_j}} \\ &= C_j \|g\|_{L^2_s} \|h\|_{L^2_s} \\ &= C_j \ . \end{aligned}$$

By the three lines theorem, if $\Re e z = (1-\theta)\sigma_0 + \theta\sigma_1 = 0$,

$$|F(z)| \le C_0^{1-\theta} C_1^\theta \ .$$

For $z = 0$ this gives (7.1). □

Lemma 7.3. *Given a family of dilations on $\mathbb{R}^n$, with homogeneous dimension Q. Let $\langle x \rangle$ be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, homogeneous of degree 1 with respect to the given dilations, and strictly positive for $x \neq 0$ (e.g. a homogeneous norm*[11]*). Let S be the set where $\langle x \rangle = 1$. There is a positive Borel measure σ on S such that*

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_S f(r \cdot x)\, d\sigma(x)\, r^{Q-1}\, dr \ ,$$

for every integrable function f.

Proof. If E is a Borel subset of S, let

$$E^\sharp = \{ r \cdot x : x \in E\ ,\ r \le 1 \} \ ,$$

and define

$$\sigma(E) = Q m(E^\sharp) \ .$$

For $0 < a < b$, let

$$E_{a,b} = \{ r \cdot x : x \in E\ ,\ a < r \le b \} = (b \cdot E^\sharp) \setminus (a \cdot E^\sharp) \ .$$

Then

$$m(E_{a,b}) = \frac{b^Q - a^Q}{Q} \sigma(E) = \int_{E \times [a,b]} r^{Q-1} dr\, d\sigma \ .$$

Standard measure-theoretic arguments give the conclusion. □

Proposition 7.4. *Let $f \in H^s(\mathbb{R})$, $s \ge 0$, be supported on interval $[b, 2b]$, with $b > 0$. If P satisfies* (1), (2)*, then $f \circ P \in H^s(\mathbb{R}^n)$, and*

$$\|f \circ P\|_{H^s(\mathbb{R}^n)} \le C(b, P) \|f\|_{H^s(\mathbb{R})} \ .$$

Proof. If $s = m \in \mathbb{N}$, we use the characterization of H^m as the space of L^2 functions with derivatives in L^2 up to order m.

Set $\langle \xi \rangle = P(\xi)^{\frac{1}{k}}$. By Lemma 7.3,

$$\|f \circ P\|_2^2 = \sigma(S) \int_{b^{1/k}}^{(2b)^{1/k}} \big| f(r^k) \big|^2 r^{Q-1}\, dr = C \int_b^{2b} \big| f(r) \big|^2 r^{\frac{Q}{k} - 1}\, dr \le C b^{\frac{Q}{k} - 1} \|f\|_2^2 \ .$$

[11] The missing hypothesis is that $\langle -x \rangle = \langle x \rangle$.

Similar estimates hold for the L^2 norms of $\partial^\alpha(f\circ P)$, with $|\alpha|\le m$, by the chain rule and Leibniz's rule.

Assume now that $m<s<m+1$. Let ω be a smooth function on the positive half-line, equal to 1 on $[b,2b]$ and with compact support. Define the operator

$$Tg(\xi)=(g\omega)\big(P(\xi)\big)\ ,$$

mapping functions on $\mathbb{R}$ to functions on $\mathbb{R}^n$. By the first part of the proof and Lemma 6.3,

$$\|Tg\|_{H^m(\mathbb{R}^n)}\le C(b,P)\|g\omega\|_{H^m(\mathbb{R})}\le C(b,P,\omega)\|g\|_{H^m(\mathbb{R})}\ ,$$

and similarly for H^{m+1}. By Lemma 7.2,

$$\|Tg\|_{H^s(\mathbb{R}^n)}\le C(b,P,\omega)\|g\|_{H^s(\mathbb{R})}\ .$$

If $\operatorname{supp} f\subseteq[b,2b]$, then $Tf=f\circ P$ and does not depend on the choice of ω. □

We can prove now Theorem 7.1.

Proof of Theorem 7.1. Let $\tilde m(\xi)=m\big(P(\xi)\big)$. If η satisfies (i) and (ii) of Section 6 on $\mathbb{R}^+$, then $\eta\circ P$ satisfies the same conditions on $\mathbb{R}^n$. Since

$$\tilde m_r(\xi)=m\big(P(r^{-1}\cdot\xi)\big)=m\big(r^{-1}P(\xi)\big)=m_r\big(P(\xi)\big)\ ,$$

we have that

$$\|\tilde m_r(\eta\circ P)\|_{H^s(\mathbb{R}^n)}=\|(m_r\eta)\circ P\|_{H^s(\mathbb{R}^n)}\le C\|m_r\eta\|_{H^s(\mathbb{R})}\ .$$

Therefore $\tilde m\in MH_s(\mathbb{R}^n)$. □

Here is a corollary which shows the importance of looking for minimal assumptions on the multipliers (in terms of the Sobolev spaces they must belong to).

Corollary 7.5. *Let $\gamma\in\mathbb{R}$ and $1<p<\infty$. Then $L^{i\gamma}$ is bounded on $L^p(\mathbb{R}^n)$ and, for every $\varepsilon>0$,*

$$\|L^{i\gamma}\|_{L^p\to L^p}\le C_{p,\varepsilon}\big(1+|\gamma|\big)^{(n+\varepsilon)\left|\frac12-\frac1p\right|}\ .$$

Proof. We apply Theorem 7.1 to $m(\lambda)=\lambda^{i\gamma}$. If $\eta\in\mathcal{D}(\mathbb{R}^+)$ and $r>0$,

$$\|m_r\eta\|_{H^s}=\|m\eta\|_{H^s}\ .$$

If $s=k\in\mathbb{N}$, $\|m\eta\|_{H^k}\le C_k(1+|\gamma|)^k$, by estimating L^2-norms of derivatives. Assume now that $k<s<k+1$, $s=\theta k+(1-\theta)(k+1)$. Setting $u=\widehat{m\eta}$ and using Hölder's inequality,

$$\begin{aligned}\|m\eta\|_{H^s}^2&=\int_{\mathbb{R}}|u(\tau)|^2(1+|\tau|^2)^s\,d\tau\\&\le\Big(\int_{\mathbb{R}}|u(\tau)|^2(1+|\tau|^2)^k\,d\tau\Big)^\theta\Big(\int_{\mathbb{R}}|u(\tau)|^2(1+|\tau|^2)^{k+1}\,d\tau\Big)^{1-\theta}\\&=\|m\eta\|_{H^k}^{2\theta}\|m\eta\|_{H^{k+1}}^{2(1-\theta)}\\&\le C_s\big(1+|\gamma|\big)^{2s}\ .\end{aligned}$$

Then $\|m\|_{MH_s} \le C_s(1+|\gamma|)^s$. By Theorem 7.1, $\|L^{i\gamma}\|_{L^p\to L^p} \le C(1+|\gamma|)^s$ for every $p \in (1,+\infty)$. This is not yet the required estimate, but we shall use this partial result to complete the proof.

It is sufficient to take $p > 2$, by duality. The Plancherel formula gives that $\|L^{i\gamma}\|_{L^2\to L^2} = 1$. Take $p_0 > p$, and let $\theta \in (0,1)$ be such that

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{2} = \frac{1}{2} - \theta\Big(\frac{1}{2} - \frac{1}{p_0}\Big) .$$

By the Riesz interpolation theorem,

$$\|L^{i\gamma}\|_{L^p\to L^p} \le C_p(1+|\gamma|)^{s\theta} = C_p(1+|\gamma|)^{s\frac{\frac12-\frac1p}{\frac12-\frac1{p_0}}} .$$

If we let p_0 tend to ∞, the exponent decreases to $2s\big(\frac12 - \frac1p\big)$, and if s tends to $n/2$, this quantity dcereases to $n\big(\frac12 - \frac1p\big)$. Given $\varepsilon > 0$, it is then possible to find s and p_0 such that the required estimates holds. $\square$

Our second application concerns a-priori estimates. We keep condition (1) at the beginning of this Section, and replace (2) by

(2') $P(\xi) \neq 0$ for $\xi \neq 0$.

Theorem 7.6. *Assume that P satisfies (1) and (2'), and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with $d(\alpha) \le k$. Then for every $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 < p < \infty$,*

$$\|\partial^\alpha f\|_p \le C_p\big(\|f\|_p + \|Lf\|_p\big) .$$

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ on some neighborhood of 0. Then

$$\begin{aligned}\widehat{\partial^\alpha f}(\xi) &= (i\xi)^\alpha \varphi(\xi)\hat f(\xi) + \frac{(i\xi)^\alpha\big(1-\varphi(\xi)\big)}{P(\xi)}\widehat{Lf}(\xi)\\ &= m_1(\xi)\hat f(\xi) + m_2(\xi)\widehat{Lf}(\xi) .\end{aligned}$$

The multiplier m_1 is in $\mathcal{D}(\mathbb{R}^n)$, so that $u = \mathcal{F}^{-1}m_1 \in \mathcal{S}(\mathbb{R}^n)$ and

$$\big\|\mathcal{F}^{-1}(m_1\hat f)\big\|_p \le \|f * u\|_p \le \|u\|_1\|f\|_p .$$

We verify now that m_2 satisfies (6.2) for arbitrary multi-indices β. Because m_2 is smooth, these estimates are trivial when ξ is in the support of φ. We can then restrict ourselves to ξ large enough so that $\varphi(\xi) = 0$.

In this region, m_2 is homogeneous of degree $-k + d(\alpha) \le 0$, hence it is bounded. Any derivative $\partial^\beta m_2$ is homogeneous of degree $-k + d(\alpha) - d(\beta) \le -d(\beta)$. Hence

$$\big|\partial^\beta m_2(\xi)\big| \le C_\beta|\xi|^{-d(\beta)} .$$

It follows that

$$\big\|\mathcal{F}^{-1}(m_2\hat f)\big\|_p \le C_p\|f\|_p . \qquad \square$$

CHAPTER III
LITTLEWOOD-PALEY THEORY AND MARCINKIEWICZ MULTIPLIERS

1. Square functions

Sia $I = [0,1]$. La *funzione di Rademacher* $r_n \in L^2(I)$ è definita, per $n \geq 0$, da

$$r_n(t) = (-1)^{[2^n t]} \ .$$

In altri termini, decomponendo I nell'unione degli intervalli

$$[j2^{-n}, (j+1)2^{-n}] \ , \qquad j = 0, \dots, 2^n - 1 \ ,$$

r_n assume il valore costante $(-1)^j$ su ciascun intervallo.

Lemma 1.1. *Se $k \geq 1$ e $0 < n_1 < n_2 < \cdots < n_k$, allora*

$$\int_0^1 r_{n_1}(t) r_{n_2}(t) \cdots r_{n_k}(t)\, dt = 0 \ ,$$

e le funzioni di Rademacher formano un sistema ortonormale, ma non completo, in $L^2(I)$.

Proof. Il primo asserto è ovvio per $k = 1$. Supponiamo dunque $k \geq 2$. Su ognuno degli intervalli $[j2^{-n_{k-1}}, (j+1)2^{-n_{k-1}}]$ il prodotto $r_{n_1}(t) r_{n_2}(t) \cdots r_{n_{k-1}}(t)$ è costante, mentre $r_{n_k}(t)$ assume i valori ± 1 su sottoinsiemi di uguale misura. Quindi l'integrale dell'intero prodotto è nullo su ciascuno di tali intervalli.

L'ortonormalità è dunque ovvia. Si osservi infine che la funzione $r_1 r_2$ è ortogonale a tutte le r_n per concludere che il sistema non è completo. □

La rilevanza delle funzioni di Rademacher è dovuta al seguente risultato, noto come *teorema di Khintchin*.

Teorema 1.2. *Sia $f(t) = \sum_{n=0}^\infty a_n r_n(t) \in L^2(I)$. Allora per ogni $p < \infty$, la norma di f in L^p è equivalente alla norma di f in L^2, ossia*

$$c_p \Big(\sum_{n=0}^\infty |a_n|^2 \Big)^{1/2} \leq \|f\|_p \leq C_p \Big(\sum_{n=0}^\infty |a_n|^2 \Big)^{1/2} \ .$$

Proof. Supponiamo inizialmente $p > 2$. Per la disuguaglianza di Hölder, $\|f\|_2 \leq \|f\|_p$. Per dimostrare la disuguaglianza opposta, è sufficiente prendere $p = 2k$ ed f reale. Si ha

$$\begin{aligned} \|f\|_{2k}^{2k} &= \int_0^1 \left(\sum_{n=0}^{\infty} a_n r_n(t) \right)^{2k} dt \\ &= \sum_{(n_1,\dots,n_{2k})\in\mathbb{N}^{2k}} \int_0^1 a_{n_1} a_{n_2} \cdots a_{n_{2k}} r_{n_1}(t) r_{n_2}(t) \cdots r_{n_{2k}}(t)\, dt \ . \end{aligned} \tag{1.1}$$

Per il Lemma 1.1, gli addendi non nulli nella (1.1) possono essere solo quelli in cui uno stesso indice compare un numero pari di volte. In tal caso l'integrando è una costante. Pertanto

$$\|f\|_{2k}^{2k} \leq C_k \sum_{n_1 \leq \cdots \leq n_k} a_{n_1}^2 a_{n_2}^2 \cdots a_{n_k}^2 \ ,$$

dove C_k è un maggiorante del numero di elementi di $\mathbb{N}^{2k}$ in cui compaiono ripetuti due volte gli indici $n_1 \leq \cdots \leq n_k$.

Ma allora

$$\begin{aligned} \|f\|_{2k}^{2k} &\leq C_k \sum_{(n_1,\dots,n_k)\in\mathbb{N}^k} a_{n_1}^2 a_{n_2}^2 \cdots a_{n_k}^2 \\ &= C_k \left(\sum_{n=0}^{\infty} a_n^2 \right)^k \ , \end{aligned}$$

che fornisce la tesi per $p > 2$.

Se $1 < p < 2$, dalla disuguaglianza di Hölder segue che $\|f\|_p \leq \|f\|_2$. Sempre per la disuguaglianza di Hölder, e per la parte precedente della dimostrazione,

$$\|f\|_2^2 \leq \|f\|_p \|f\|_{p'} \leq C_{p'} \|f\|_p \|f\|_2 \ .$$

Quindi $\|f\|_2 \leq C_{p'} \|f\|_p$.

Rimane da considerare il caso $p = 1$. Procedendo come sopra,

$$\|f\|_{4/3}^2 \leq \|f\|_1 \|f\|_2 \leq C \|f\|_1 \|f\|_{4/3} \ ,$$

da cui $\|f\|_{4/3} \leq C\|f\|_1$. Essendo anche $\|f\|_1 \leq \|f\|_{4/3}$ per la disuguaglianza di Hölder, la dimostrazione è completata. □

Corollario 1.3. *Sia T_n una successioni di operatori limitati su $L^p(X)$, dove X è uno spazio di misura e $p < \infty$. Se esiste una costante A tale che per ogni scelta possibile dei segni $\varepsilon_n = \pm 1$, risulta*

$$\| \sum_{n=0}^{\infty} \varepsilon_n T_n \|_{pp} \leq A \ , \tag{1.2}$$

allora vale la maggiorazione

$$\left\| \left(\sum_{n=0}^{\infty} |T_n f|^2 \right)^{1/2} \right\|_p \leq C_p A \|f\|_p \ .$$

Proof. Preso $t \in [0,1]$, si consideri l'operatore

$$T_t = \sum_{n=0}^{\infty} r_n(t) T_n \ ,$$

dove r_n è l'n-esima funzione di Rademacher. Per ipotesi,

$$\|T_t f\|_p^p \leq A^p \|f\|_p^p \ .$$

Allora anche

$$\int_0^1 \int_X |T_t f(x)|^p \, dx \, dt = \int_0^1 \|T_t f\|_p^p \, dt \leq A^p \|f\|_p^p \ .$$

Cambiando ordine di integrazione, si ha internamente

$$\begin{aligned}\int_0^1 |T_t f(x)|^p \, dt &= \int_0^1 \Big| \sum_{n=0}^{\infty} r_n(t) T_n f(x) \Big|^p \, dt \\ &\geq c_p \Big(\sum_{n=0}^{\infty} |T_n f(x)|^2 \Big)^{p/2}\end{aligned}$$

per il Teorema 1.2. Quindi

$$\begin{aligned}\int_X \Big(\sum_{n=0}^{\infty} |T_n f(x)|^2 \Big)^{p/2} dx &\leq c_p^{-1} \int_X \int_0^1 |T_t f(x)|^p \, dt \, dx \\ &\leq c_p^{-1} A^p \|f\|_p^p \ ,\end{aligned}$$

come da dimostrarsi. □

Il Corollario 1.3 può essere visto nel modo seguente. Si consideri lo spazio $L^p(\ell^2)$ costituito dalle successioni $F = \{f_n\}$ di funzioni misurabili su X tali che $F(x) = \{f_n(x)\} \in \ell^2$ per quasi ogni $x \in X$ e inoltre

$$\|F\|_{L^p(\ell^2)} = \Big(\int_X \|F(x)\|_{\ell^2}^p \, dx \Big)^{1/p} < \infty \ .$$

Il Corollario 1.3 afferma che, sotto l'ipotesi (1.2), l'operatore

$$\mathbf{T} f = \{T_n f\}$$

è limitato da L^p a $L^p(\ell^2)$. Per dualità si ha allora il seguente corollario.

Corollario 1.4. *Sia T_n una successione di operatori che soddisfino la* (1.2)*, e sia $1 < p < \infty$. Allora, data $F = \{f_n\} \in L^p(\ell^2)$, risulta*

$$\Big\| \sum_{n=0}^{\infty} T_n f_n \Big\|_p \leq C_p A \|F\|_{L^p(\ell^2)} \ .$$

Proof. La (1.2) implica la stessa maggiorazione per gli operatori T_n^* e con p' al posto di p. Essendo $p > 1$, p' è finito. Dunque l'operatore $\mathbf{U}f = \{T_n^* f\}$ è limitato da $L^{p'}$ a $L^{p'}(\ell^2)$. Di conseguenza, $\mathbf{U}^*$ è limitato dal duale di $L^{p'}(\ell^2)$ a L^p.

Data $G \in L^p(\ell^2)$, si ponga

$$\langle F|G\rangle = \int_X \sum_{n=0}^{\infty} f_n(x)\overline{g_n(x)}\,dx \ .$$

Si verifica facilmente che le applicazioni lineari $F \mapsto \langle F, G\rangle$ sono tutti e soli i funzionali continui su $L^{p'}(\ell^2)$, ossia lo spazio duale di $L^{p'}(\ell^2)$ si identifica con $L^p(\ell^2)$. Inoltre

$$\begin{aligned}\langle \mathbf{U}^* F|g\rangle &= \langle F|\mathbf{U}g\rangle \\ &= \sum_{n=0}^{\infty} \int_X f_n(x)\overline{T_n^* g(x)}\,dx \\ &= \sum_{n=0}^{\infty} \int_X T_n f_n(x)\overline{g(x)}\,dx \\ &= \int_X \left(\sum_{n=0}^{\infty} T_n f_n\right)\overline{g(x)}\,dx \ .\end{aligned}$$

Quindi $\mathbf{U}^* F = \sum_{n=0}^{\infty} T_n f_n$, da cui la segue la tesi. □

2. Littlewood-Paley functions

Combining the results in the previous Section with the Calderón-Zygmund theory, we shall obtain the basic properties of *Littlewood-Paley functions*.

on $\mathbb{R}^n$ we consider a family of dilations $x \longmapsto r \cdot rx$, and call Q the resulting homogeneous dimension of $\mathbb{R}^n$. If f is defined on $\mathbb{R}^n$, we set $f^{(j)}(x) = 2^{-Qj} f(2^{-j} \cdot x)$.

We shall take at various stages functions $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$0 \le \hat{\psi} \in \mathcal{D}(\mathbb{R}^n) \ \text{ and } 0 \notin \operatorname{supp} \hat{\psi} \ ; \tag{2.1}$$

sometimes we shall also impose one of the two following conditions:

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j \cdot \xi) > 0 \ , \ \text{for } \xi \neq 0 \ , \tag{2.2}$$

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j \cdot \xi) = 1 \ , \ \text{for } \xi \neq 0 \ , \tag{2.2'}$$

Observe that condition (2.2) can be obtained by imposing that $\hat{\psi}(\xi) > 0$ for $1 \le |\xi| \le 2$, if $|\ |$ is a homogeneous norm for the given dilations.

There are different ways to obtain a ψ satisfying (2.1) and (2.2'). Starting with ψ_0 satisfying (2.1) and (2.2), and denoting by $s(\xi)$ the sum in (2.2), set $\psi = \mathcal{F}^{-1}(\hat{\psi}_0/s)$. Another way is to take $\varphi \in \mathcal{D}(\mathbb{R}^n)$ so that $\varphi(\xi) = 1$ on a neighborhood of 0, and set

$\hat{\psi}(\xi) = \varphi(\xi) - \varphi(2 \cdot \xi)$. We remark that, conversely, if ψ satisfies (2.1) and (2.2'), the function

$$\varphi(\xi) = \begin{cases} \sum_{j \geq 0} \hat{\psi}(2^j \cdot \xi) & \text{if } \xi \neq 0 , \\ 1 & \text{if } \xi = 0 , \end{cases} \tag{2.3}$$

is in $\mathcal{D}(\mathbb{R}^n)$, $\varphi(\xi) = 1$ on a neighborhood of 0, and $\hat{\psi}(\xi) = \varphi(\xi) - \varphi(2 \cdot \xi)$.

Proposition 2.1. *Suppose ψ satisfies (2.1) and (2.2′). If $f \in L^p(\mathbb{R}^n)$, and $1 < p < \infty$, the series $\sum_{j \in \mathbb{Z}} f * \psi^{(j)}$ converges to f in L^p.*

Proof. Let φ be the function in (2.3) and let $u = \mathcal{F}^{-1}\varphi$. The $u^{(j)}$ form an approximate identity for $j \to -\infty$, so that

$$\lim_{j \to -\infty} f * u^{(j)} = f$$

in L^p. We prove now that

$$\lim_{j \to \infty} \|f * u^{(j)}\|_p = 0 \ .$$

For f continuous with compact support, this follows from

$$\|f * u^{(j)}\|_p \leq \|f\|_1 \|u^{(j)}\|_p \leq C 2^{-Qj/p'} \ .$$

For a general $f \in L^p$, given $\delta > 0$, take g continuous with compact support such that $\|f - g\|_p < \delta$. If j is large enough, $\|g * u^{(j)}\|_p < \delta$, so that

$$\begin{aligned} \|f * u^{(j)}\|_p &\leq \|(f - g) * u^{(j)}\|_p + \|g * u^{(j)}\|_p \\ &\leq \|f - g\|_p \|u^{(j)}\|_1 + \|g * u^{(j)}\|_p \\ &< 2\delta \ . \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{j=-M}^{N} f * \psi^{(j)} &= \sum_{j=-M}^{N} f * (u^{(j)} - u^{(j+1)}) \\ &= f * u^{(-M)} - f * u^{(N+1)} \ , \end{aligned}$$

and the conclusion follows. $\square$

If $\{f_j\}_{j \in \mathbb{Z}}$ is a sequence of L^p-functions on $\mathbb{R}^n$, we set

$$\big\|\{f_j\}\big\|_{L^p(\ell^2)} = \left(\int_{\mathbb{R}^n} \Big(\sum_{j \in \mathbb{Z}} |f_j(x)|^2 \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} ,$$

i.e. the norm in $L^p\big(\mathbb{R}^n, \ell^2(\mathbb{Z})\big)$.

Teorema 2.2. *Assume that $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.1). Then, if $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$,*

$$\|\{f * \psi^{(j)}\}\|_{L^p(\ell^2)} \le C(\psi, p)\|f\|_p \ . \tag{2.4}$$

If, in addition, ψ also satisfies (2.2), then

$$\|f\|_p \sim \|\{f * \psi^{(j)}\}\|_{L^p(\ell^2)} \ , \tag{2.5}$$

i.e. the two norms are equivalent for $f \in L^p(\mathbb{R}^n)$. In other words, defining the Littlewood-Paley function

$$Sf(x) = \Big(\sum_{j\in\mathbb{Z}} \big|f * \psi^{(j)}(x)\big|^2\Big)^{\frac{1}{2}} \ ,$$

we have $\|Sf\|_p \sim \|f\|_p$.

Proof. We apply Corollary 1.3 to the operators $T_j f = f * \psi^{(j)}$. By Theorem 5.3 of Chapter II, for every choice of the signs ε_j, the series $\sum_{j\in\mathbb{Z}} \varepsilon_j \psi^{(j)}$ converges to a Calderón-Zygmund kernel, provided ψ satisfies (2.1). Since the constants appearing in the estimates do not depend on the choice of signs,, the operator norms of the $\sum_{j\in\mathbb{Z}} \varepsilon_j T_j$ are uniformly bounded. This gives (2.4).

Assume now that ψ also satisfies (2.2) and consider the function

$$b(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\psi^{(j)}}(\xi)^2 = \sum_{j\in\mathbb{Z}} \hat{\psi}(2^j \cdot \xi)^2 \ , \tag{2.6}$$

By (2.1), the support of $\hat{\psi}$ is contained in a rim $0 < a \le |\xi| \le b$, so that for every $\xi \ne 0$ at most $N \sim \log_2(b/a)$ terms in the series (2.7) are different from 0. This implies that $b(\xi)$ is smooth for $\xi \ne 0$. Since $b(2 \cdot \xi) = b(\xi)$,

$$\inf_{\xi\ne 0} b(\xi) = \min_{1\le|\xi|\le 2} b(\xi) > 0 \ .$$

Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ be the function such that

$$\hat{\eta}(\xi) = \frac{\hat{\psi}(\xi)}{b(\xi)} \ .$$

Then η too satisfies (2.1), so that, arguing as before, the operators $T'_j f = f * \eta^{(j)}$ satisfy (1.2). By Corollary 1.4,

$$\Big\|\sum_{j\in\mathbb{Z}} f * \psi^{(j)} * \eta^{(j)}\Big\|_p \le C\|\{f * \psi^{(j)}\}\|_{L^p(\ell^2)} \ .$$

Consider finally $\psi * \eta$. Since

$$\sum_{j\in\mathbb{Z}} \widehat{\psi * \eta}(2^j \cdot \xi) = \sum_{j\in\mathbb{Z}} \frac{\hat{\psi}(2^j \cdot \xi)^2}{b(2^j \cdot \xi)} = 1 \ ,$$

it satisfies (2.2') together with (2.1), so that

$$\sum_{j\in\mathbb{Z}} f*(\psi*\eta)^{(j)} = \sum_{j\in\mathbb{Z}} f*\psi^{(j)}*\eta^{(j)} = f\ .$$

This proves (2.5). □

We shall prove below a multi-parameter version of Theorem 2.2. But before that, we present the general aspects of the "multi-parameter theory".

The Calderón-Zygmund theory is often referred to as the *one-parameter* singular integral theory, because the assumptions made on the kernels are adapted to a given family of dilations depending on one parameter $r>0$. From the point of view of Fourier multipliers, the same can be said for the Mihlin-Hörmander condition.

In the *multi-parameter* theory (also called *product theory*) one has a finite family of spaces $\mathbb{R}^{n_i}$, each with its own dilations $x_i \longmapsto r\cdot x_i$, and on the product $\mathbb{R}^N = \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}$ one defines

$$\mathbf{r}(x_1,\dots,x_k) = (r_1\cdot x_1,\dots,r_k\cdot x_k)\ , \tag{2.7}$$

for $\mathbf{r}=(r_1,\dots,r_k)\in(\mathbb{R}^+)^k$.

The simplest example of an operator arising in the multi-parameter theory is the convolution operator $Tf = f*K$, where K is the tensor product

$$K(x) = K_1(x_1)\cdots K_k(x_k) \tag{2.8}$$

of Calderón-Zygmund kernels on the various $\mathbb{R}^{n_i}$. Since each K_i is singular (i.e. non-locally integrable) only at the origin, the product kernel K is singular on the union of the "coordinate subspaces" $x_i=0$.

However, we shall not discuss product kernels, but we shall instead restrict ourselves to the Fourier multipliers connected with the product theory, the *Marcinkiewicz multipliers*. The simplest example is the product

$$m(\xi) = m_1(\xi_1)\cdots m_k(\xi_k) \tag{2.9}$$

of Mihlin-Hörmander multipliers on the $\mathbb{R}^{n_i}$.

The general product theory is however less trivial than what these examples maight suggest[12]. The proofs are based on the one-parameter theory and on Littlewood-Paley decompositions.

On each $\mathbb{R}^{n_i}$ fix a ψ_i satisfying (2.1), and, for $J=(j_1,\dots,j_k)\in\mathbb{Z}^k$, let

$$\psi^{(J)}(x) = \psi_1^{(j_1)}(x_1)\cdots\psi_k^{(j_k)}(x_k)\ .$$

We then construct

$$Sf(x) = \Big(\sum_{J\in\mathbb{Z}^k} \big|f*\psi^{(J)}(x)\big|^2\Big)^{\frac12}\ ,$$

[12] In fact, there is no difficulty in proving that operators $f\longmapsto f*K$, with K as in (2.8), or $f\longmapsto\mathcal{F}^{-1}(m\hat f)$, with m as in (2.9), are bounded on $L^p(\mathbb{R}^N)$ for $1<p<\infty$. It is a general fact that if T_i is a bounded operator on $L^p(X_i,\mu_i)$, then $T=T_1\otimes\cdots\otimes T_k$ is bounded on $L^p(X_1\times,\times X_k,\mu_1\times\cdots\times\mu_k)$.

Theorem 2.3. *For $1 < p < \infty$, $\|Sf\|_p \le C(\psi,p)\|f\|_p$. If, in addition, the ψ_i satisfy* (2.2)*, f and Sf have equivalent L^p-norms.*

Proof. The proof is based on an iteration argument. We only discuss the case when (2.2) is satisfied.

For each i and j, let $\tilde\psi_i^{(j)}$ be the measure on $\mathbb{R}^N$ obtained by tensoring the function $\psi_i^{(j)}(x_i)$ on $\mathbb{R}^{n_i}$ with the Dirac measure δ_0 on the other $\mathbb{R}^{n_{i'}}$.

Consider the operator T_1 mapping f into the sequence $\{f * \tilde\psi_1^{(j_1)}\}_{j_1\in\mathbb{Z}}$. For a.e. $(x_2,\dots,x_k)$, denoting by $*_{\mathbb{R}^m}$ convolution in $\mathbb{R}^m$, we have

$$\big(f *_{\mathbb{R}^N} \tilde\psi_1^{(j_1)}\big)(x) = \big(f(\cdot,x_2,\dots,x_k) *_{\mathbb{R}^{n_1}} \psi_1^{(j_1)}\big)(x_1)\ ,$$

so that, by Theorem 2.2,

$$\int_{\mathbb{R}^{n_1}} \Big(\sum_{j_1\in\mathbb{Z}} \big|f * \tilde\psi_1^{(j_1)}(x)\big|^2\Big)^{\frac{p}{2}} dx_1 \sim \int_{\mathbb{R}^{n_1}} \big|f(x)\big|^p\, dx_1\ .$$

Integrating in the other variables, we obtain that T_1 is bounded from $L^p(\mathbb{R}^N)$ to $L^p\big(\mathbb{R}^N,\ell^2(\mathbb{Z})\big)$, with equivalence between the norm of f and $T_1 f$.

Consider next T_2, mapping a sequence $\{g_{j_1}\}_{j_1\in\mathbb{Z}}$ into the double sequence $\{g_{j_1} * \tilde\psi_2^{(j_2)}\}_{(j_1,j_2)\in\mathbb{Z}^2}$. The same argument as before shows that T_2 is bounded from $L^p\big(\mathbb{R}^N,\ell^2(\mathbb{Z})\big)$ to $L^p\big(\mathbb{R}^N,\ell^2(\mathbb{Z}^2)\big)$, with equivalence between the norms of $\{g_{j_1}\}$ and $T_2\big(\{g_{j_1}\}\big)$.

Iterating this argument k times and considering the composition $T_kT_{k-1}\cdots T_1$, we obtain that

$$\int_{\mathbb{R}^N} \Big(\sum_{j\in\mathbb{Z}^k} \big|f * \tilde\psi_1^{(j_1)} * \cdots * \tilde\psi_k^{(j_k)}(x)\big|^2\Big)^{\frac{p}{2}} dx \sim \int_{\mathbb{R}^N} \big|f(x)\big|^p\, dx\ .$$

But this is the required estimate, because $\tilde\psi_1^{(j_1)} * \cdots * \tilde\psi_k^{(j_k)} = \psi^{(J)}$. □

3. Marcinkiewicz multipliers

Given $\mathbf{s} = (s_1,\dots,s_k) \in \mathbb{R}^k$, we define the *product Sobolev space* $H^{\mathbf{s}}(\mathbb{R}^N)$ as the space of tempered distributions f such that $\hat f$ is locally integrable and

$$\|f\|^2_{H^{\mathbf{s}}} = \int_{\mathbb{R}^N} \big(1+\|\tau_1\|^2\big)^{s_1}\cdots\big(1+\|\tau_k\|^2\big)^{s_k}|\hat f(\tau)|^2\, d\tau < +\infty\ ,$$

where, as before, $\tau_i \in \mathbb{R}^{n_i}$ and $\sum_{i=1}^k n_i = N$. Lemmas 6.1 and 6.2 of Chapter II have the following analogues.

Lemma 3.1. *If $s_i \in \mathbb{N}$ for every i, $H^{\mathbf{s}}(\mathbb{R}^N)$ consists of the L^2-functions f such that $\partial_{x_1}^{\alpha_1}\cdots\partial_{x_k}^{\alpha_k} f \in L^2(\mathbb{R}^N)$ for every choice of the multi-indices α_i with $|\alpha_i| \le s_i$ for every i.*

Lemma 3.2. *Let φ be such that $\hat\varphi \in H^{\mathbf{s}}(R^N)$ with $s_i > \frac{n_i}{2}$ for every i. Then $\varphi \in L^1(R^N)$ and, if $0 \le \varepsilon < s_i - \frac{n_i}{2}$ for every i,*

$$\int_{\mathbb{R}^N} |\varphi(x)|\big(1+\|x_1\|\big)^{\varepsilon} \cdots \big(1+\|x_k\|\big)^{\varepsilon}\, dx \le C_\varepsilon \|\hat\varphi\|_{H^{\mathbf{s}}} \ .$$

In particular, under these hypotheses on $\mathbf{s}$, $H^s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$.

The proofs are direct adaptations of those given in Chapter II.

On each $\mathbb{R}^{n_i}$ we fix a homogeneous norm (denoted by $|\ |$ for each i), and a function $\eta_i \in \mathcal{D}(R^{n_i})$ such that

(i) $\operatorname{supp} \eta_i \subseteq \{\xi_i : a_0 \le |\xi_i| \le b_0\}$,

(ii) $\eta_i(\xi_i) \ge 0$ for every ξ_i e $\eta_i(\xi_i) > 0$ for $a_1 \le |\xi_i| \le b_1$,

with $0 < a_0 < a_1 < b_1 < b_0$ are given constants.

We set $\eta(\xi) = \eta_1(\xi_1)\cdots\eta_k(\xi_k)$. For $\mathbf{r} \in (\mathbb{R}^+)^k$, we also set $m_{\mathbf{r}}(\xi) = m(\mathbf{r}\cdot\xi)$, with the notation of (2.7).

Definition. *A* Marcinkiewicz multiplier *on* $\mathbb{R}^N$, *adapted to the k-parameter dilations* (2.7), *is a function m such that*

$$\sup_{\mathbf{r}\in(\mathbb{R}^+)^k} \|m_{\mathbf{r}}\eta\|_{H^{\mathbf{s}}} = \|m\|_{M_{\mathbf{s}}} < \infty \ , \tag{3.1}$$

for some $\mathbf{s}$ *with* $s_i > \frac{n_i}{2}$ *for every* i.

We shall denote by $M_{\mathbf{s}}(\mathbb{R}^N)$ the class of such multipliers. A simpler pointwise condition implying that m is a Marcinkiewicz multiplier is that for some $\mathbf{s} \in \mathbb{N}^k$ with $s_i > \frac{n_i}{2}$ for every i, and for every $\xi \in \mathbb{R}^N$ with $\xi_i \ne 0$ for every i,

$$\big|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_k}^{\alpha_k} m(\xi)\big| \le C_\alpha |\xi_1|^{-d_1(\alpha_1)} |\xi_2|^{-d_2(\alpha_2)} \cdots |\xi_k|^{-d_k(\alpha_k)} \ , \tag{3.2}$$

where the d_i are the degrees of the multi-indices α_i w.r. to the dilations in $\mathbb{R}^{n_i}$.

Condition (3.1) does not depend on the choice of the η_i, and we shall choose the η_i in such a way that for every i

$$\sum_{j\in\mathbb{Z}} \eta_i(2^j \cdot \xi_i) > 0 \tag{3.3}$$

for $\xi_i \ne 0$.

Theorem 3.3. *Let m be a Marcinkiewicz multiplier on $\mathbb{R}^N$. Then the operator $S_m = m(i^{-1}\partial)$ is bounded on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$.*

Proof. Let $\psi_i = \mathcal{F}^{-1}\eta_i$, $\psi(x) = \prod_{i=1}^k \psi_i(x_i)$. We write $2^J \cdot \xi$ for $(2^{j_1}\cdot\xi_1, \dots, 2^{j_k}\cdot\xi_k)$, with $J \in \mathbb{Z}^k$. Define

$$m_J(\xi) = m(2^{-J}\cdot\xi)\eta(\xi) \ ,$$

and $K_J = \mathcal{F}^{-1} m_J$. The Marcinkiewicz condition implies that

$$\int_{\mathbb{R}^N} |K_J(x)|^2 \prod_{i=1}^k \big(1 + \|x_i\|^2\big)^{s_i}\, dx \le C \tag{3.4}$$

uniformly in J.

Let

$$T_J f = (S_m f) * \psi^{(J)} = S_m(f * \psi^{(J)}) = \mathcal{F}^{-1}\big(\hat{f} m \widehat{\psi^{(J)}}\big) = f * K_J^{(J)} ,$$

where, consistently with our previous notation and calling Q_i the homogeneous dimension of $\mathbb{R}^{n_i}$ w.r. to the given dilations,

$$K_J^{(J)}(x) = 2^{-Q_1 j_1} \cdots 2^{-Q_k j_k} K_J(2^{-j_1} \cdot x_1, \ldots, 2^{-j_k} x_k) .$$

Since each $\psi_i * \psi_i$ satisfies (2.1) and (2.2), we have, assuming $2 \le p < \infty$,

$$\begin{aligned} \|S_m f\|_p^2 &\le C \Big\| \Big(\sum_{J \in \mathbb{Z}^k} |(S_m f) * \psi^{(J)} * \psi^{(J)}|^2) \Big)^{\frac12} \Big\|_p^2 \\ &= C \Big\| \Big(\sum_{J \in \mathbb{Z}^k} \big| T_J(f * \psi^{(J)}) \big|^2 \Big)^{\frac12} \Big\|_p^2 \\ &= C \Big\| \sum_{J \in \mathbb{Z}^k} \big| T_J(f * \psi^{(J)}) \big|^2 \Big\|_{p/2} . \end{aligned}$$

Call $f_J = f * \psi^{(J)}$ and $w(x) = \prod_{i=1}^k \big(1 + \|x_i\|^2\big)^{-s_i}$. By Hölder's inequality and (3.4),

$$\begin{aligned} \big| T_J f_J(x) \big|^2 &= \Big| \int_{\mathbb{R}^N} K_J^{(J)}(x-y) f_J(y)\, dy \Big|^2 \\ &\le \Big(\int_{\mathbb{R}^N} \frac{\big|K_J^{(J)}(x-y)\big|^2}{w^{(J)}(x-y)}\, dy \Big) \Big(\int w^{(J)}(x-y) \big| f_J(y) \big|^2\, dy \Big) \\ &\le C\big(|f_J|^2 * w^{(J)} \big)(x) . \end{aligned}$$

Take $g \in L^{(p/2)'}(\mathbb{R}^N)$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} \sum_{J \in \mathbb{Z}^k} \big| T_J f_J(x) \big|^2 g(x)\, dx &\le C \sum_{J \in \mathbb{Z}^k} \int_{\mathbb{R}^N} \big(|f_J|^2 * w^{(J)} \big)(x) |g(x)|\, dx \\ &= C \sum_{J \in \mathbb{Z}^k} \int_{\mathbb{R}^N} |f_J(x)|^2 \big(w^{(J)} * |g| \big)(x)\, dx \\ &\le C \int_{\mathbb{R}^N} \Big(\sum_{J \in \mathbb{Z}^k} |f_J(x)|^2 \Big) \Big(\sup_{J \in \mathbb{Z}^k} \big(w^{(J)} * |g| \big)(x) \Big)\, dx \\ &\le C \Big\| \sum_{J \in \mathbb{Z}^k} |f_J|^2 \Big\|_{p/2} \Big\| \sup_{J \in \mathbb{Z}^k} w^{(J)} * |g| \Big\|_{(p/2)'} . \end{aligned}$$

If we prove that the maximal operator

$$M_w f(x) = \sup_{J \in \mathbb{Z}^k} \big(w^{(J)} * |f| \big)(x) \tag{3.5}$$

is bounded on $L^{(p/2)'}(\mathbb{R}^N)$, it follows that

$$\Big\| \sum_{J\in\mathbb{Z}^k} \big|T_J f_J(x)\big|^2 \Big\|_{p/2} \le C \Big\| \sum_{J\in\mathbb{Z}^k} \big|f_J(x)\big|^2 \Big\|_{p/2}$$

and we can conclude that

$$\begin{aligned}\|S_m f\|_p^2 &\le C \Big\| \sum_{J\in\mathbb{Z}^k} \big|T_J f_J(x)\big|^2 \Big\|_{p/2} \\ &\le C' \Big\| \sum_{J\in\mathbb{Z}^k} \big|f_J(x)\big|^2 \Big\|_{p/2} \\ &= C' \Big\| \Big(\sum_{J\in\mathbb{Z}^k} \big|f_J(x)\big|^2 \Big)^{\frac{1}{2}} \Big\|_p^2 \\ &\le C'' \|f\|_p^2 .\end{aligned}$$

We have used the fact that $f_j = f * \psi^{(J)}$ and that the ψ_i satisfies (2.1).

The proof of L^p-boundedness of (3.5) is given as a separate lemma. □

Lemma 3.4. *The maximal operator M_w is bounded on $L^p(\mathbb{R}^N)$ for $1 < p \le \infty$.*

Proof. We need at this stage to introduce *scalar* coordinates $x = (t_1, \dots, t_N)$ in $\mathbb{R}^N$, and we do this in such a way that the first n_1 coordinates determine the component of x in $\mathbb{R}^{n_1}$, etc. Following the notation used in the rest of this chapter, this means that

$$(t_1, \dots, t_{n_1}) = x_1 , \qquad \cdots\cdots \qquad (t_{N-n_k+1}, \dots, t_N) = x_k .$$

Inside each $\mathbb{R}^{n_i}$ we choose the coordinates so that the dilations are diagonal. This implies that for each index $\ell \in \{1, \dots, N\}$ there are $i = i(\ell) \in \{1, \dots, k\}$ and $\lambda_\ell > 0$ such that

$$2^J \cdot x = (2^{j_{i(\ell)}\lambda_\ell} x_\ell)_{\ell=1,\dots,N} .$$

Write $s_i = \frac{n_i}{2}(1+\varepsilon_i)$ with $\varepsilon_i > 0$. Then, taking $i = 1$ to simplify the notation,

$$\big(1+\|x_1\|^2\big)^{s_1} \ge \prod_{\ell=1}^{n_1} \big(1+|t_\ell|^2\big)^{\frac{1}{2}(1+\varepsilon_1)} \ge C \prod_{\ell=1}^{n_1} \big(1+|t_\ell|\big)^{1+\varepsilon_1} .$$

Therefore, if $\varepsilon = \min_i \varepsilon_i$,

$$w(x) = \prod_{i=1}^{k} \big(1+\|x_i\|^2\big)^{-s_i} \le C \prod_{\ell=1}^{N} \big(1+|t_\ell|\big)^{-1-\varepsilon} = C\tilde w(x) .$$

Therefore $M_w f(x) \le C M_{\tilde w} f(x)$ for every f and every x, where

$$\begin{aligned} M_{\tilde w} f(x) &= \sup_{J\in\mathbb{Z}^k} \big(\tilde w^{(J)} * |f|\big)(x) \\ &= \sup_{J\in\mathbb{Z}^k} \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} 2^{-j_{i(\ell)}\lambda\ell} \big(1+|2^{-j_{i(\ell)}\lambda\ell} t'_\ell|\big)^{-1-\varepsilon} \big|f(t_1-t'_1, \dots, t_N-t'_N)\big|\, dt' \\ &\le \sup_{J\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} 2^{-j_\ell\lambda_\ell} \big(1+|2^{-j_\ell\lambda_\ell} t'_\ell|\big)^{-1-\varepsilon} \big|f(t_1-t'_1, \dots, t_N-t'_N)\big|\, dt' .\end{aligned}$$

Consider first the one-dimensional integral

$$I_j(t)=\int_{\mathbb{R}} 2^{-j\lambda}\big(1+|2^{-j\lambda}t'|\big)^{-1-\varepsilon}\big|f(t-t')\big|\,dt'\ .$$

We decompose the real line as the union of the interval $[-2^{j\lambda},2^{j\lambda}]$, where $1+|2^{-j\lambda}t'|$ can be bounded by 1 from below, and the outer dyadic regions $\{t:2^{j'\lambda}<|t'|<2^{(j'+1)\lambda}\}$, where $1+|2^{-j\lambda}t'|$ can be bounded from below by $2^{(j'-j)\lambda}$. Each partial integral can be estimated by the Hardy-Littlewood maximal function Mf, using the trivial inequality

$$\int_{|t'|<r}|f(t-t')|\,dt'\le 2rMf(t)\ .$$

Therefore

$$\begin{aligned}
I_j(t)&\le 2^{-j\lambda}\int_{|t'|<2^{j\lambda}}\big|f(t-t')\big|\,dt'\\
&\qquad+\sum_{j'\ge j}2^{-j\lambda}\int_{2^{j'\lambda}<|t'|<2^{(j'+1)\lambda}}2^{(1+\varepsilon)(j-j')\lambda}\big|f(t-t')\big|\,dt'\\
&\le 2\cdot 2^{-j\lambda}2^{j\lambda}Mf(t)+2\sum_{j'\ge j}2^{-j\lambda}2^{(j'+1)\lambda}2^{(1+\varepsilon)(j-j')\lambda}Mf(t)\\
&\le C_{\lambda,\varepsilon}Mf(t)\ .
\end{aligned}$$

For each ℓ, define $M_\ell f$ as the one-dimensional Hardy-Littlewood maximal function of f regarded as a function of t_ℓ only:

$$M_\ell f(t_1,\dots,t_N)=\sup_{a<t_\ell<b}\frac{1}{b-a}\int_a^b|f(t_1,\dots,t_{\ell-1},t,t_{\ell+1},\dots,t_N)|\,dt\ .$$

The estimate obtained for $I_j(t)$ implies that

$$\begin{aligned}
&\int_{\mathbb{R}}\prod_{\ell=1}^N 2^{-j_\ell\lambda_\ell}\big(1+|2^{-j_\ell\lambda_\ell}t'_\ell|\big)^{-1-\varepsilon}\big|f(t_1-t'_1,\dots,t_N-t'_N)\big|\,dt'_1\le\\
&\qquad\le C\prod_{\ell=2}^N 2^{-j_\ell\lambda_\ell}\big(1+|2^{-j_\ell\lambda_\ell}t'_\ell|\big)^{-1-\varepsilon}M_1f(t_1,t_2-t'_2\dots,\dots,t_N-t'_N)\ .
\end{aligned}$$

Integrating one variable at the time, we find that

$$\begin{aligned}
\int_{\mathbb{R}^N}\prod_{\ell=1}^N 2^{-j_\ell\lambda_\ell}\big(1+|2^{-j_\ell\lambda_\ell}t'_\ell|\big)^{-1-\varepsilon}\big|f(t_1-t'_1,\dots,t_N-t'_N)\big|\,dt'\le\\
\le CM_NM_{N-1}\cdots M_1f(x)\ .
\end{aligned}$$

Taking the supremum over $J\in\mathbb{Z}^k$, we obtain that

$$M_wf(x)\le CM_{\tilde w}f(x)\le C'M_NM_{N-1}\cdots M_1f(x)\ .$$

By Corollary 2.5 in Chapter II, if $1 < p \le \infty$,

$$\int_{-\infty}^{+\infty} M_\ell f(x)^p \, dx_\ell \le C \int_{-\infty}^{+\infty} |f(x)|^p \, dx_\ell \ ,$$

for a.e. choice of the $x_{\ell'}$ with $\ell' \neq \ell$. Integrating in the remaining variables, we obtain that

$$\|M_\ell f\|_p \le C\|f\|_p \ .$$

Therefore

$$\|M_w f\|_p \le C\|M_N M_{N-1} \cdots M_1 f\|_p \le C'\|f\|_p \ . \qquad \square$$

4. Applications

We discuss two different applications of the product theory to differential operators. One is to estimates of the kind given in Theorem 7.6 of Chapter II, involving fractional powers of differential operators, the other is to multipliers of a system of constant coefficient differential operators.

If $L = P(i^{-1}\partial)$ is a constant coefficient differential operator on $\mathbb{R}^n$, define, for $\gamma \in \mathbb{C}$,

$$|L|^\gamma f = \mathcal{F}^{-1}(|P|^\gamma \hat{f}) \ .$$

In particular, the *fractional derivative of order* γ *in* x_j (x_j being a scalar component of x) is

$$D_j^\gamma f = \mathcal{F}^{-1}(|\xi_j|^\gamma \hat{f}) \ .$$

The statement of Theorem 7.6 in Chapter II can then be completed as follows.

Theorem 4.1. *In the notation of Section 7 in Chapter II, assume that P satisfies* (1) *and* (2′)*, and let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be such that $\sum_j \lambda_j \Re e \alpha_j \le k$. Then for every $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 < p < \infty$,*

$$\|D_1^{\alpha_1} \cdots D_n^{\alpha_n} f\|_p \le C_p\big(\|f\|_p + \|Lf\|_p\big) \ .$$

Proof. We give the proof for $n = 2$, the general case being essentially the same, with the extra disadvantage of more complicated notations.

Let $\varphi \in \mathcal{D}(\mathbb{R})$ be equal to 1 on a neighborhood of 0, and write the Fourier transform of $D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} f$ as

$$\begin{aligned}
\widehat{D^\alpha f}(\xi) &= \big(|\xi_1|^{\alpha_1}\varphi(\xi_1)\big)\big(|\xi_2|^{\alpha_2}\varphi(\xi_2)\big)\hat{f}(\xi) \\
&\qquad + \frac{|\xi_1|^{\alpha_1}\big(1-\varphi(\xi_1)\big)}{P(\xi_1,0)}\big(|\xi_2|^{\alpha_2}\varphi(\xi_2)\big)P(\xi_1,0)\hat{f}(\xi) \\
&\qquad + \big(|\xi_1|^{\alpha_1}\varphi(\xi_1)\big)\frac{|\xi_2|^{\alpha_2}\big(1-\varphi(\xi_2)\big)}{P(0,\xi_2)}P(0,\xi_2)\hat{f}(\xi) \\
&\qquad + \frac{|\xi_1|^{\alpha_1}\big(1-\varphi(\xi_1)\big)|\xi_2|^{\alpha_2}\big(1-\varphi(\xi_2)\big)}{P(\xi)}P(\xi)\hat{f}(\xi) \\
&= m_1(\xi)\hat{f}(\xi) + m_2(\xi)P(\xi_1,0)\hat{f}(\xi) + m_3(\xi)P(0,\xi_2)\hat{f}(\xi) + m_4(\xi)P(\xi)\hat{f}(\xi) \ .
\end{aligned}$$

We begin with m_2 and verify condition (3.2), taking $s_1 = s_2 = 1$. Because m_2 is the product of two factors in splitted variables, we need to observe that

(1) m_2 is bounded (the hypotheses on P imply that $|P(\xi_1, 0)| \geq C|\xi_1|^{k/\lambda_1}$);
(2) the derivative of $|\xi_2|^{\alpha_2}\varphi(\xi_2)$ is bounded by a constant times $|\xi_2|^{-1}$ for $\xi_2 \neq 0$;
(3) the derivative of $\frac{|\xi_1|^{\alpha_1}\big(1-\varphi(\xi_1)\big)}{P(\xi_1,0)}$ is smooth and, away from the support of φ, it is homogeneous (in the ordinary sense) of degree one less than the degree of this fraction itself (this is $\alpha_1 - \frac{k}{\lambda_1}$, which has a non-positive real part); hence the derivative is bounded by a constant times $|\xi_1|^{-1}$ for $\xi_1 \neq 0$.

The same remarks imply that also m_1 and m_3 satisfy (3.2) with $s_1 = s_2 = 1$.

The verification that m_4 also verifies (3.2) with $s_1 = s_2 = 1$ cannot be done by separation of variables.

Observe that m_4 equals $\big(1-\varphi(\xi_1)\big)\big(1-\varphi(\xi_2)\big)$ times a function which is continuous and homogeneous, w.r. to the non-isotropic dilations, of degree $\lambda_1\alpha_1+\lambda_2\alpha_2-k$, whose real part is non-positive. Since ξ_1 and ξ_2 are bounded away from 0 on the support of m_4, we conclude that m_4 is bounded.

We skip now to estimating that $\partial_{\xi_1}\partial_{\xi_2}m_4$ is bounded by $|\xi_1|^{-1}|\xi_2|^{-1}$ times a constant. We apply Leibniz's rule.

If both derivatives fall on $\big(1 - \varphi(\xi_1)\big)\big(1 - \varphi(\xi_2)\big)$, this term is supported on a compact set and the estimate is trivial. Consider then the case where the ξ_1-derivative falls on $1 - \varphi(\xi_1)$ and the ξ_2-derivative on the homogeneous function. The derivation in ξ_2 reduces the non-isotropic degree of homogeneity by λ_2. Hence we obtain $-\varphi'(\xi_1)$ times a function whose non-isotropic degree of homogeneity is $\lambda_1\alpha_1 + \lambda_2(\alpha_2 - 1) - k$, whose real part is $\leq -\lambda_2$.

Considering that this term is supported on a set where $a < |\xi_1| < b$ and $|\xi_2| > c$ for appropriate $a, b, c > 0$, this term is bounded by a constant times

$$|\xi|^{-\lambda_2} \sim \big(|\xi_1|^{1/\lambda_1} + |\xi_2|^{1/\lambda_2}\big)^{-\lambda_2} \sim |\xi_2|^{-1} \sim |\xi_1|^{-1}|\xi_2|^{-1} .$$

The same applies if the rôle of ξ_1 and ξ_2 is interchanged. Suppose finally that both derivatives fall on the homogeneous factor. We then obtain a function homogeneous of degree $\lambda_1(\alpha_1 - 1) + \lambda_2(\alpha_2 - 1) - k$, whose real part is $\leq -\lambda_1 - \lambda_2$. Considering that the other factor restrict the support to $|\xi_1|, |\xi_2| > c > 0$, we get the bound

$$|\xi|^{-\lambda_1-\lambda_2} \sim \big(|\xi_1|^{1/\lambda_1} + |\xi_2|^{1/\lambda_2}\big)^{-\lambda_1-\lambda_2} \leq C|\xi_1|^{-1}|\xi_2|^{-1} .$$

The remaining verifications follow the same lines. Therefore all the m_i are Marcinkiewicz multipliers. If we call $L_1 = P(i^{-1}\partial_{x_1}, 0)$, and $L_2P(0, i^{-1}\partial_{x_2})$, we have proved that

$$\|D^\alpha f\|_p \leq C_p\big(\|f\|_p + \|L_1 f\|_p + \|L_2 f\|_p + \|Lf\|_p\big) ,$$

for $1 < p < \infty$. But we can now apply Theorem 7.6 in Chapter II directly to show that $\|L_i f\|_p \leq C\big(\|f\|_p + \|Lf\|_p\big)$ for $i = 1, 2$. □

We pass now to the second application. On $\mathbb{R}^n$ are given k differential operators $L_i = P_i(i^{-1}\partial)$, where

(1) for each i there is a subspace V_i of $\mathbb{R}^n$ of dimension $n_i \leq n$ such that $P_i(\xi) = P_i(\xi_i)$, where ξ_i is the orthogonal projection of ξ on V_i;

(2) $P_i(\xi_i) > 0$ for $\xi_i \in V_i \setminus \{0\}$;
(3) each P_i is homogeneous of degree k_i w.r. to some non-isotropic 1-parameter dilations on V_i.

Theorem 4.2. *Let $m(\lambda_1, \dots, \lambda_k)$ be a Marcinkiewicz multiplier on $\mathbb{R}^k = \mathbb{R} \times \cdots \times \mathbb{R}$, with the standard dilations on each coordinate line, of order $\mathbf{s} = (s_1, \dots, s_k)$ with $s_i > \frac{n_i}{2}$ for every i. Then $m(L_1, \dots, L_k)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

We point out that this is not the optimal result, but is what can be obtained by our method of proof. The process of lifting to a higher–dimensional space, as described below, forces us to impose the Marcinkiewicz conditions adapted to the higher dimension.

Proof. On each V_i we fix coordinates that reduce the dilations to diagonal form. For each i, this choice of coordinates determines a linear bijection $\tau_i : \mathbb{R}^{n_i} \to V_i$. Let $\tilde{P}_i = P_i \circ \tau_i$.

On $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} = \mathbb{R}^N$ we consider the operators

$$\tilde{L}_i = \tilde{P}_i(i^{-1}\partial_{x_i}) \ ,$$

where ∂_{x_i} is the gradient in $R^{n_i} \sim V_i$. For convenience, we use the symbol $\eta_i \in \mathbb{R}^{n_i}$ to denote the variable for Fourier transforms on $\mathbb{R}^{n_i}$.

We claim that the Fourier multiplier on $\mathbb{R}^N$

$$\tilde{m}(\eta_1, \dots, \eta_k) = m\big(\tilde{P}_1(\eta_1), \dots, \tilde{P}_k(\eta_k)\big)$$

is a Marcinkiewicz multiplier of order $\mathbf{s}$ w.r. to the k-parameter dilations on $\mathbb{R}^N$ induced by the 1-parameter dilations on each $\mathbb{R}^{n_i}$ (we skip the details of the proof, which is a refinement of the proofs of Theorem 7.1 and Proposition 7.4 in Chapter II). Therefore $m(\tilde{L}_1, \dots, \tilde{L}_k)$ is bounded on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$.

Call $\sigma_i = \tau_i^{-1} \circ \pi_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, where π_i is the orthogonal projection onto V_i in $\mathbb{R}^n$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^N$ the map

$$\sigma(\xi) = \big(\sigma_i(\xi), \dots, \sigma_k(\xi)\big) \ .$$

Then the Fourier multiplier of $m(L_1, \dots, L_k)$ is

$$m\big(P_1(\xi), \dots, P_k(\xi)\big) = \tilde{m} \circ \sigma(\xi) \ .$$

We give the conclusion after the next lemma. □

What follows is part of a general principle, called *transference*, whose general idea is that an L^p-estimate for a convolution operator (or, equivalently, for a Fourier multiplier) induces other L^p-estimates for a certain class of induced (or transferred) operators. The statement we give concerns transference from Euclidean space to another, by means of a linear map.

Lemma 4.3. *Let $\sigma : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map, and let m be a Fourier L^p-multiplier on $\mathbb{R}^d$, continuous on the image of A. Then $m \circ \sigma$ is a Fourier L^p-multiplier on $\mathbb{R}^n$, and, if S_m and $S_{m\circ\sigma}$ are the corresponding operators,*

$$\|S_{m\circ\sigma}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le \|S_m\|_{\mathcal{L}(L^p(\mathbb{R}^d))} \ .$$

Proof. Changing coordinates on $\mathbb{R}^d$ and $\mathbb{R}^n$ if necessary, we can assume that

(1) $\mathbb{R}^d = \mathbb{R}^\nu \times \mathbb{R}^{d'}$, and $\operatorname{im}\sigma = \mathbb{R}^\nu \times \{0\}$;
(2) $\mathbb{R}^n = \mathbb{R}^\nu \times \mathbb{R}^{n'}$, and $\ker\sigma = \{0\} \times \mathbb{R}^{n'}$.

Introducing coordinates (η,η') on $\mathbb{R}^d$ and (ξ,ξ') on $\mathbb{R}^n$ compatible with these splittings, we then have $\sigma(\xi,\xi')=(A\xi,0)$ for some invertible $\nu\times\nu$ matrix A.

The proof follows from the combination of three facts.

First fact: if $m(\eta,\eta')$ is a Fourier L^p-multiplier on $\mathbb{R}^d$, continuous on $\mathbb{R}^\nu\times\{0\}$, then $m_0(\eta,\eta')=m(\eta,0)$ is also a Fourier L^p-multiplier on $\mathbb{R}^d$, with no increase in the norm. In order to see this, for $\varepsilon>0$ consider $m_\varepsilon(\eta,\eta')=m(\eta,\varepsilon\eta')$. Then, for $f,g\in\mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}\langle S_{m_\varepsilon}f|g\rangle&=(2\pi)^{\frac d2}\int_{\mathbb{R}^d}m(\eta,\varepsilon\eta')\hat f(\eta,\eta')\overline{\hat g(\eta,\eta')}\,d\eta\,d\eta'\\&=(2\pi)^{\frac d2}\int_{\mathbb{R}^d}m(\eta,\eta')\varepsilon^{-\frac{d'}{p'}}\hat f(\eta,\varepsilon^{-1}\eta')\varepsilon^{-\frac{d'}{p}}\overline{\hat g(\eta,\varepsilon^{-1}\eta')}\,d\eta\,d\eta'\\&=\langle S_m f_\varepsilon|g_\varepsilon\rangle\ ,\end{aligned}$$

where $f_\varepsilon(y,y')=\varepsilon^{\frac{d'}{p}}f(x,\varepsilon x')$ and $g_\varepsilon(y,y')=\varepsilon^{\frac{d'}{p'}}g(x,\varepsilon x')$. Therefore

$$\big|\langle S_{m_\varepsilon}f|g\rangle\big|\le\|S_m\|_{\mathcal{L}(L^p(\mathbb{R}^d))}\|f_\varepsilon\|_p\|g_\varepsilon\|_{p'}=\|S_m\|_{\mathcal{L}(L^p(\mathbb{R}^d))}\|f\|_p\|g\|_{p'}\ .$$

If ε tends to 0, $\langle S_{m_\varepsilon}f|g\rangle$ tends to $\langle S_{m_0}f|g\rangle$, and this proves the first fact.

Second fact: a bounded function $m(\eta,\eta')=\mu(\eta)$ is a Fourier L^p-multiplier on $\mathbb{R}^d$ if and only if μ is a Fourier L^p-multiplier on $\mathbb{R}^\nu$ and the two norms coincide. To see this, let $k=\mathcal{F}^{-1}\mu\in\mathcal{S}'(\mathbb{R}^\nu)$. Since $m=\mu\otimes 1$, we have $\mathcal{F}^{-1}m=k\otimes\delta_0\in\mathcal{S}'(\mathbb{R}^d)$. Calling $f^{y'}(y)=f(y,y')$, we then have

$$S_mf(y,y')=f*_{\mathbb{R}^d}(k\otimes\delta_0)(y,y')=f^{y'}*_{\mathbb{R}^\nu}k(y)=S_\mu f^{y'}(y)\ .$$

Therefore,

$$\begin{aligned}\|S_mf\|^p_{L^p(\mathbb{R}^d)}&=\int_{\mathbb{R}^{d'}}\|S_\mu f^{y'}\|^p_{L^p(\mathbb{R}^\nu)}\,dy'\\&\le\|S_\mu\|^p_{\mathcal{L}(L^p(\mathbb{R}^\nu))}\int_{\mathbb{R}^{d'}}\|f^{y'}\|^p_{L^p(\mathbb{R}^\nu)}\,dy'\\&\le\|S_\mu\|^p_{\mathcal{L}(L^p(\mathbb{R}^\nu))}\|f\|^p_p\ .\end{aligned}$$

This proves that $\|S_mf\|_{L^p(\mathbb{R}^d)}\le\|S_\mu\|_{\mathcal{L}(L^p(\mathbb{R}^\nu))}$. For the opposite inequality, take $f(y,y')=f_1(y)f_2(y')$. Then $S_mf(y,y')=(S_\mu f_1)(y)f_2(y')$. With $f_2\neq0$ fixed,

$$\|S_\mu f_1\|_p=\frac{\|S_mf\|_p}{\|f_2\|_p}\le\|S_mf\|_{L^p(\mathbb{R}^d)}\|f_1\|_p\ .$$

Third fact: the statement is true if $d=n=\nu$. In This case $\tilde m(\xi)=m(A\xi)$ with A invertible. Taking again $f,g\in\mathcal{S}(\mathbb{R}^\nu)$, we have

$$\begin{aligned}\langle S_{\tilde m}f|g\rangle&=(2\pi)^{\frac\nu2}\int_{\mathbb{R}^\nu}m(A\xi)\hat f(\xi)\overline{\hat g(\xi)}\,d\xi\\&=(2\pi)^{\frac\nu2}|\det A|^{-1}\int_{\mathbb{R}^\nu}m(\eta)\hat f(A^{-1}\eta)\overline{\hat g(A^{-1}\eta)}\,d\eta\\&=|\det A|^{-1}\langle S_mf_A|g_A\rangle\ ,\end{aligned}$$

where $f_A(x) = |\det A| f(Ax)$, $g_A(x) = |\det A| g(Ax)$. Then

$$\big|\langle S_{\tilde m} f|g\rangle\big| \le |\det A|^{-1}\|S_m\|_{\mathcal{L}(L^p(\mathbb{R}^\nu))}\|f_A\|_p\|g_A\|_{p'} = \|S_m\|_{\mathcal{L}(L^p(\mathbb{R}^\nu))}\|f\|_p\|g\|_{p'} \ .$$

Going back to the general statement, from the L^p-boundedness of S_m on $\mathbb{R}^d$ we deduce the L^p-boundedness of $\mu(\eta) = m(\eta, 0)$ on $\mathbb{R}^\nu$ (facts 1 and 2). From this we deduce the L^p-boundedness of $S_{\mu\circ A}$ on $\mathbb{R}^\nu$ (fact 3), and from this the L^p-boundedness of $S_{m\circ\sigma}$ on $\mathbb{R}^n$ (fact 2 again). □

End of the proof of Theorem 4.2. Lemma 4.3 implies that $m(L_1, \dots, L_k)$ is bounded on L^p for $1 < p < \infty$ when m is a Marcinkiewicz multiplier, continuous on $\mathbb{R}^k$. Let m be a generic Marcinkiewicz multiplier on $\mathbb{R}^k$, and observe that Lemma 3.2 implies that m is continuous when $\lambda_i \neq 0$ for every i.

Given $\varepsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-2, 2]$ and $\varphi = 1$ on $[-1, 1]$, let

$$m_\varepsilon(\lambda) = m(\lambda)\prod_{i=1}^k \Big(1 - \varphi\big(\frac{\lambda_i}{\varepsilon}\big)\Big) \ .$$

Then m_ε is continuous. If we prove that the Marcinkiewicz norm (3.1) of the m_ε is uniformly bounded in ε, we can conclude that the operator norms of $m_\varepsilon(L_1, \dots, L_k)$ are uniformly bounded. It is then easy to observe, using the Plancherel formula, that, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle m(L_1, \dots, L_k)f|g\rangle = \lim_{\varepsilon\to 0}\langle m_\varepsilon(L_1, \dots, L_k)f|g\rangle \ ,$$

which would allow to conclude. Fix $\eta \in \mathcal{D}(\mathbb{R})$ satisfying (i) and (ii) in Section 3. We must estimate the $H^{\mathbf{s}}$-norm of

$$m(r_1\lambda_1, \dots, r_k\lambda_k)\prod_{i=1}^k \Big(1 - \varphi\big(\frac{r_i\lambda_i}{\varepsilon}\big)\Big)\prod_{i=1}^k \eta(\lambda_i) \ .$$

Observe that $\big(1 - \varphi(r_i\lambda_i/\varepsilon)\big)\eta(\lambda_i)$ is identically zero if $r_i/\varepsilon \le 1/4$, and it equals $\eta(\lambda_i)$ if $r_i/\varepsilon \ge 4$. Matters reduce therefore to verifying that multiplication by $\varphi(t\lambda_i)$ is a continuous operation on $H^{\mathbf{s}}$, uniformly for $1/4 \le t \le 4$. This fact can be proved by adapting the proof of Lemma 6.3 in Chapter II. □

CHAPTER IV
FOURIER ANALYSIS ON THE HEISENBERG GROUP

1. The Heisenberg group

On $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ consider the composition law

$$(x,y,t)(x',y',t') = \Big(x+x', y+y', t+t' + \frac{1}{2}(x\cdot y' - y\cdot x')\Big) \ .$$

In complex coordinates $(z,t)\in\mathbb{C}^n\times\mathbb{R}$, this becomes

$$(z,t)(z',t') = \Big(z+z', t+t' - \frac{1}{2}\Im\mathrm{m}\langle z|z'\rangle\Big) \ ,$$

where $\langle z|z'\rangle = \sum_{j=1}^n z_j\overline{z_j'}$ is the Hermitean product on $\mathbb{C}^n$.

One can verify that this is a non-commutative group law, with neutral element $(0,0)$, and with inverse $(z,t)^{-1} = (-z,-t)$. This is the *Heisenberg group* H_n.

The elements $(0,t)$ form the center $Z(H_n)$ of H_n, and $H_n/Z(H_n)$ is isomorphic to the additive group $\mathbb{R}^{2n}$. The following transformations are automorphisms of H_n:

(1) the *dilations* $(z,t)\longmapsto(rz,r^2t) = r\cdot(z,t)$, with $r>0$;
(2) the *rotations* $(z,t)\longmapsto(Uz,t)$, with U a unitary transformation of $\mathbb{C}^n$ (i.e. $UU^*=I$);
(3) the *conjugation* $(z,t)\longmapsto(\bar z,-t)$.

One finds the vector fields X_j, Y_j in (2.1), Chapter I, by means of the following operations:

$$\begin{aligned}
X_jf(x,y,t) &= \frac{d}{ds}_{|_{s=0}} f\big((x,y,t)(se_j,0,0)\big)\\
&= \frac{d}{ds}_{|_{s=0}} f\Big(x+se_j, y, t-\frac{1}{2}sy_j\Big)\\
&= \partial_{x_j}f(x,y,t) - \frac{y_j}{2}\partial_t f(x,y,t) \ ,\\
Y_jf(x,y,t) &= \frac{d}{ds}_{|_{s=0}} f\big((x,y,t)(0,se_j,0)\big)\\
&= \frac{d}{ds}_{|_{s=0}} f\Big(x, y+se_j, t+\frac{1}{2}sx_j\Big)\\
&= \partial_{y_j}f(x,y,t) + \frac{x_j}{2}\partial_t f(x,y,t) \ .
\end{aligned} \tag{1.1}$$

Therefore these vector fields are *left-invariant*: denoting by

$$L_{(z',t')}f(z,t) = f\big((z',t')^{-1}(z,t)\big)$$

the left-translate of f by (z',t'), then

$$X_j\big(L_{(z',t')}f\big) = L_{(z',t')}(X_jf) \ , \qquad Y_j\big(L_{(z',t')}f\big) = L_{(z',t')}(Y_jf) \ .$$

It follows that also

$$[X_j, Y_j] = T = \partial_t$$

is left-invariant and that any left-invariant vector field on H_n is a linear combination of the X_j, Y_j, T. The other Lie brackets are

$$[X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0 \ , \qquad [X_j, Y_k] = \delta_{j,k}T \ .$$

More generally, one says that a linear operator T acting on functions on H_n is left-invariant if

$$T(L_{(z,t)}f) = L_{(z,t)}(Tf)$$

for all $(z,t) \in H_n$ and every f. We impose here and in the sequel that T maps[13] $\mathcal{S}(H_n)$ into $\mathcal{S}'(H_n)$ boundedly, in the sense that the map[14]

$$(f,g) \longmapsto \langle Tf, g\rangle$$

is continuous on $\mathcal{S}(H_n) \times \mathcal{S}(H_n)$. This is a very mild initial assumption. It is satisfied, e.g., by the *convolution operators*

$$Tf = f * k(z,t) = \int_{H_n} f\big((z,t)(w,u)^{-1}\big)k(w,u)\, dw\, du \ , \tag{1.2}$$

with $k \in L^1(H_n)$ (which must be understood w.r. to the Lebesgue measure).

Convolution on H_n is a non-commutative operation, and in order to have a left-invariant operator, the kernel must be on the right of f. The convolution (1.1) can be extended to distributional kernels $k \in \mathcal{S}'(H_n)$, provided $f \in \mathcal{S}(H_n)$; the integral on the right-hand side of (1.2) must then be interpreted as the pairing between k and the function $(w,u) \longmapsto f\big((z,t)(w,u)^{-1}\big)$.

It follows from the Schwartz kernel theorem that any left-invariant operator T mapping $\mathcal{S}(H_n)$ boundedly into $\mathcal{S}'(H_n)$ has the form (1.2) for some $k \in \mathcal{S}'(H_n)$.

The sub-Laplacian $L = -\sum_{j=1}^n (X_j^2 + Y_j^2)$ studied in Chapter I is also left-invariant, and such are its spectral projections, the joint spectral projections of L and $i^{-1}T$, and the operators defined by spectral multipliers of L and $i^{-1}T$. Therefore they can all be realized as convolution operators on H_n.

[13] $\mathcal{S}(H_n)$ is the same as $\mathcal{S}(\mathbb{R}^{2n+1})$, and similarly for $\mathcal{S}'$.

[14] Recall that the notation $\langle\ ,\ \rangle$ stands for the *bilinear* pairing.

Obviously, besides the left-invariant vector fields, one has the *right-invariant* ones. We shall put a superscript (r) to denote the right-invariant versions of the X_j, Y_j:

$$
\begin{aligned}
X_j^{(r)} f(x,y,t) &= \frac{d}{ds}_{|_{s=0}} f\big((se_j,0,0)(x,y,t)\big) \\
&= \frac{d}{ds}_{|_{s=0}} f\Big(x+se_j, y, t+\frac{1}{2}sy_j\Big) \\
&= \partial_{x_j} f(x,y,t) + \frac{y_j}{2}\partial_t f(x,y,t)\ , \\
Y_j^{(r)} f(x,y,t) &= \frac{d}{ds}_{|_{s=0}} f\big((0,se_j,0)(x,y,t)\big) \\
&= \frac{d}{ds}_{|_{s=0}} f\Big(x, y+se_j, t-\frac{1}{2}sx_j\Big) \\
&= \partial_{y_j} f(x,y,t) - \frac{x_j}{2}\partial_t f(x,y,t)\ .
\end{aligned}
\tag{1.3}
$$

Observe that T is both right- and left-invariant, and that

$$
[X_j^{(r)}, Y_j^{(r)}] = -T\ .
$$

We shall also write

$$
L^{(r)} = -\sum_{j=1}^{n} \big((X_j^{(r)})^2 + (Y_j^{(r)})^2\big)\ .
$$

One easily verifies that, if $\check f(z,t) = f\big((z,t)^{-1}\big) = f(-z,-t)$, then

$$
X_j^{(r)} f = -(X_j \check f)\check{}\ ,
$$

(similarly for the Y_j) and that

$$
L^{(r)} f = (L\check f)\check{}\ .
$$

2. The group Fourier transform

For a few sections we shall restrict ourselves to $H_1 = \mathbb{C}\times\mathbb{R}$. We recall that in Chapter I we have introduced the functions

$$
h_{j,k}(z) = \bar{\mathcal{Z}}^k\big(\bar z^j e^{-\frac{|z|^2}{4}}\big) = e^{\frac{|z|^2}{4}}\partial_{\bar z}^k\big(\bar z^j e^{-\frac{|z|^2}{2}}\big)
\tag{2.1}
$$

on $\mathbb{C}$, with $j,k\in\mathbb{N}$, and discussed their rôle in the spectral analysis of the sub-Laplacian L. As we shall see, these functions are the key ingredients to construct the *group Fourier transform* on H_1 (later on we shall extend all this to H_n).

The general notion of Fourier transform on locally compact groups involves notions of abstract representation theory that we do not want to develop here[15].

[15] See the notes of the course “Analisi di Fourier non commutativa”.

We shall use instead the spectral analysis of L to study the specific case of the Heisenberg group.

To begin with, it is appropriate to normalize the $h_{j,k}$ in $L^2(\mathbb{C})$. From (3.8) in Chapter I, we deduce that

$$\begin{aligned}\|h_{j,k}\|_2^2 &= \frac{k!}{2^k}\|h_{j,0}\|_2^2\\ &= \frac{k!}{2^k}2\pi\int_0^\infty r^{2j+1}e^{-\frac{r^2}{2}}\,dr\\ &= (2\pi)j!k!2^{j-k}\ .\end{aligned}$$

We then define

$$\varphi_{j,k}(z) = \frac{1}{\sqrt{j!k!2^{j-k}}}h_{j,k}(z)\ . \tag{2.2}$$

By Proposition 3.2 in Chapter 1, $\{(2\pi)^{-1/2}\varphi_{j,k}\}$ is an orthonormal basis of $L^2(\mathbb{C})$. From (3.10) in Chapter I, it follows that the $\varphi_{j,k}$ are in $\mathcal{S}(\mathbb{C})$.

By (3.10) and (3.12) in Chapter I,

$$\begin{aligned}&\varphi_{j,k}(0) = \delta_{j,k}\ ,\\ &\varphi_{j,k}(z) = (-1)^{j-k}\varphi_{j,k}(-z) = (-1)^{j-k}\varphi_{k,j}(\bar z) = (-1)^{j-k}\overline{\varphi_{k,j}(z)}\ .\end{aligned} \tag{2.3}$$

In analogy with Corollary 3.3 in Chapter I, we define

$$\varphi^\lambda_{j,k}(z) = \begin{cases}\varphi_{j,k}(\lambda^{\frac12}z) & \text{if } \lambda>0\ ,\\[2ex] \varphi_{j,k}\big(|\lambda|^{\frac12}\bar z\big) & \text{if } \lambda<0\ .\end{cases} \tag{2.4}$$

We also set

$$\Phi^\lambda_{j,k}(z,t) = e^{i\lambda t}\varphi^\lambda_{j,k}(z)\ . \tag{2.5}$$

Given $f\in L^1(H_1)$, we define[16], for $\lambda\neq 0$,

$$\hat f(\lambda,j,k) = \int_{H_1} f(z,t)\Phi^\lambda_{j,k}(z,t)\,dz\,dt = \langle\mathcal{F}_t f(\cdot,-\lambda)|\overline{\varphi^\lambda_{j,k}}\rangle\ . \tag{2.6}$$

Proposition 2.1. *If $f\in L^1(H_1)\cap L^2(H_1)$, then the following* Plancherel formula *holds:*

$$\|f\|_2^2 = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\sum_{j,k\in\mathbb{N}}\big|\hat f(\lambda,j,k)\big|^2\,|\lambda|\,d\lambda\ . \tag{2.7}$$

The map $f\longmapsto(2\pi)^{-1}\hat f$ extends to an isometric bijection from $L^2(H_1)$ onto $L^2\big(\mathbb{R},|\lambda|\,d\lambda,\ell^2(\mathbb{N}^2)\big)$.

---

[16] It will be clear soon (Theorem 2.3) that it is preferable not to take complex conjugates of the basis elements.

Proof. By the Plancherel formula on $\mathbb{R}$,

$$\int_{H_1} |f(z,t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{\mathbb{C}} \left|\mathcal{F}_t f(z,-\lambda)\right|^2 dz\,d\lambda\ .$$

Then, for a.e. λ, $\mathcal{F}_t f(z,-\lambda) \in L^2(\mathbb{C})$. But $\left\{(2\pi)^{-1/2}|\lambda|^{\frac{1}{2}}\overline{\varphi_{j,k}^\lambda}\right\}_{j,k}$ is an orthonormal basis of $L^2(\mathbb{C})$; hence

$$\begin{aligned}\int_{\mathbb{C}} \left|\mathcal{F}_t f(z,-\lambda)\right|^2 dz &= \frac{|\lambda|}{2\pi}\sum_{j,k\in\mathbb{N}} \left|\langle \mathcal{F}_t f(\cdot,-\lambda)\,|\,\overline{\varphi_{j,k}^\lambda}\rangle\right|^2\\ &= \frac{|\lambda|}{2\pi}\sum_{j,k\in\mathbb{N}} \left|\hat f(\lambda,j,k)\right|^2\ . \qquad \square\end{aligned}$$

We shall give a more elegant presentation of this result, introducing at the same time the representations of H_1.

Using (2.1) above and the identity (3.6) in Chapter I, we obtain the following identities:

$$\begin{aligned}\bar{\mathcal{Z}}\varphi_{j,k} &= \sqrt{\frac{k+1}{2}}\varphi_{j,k+1}\ ,\\ \mathcal{Z}\varphi_{j,k} &= \sqrt{\frac{2}{k}}\mathcal{Z}\bar{\mathcal{Z}}\varphi_{j,k-1} = -\frac{1}{4}\sqrt{\frac{2}{k}}(\mathcal{L}+I)\varphi_{j,k-1} = -\sqrt{\frac{k}{2}}\varphi_{j,k-1}\ .\end{aligned} \tag{2.8}$$

Lemma 2.2. *The following identity holds:*

$$\sum_{\ell\in\mathbb{N}} \varphi_{j,\ell}(z)\varphi_{\ell,k}(w) = e^{-\frac{i}{2}\Im\mathrm{m}(z\bar w)}\varphi_{j,k}(z+w)\ .$$

Proof. By (3.9) in Chapter I and by (2.3) above,

$$\begin{aligned}\sum_{\ell\in\mathbb{N}} \varphi_{j,\ell}(z)\varphi_{\ell,k}(w) &= \sum_{\ell\in\mathbb{N}} (-1)^{j-\ell}\,\overline{\varphi_{\ell,j}(z)}\varphi_{\ell,k}(w)\\ &= (-1)^j\sqrt{\frac{2^{j+k}}{j!k!}}\sum_{\ell\in\mathbb{N}}\frac{(-1)^\ell}{\ell!2^\ell}\overline{h_{\ell,j}(z)}h_{\ell,k}(w)\\ &= (-1)^j\sqrt{\frac{2^{j+k}}{j!k!}}e^{\frac{|z|^2+|w|^2}{4}}\sum_{\ell\in\mathbb{N}}\frac{(-1)^\ell}{\ell!2^\ell}\partial_z^j\big(z^\ell e^{-\frac{|z|^2}{2}}\big)\partial_{\bar w}^k\big(\bar w^\ell e^{-\frac{|w|^2}{2}}\big)\ .\end{aligned}$$

The series $\sum_{\ell\in\mathbb{N}}\frac{(-1)^\ell}{\ell!2^\ell}z^\ell e^{-\frac{|z|^2}{2}}\bar w^\ell e^{-\frac{|w|^2}{2}}$ converges with all its derivatives. The sum equals

$$\sum_{\ell\in\mathbb{N}}\frac{(-1)^\ell}{\ell!2^\ell}z^\ell e^{-\frac{|z|^2}{2}}\bar w^\ell e^{-\frac{|w|^2}{2}} = e^{-\frac{|z|^2+|w|^2+z\bar w}{2}} = e^{-\frac{|z+w|^2-\bar z w}{2}}\ .$$

Therefore,

$$\begin{aligned}\sum_{\ell\in\mathbb{N}}\varphi_{j,\ell}(z)\varphi_{\ell,k}(w) &= (-1)^j\sqrt{\frac{2^{j+k}}{j!k!}}e^{\frac{|z|^2+|w|^2}{4}}\partial_z^j\partial_{\bar w}^k e^{-\frac{|z+w|^2-\bar z w}{2}} \\ &= \frac{1}{\sqrt{j!k!2^{j-k}}}e^{\frac{|z|^2+|w|^2+2\bar z w}{4}}\partial_{\bar w}^k\big((\overline{z+w})^j e^{-\frac{|z+w|^2}{2}}\big) \\ &= \frac{1}{\sqrt{j!k!2^{j-k}}}e^{\frac{|z+w|^2+2i\Im\mathrm{m}\bar z w}{4}}\partial_{\bar w}^k\big((\overline{z+w})^j e^{-\frac{|z+w|^2}{2}}\big) \\ &= \frac{1}{\sqrt{j!k!2^{j-k}}}e^{-\frac12\Im\mathrm{m} z\bar w}h_{j,k}(z+w) \\ &= e^{-\frac12\Im\mathrm{m} z\bar w}\varphi_{j,k}(z+w)\ . \qquad\square\end{aligned}$$

It follows immediately that

$$\Phi^\lambda_{j,k}\big((z,t)(w,u)\big) = \sum_{\ell\in\mathbb{N}}\Phi^\lambda_{j,\ell}(z,t)\Phi^\lambda_{\ell,k}(w,u)\ , \tag{2.9}$$

for every $\lambda\neq 0$.

It is then natural to arrange the $\Phi^\lambda_{j,k}$ in the infinite matrix

$$\Phi^\lambda(z,t) = \big(\Phi^\lambda_{j,k}(z,t)\big)_{j,k}\ . \tag{2.10}$$

Theorem 2.3. *The following identities hold:*

$$\begin{aligned}&\Phi^\lambda(z,t)^*\Phi^\lambda(z,t) = \Phi^\lambda(z,t)\Phi^\lambda(z,t)^* = I \\ &\Phi^\lambda\big((z,t)^{-1}\big) = \Phi^\lambda(z,t)^* = \Phi^\lambda(z,t)^{-1}\ , \\ &\Phi^\lambda\big((z,t)(w,u)\big) = \Phi^\lambda(z,t)\Phi^\lambda(w,u)\ .\end{aligned}$$

In particular, $\Phi^\lambda(z,t)$ defines, for every $(z,t)\in H_1$ and every $\lambda\neq 0$, a unitary operator $\pi^\lambda(z,t)$ on $\ell^2=\ell^2(\mathbb{N})$. The map

$$\pi^\lambda : H_1 \longrightarrow \mathcal{L}(\ell^2)$$

is a continuous homomorphism, w.r.t. to the strong topology on $\mathcal{L}(\ell^2)$, i.e. a unitary representation of H_1.

For $f\in L^1(H_1)$, the integral

$$\pi^\lambda(f) = \int_{H_1} f(z,t)\pi^\lambda(z,t)\,dz\,dt \tag{2.11}$$

converges in the strong topology to a bounded operator on ℓ^2. The identity

$$\pi^\lambda(f*g) = \pi^\lambda(f)\pi^\lambda(g) \tag{2.12}$$

holds for every $f,g\in L^1(H_1)$.

The proof requires a few verifications that we leave to the reader. Observe that the operator $\pi^\lambda(f)$ in (2.11) is represented, w.r. to the canonical basis of ℓ^2, by the matrix

$$\int_{H_1} f(z,t)\Phi^\lambda(z,t)\,dz\,dt = \big(\hat f(\lambda,j,k)\big)_{j,k} \overset{\text{def}}{=} \hat f(\lambda)\ .$$

Then (2.12) takes the form

$$\widehat{f*g}(\lambda) = \hat f(\lambda)\hat g(\lambda)\ , \tag{2.13}$$

in analogy with the ordinary Fourier transform. These identities justify the choice of not conjugating $\Phi^\lambda_{j,k}$ in (2.6). If we had done so, the order of the two factors in (2.13) should have been changed. Recall that both convolution in H_1 and composition of linear operators on ℓ^2 are non-commutative.

We obtain the following reformulation of Proposition 2.1.

Theorem 2.4. *For $f\in L^1(H_1)\cap L^2(H_1)$, the operator $\pi^\lambda(f)$ is a Hilbert-Schmidt operator for a.e. λ, and the Plancherel formula can be written as:*

$$\|f\|_2^2 = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \|\pi^\lambda(f)\|^2_{HS}\,|\lambda|\,d\lambda\ . \tag{2.14}$$

The map $f\longmapsto (2\pi)^{-1}\pi^\lambda(f)$ extends to an isometric bijection from $L^2(H_1)$ onto $L^2\big(\mathbb{R},|\lambda|\,d\lambda, HS(\ell^2)\big)$.

This is the standard form of a Plancherel formula on a non-commutative group, invoking Hilbert-Schmidt norms of operator-valued Fourier transforms.

We write explicitely the polarized form of Plancherel's formula:

$$\begin{aligned}\int_{H_1} f(z,t)\overline{g(z,t)}\,dz\,dt &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \operatorname{tr}\big(\pi^\lambda(f)\pi^\lambda(g)^*\big)\,|\lambda|\,d\lambda\\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \operatorname{tr}\big(\hat f(\lambda)\hat g(\lambda)^*\big)\,|\lambda|\,d\lambda\\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \sum_{j,k\in\mathbb{N}} \hat f(\lambda,j,k)\overline{\hat g(\lambda,j,k)}\,|\lambda|\,d\lambda\ .\end{aligned} \tag{2.16}$$

Like in commutative Fourier analysis, the Plancherel formula has a companion inversion formula. We give the inversion formula for a narrow class of functions (the Schwartz class). It can however be extended to a larger class[17]. The proof requires a few lemmas.

Lemma 2.5. *For every j,k, $\|\Phi^\lambda_{j,k}\|_\infty\le 1$. If $L=-X^2-Y^2$ is the sub-Laplacian on H_1, and $T=\partial_t$,*

$$L\Phi^\lambda_{j,k} = |\lambda|(2k+1)\Phi^\lambda_{j,k}\ ,\qquad T\Phi^\lambda_{j,k} = i\lambda\Phi^\lambda_{j,k}\ . \tag{2.17}$$

Proof. The first identity in Theorem 2.3 implies that, for each j and each (z,t),

$$\sum_{k\in\mathbb{N}} |\Phi^\lambda_{j,k}(z,t)|^2 = 1\ . \tag{2.18}$$

This obviously proves the first statement. The first identity in (2.17) is a consequence of Corollary 3.3 in Chapter I, and the second is obvious. □

[17] The formula below shows that it is sufficient that $\pi^\lambda(f)$ be of trace class for a.e. λ and that the function $\lambda\mapsto \operatorname{tr}\big(|\pi^\lambda(f)|\big)$ be integrable on $\mathbb{R}$. The proof below shows that Schwartz functions satisfy this property.

Lemma 2.6. *The following identities hold:*

$$\widehat{Lf}(\lambda,j,k)=|\lambda|(2k+1)\hat f(\lambda,j,k)\ ,\qquad \widehat{Tf}(\lambda,j,k)=-i\lambda\hat f(\lambda,j,k)\ .$$

If $f^*(z,t)=\overline{f(-z,-t)}$ *then*

$$\widehat{f^*}(\lambda,j,k)=\overline{\hat f(\lambda,k,j)}\ ,$$

i.e.

$$\pi_\lambda(f^*)=\big(\pi_\lambda(f)\big)^*\ .$$

Proof. The first two identities follow directly from Lemma 2.5, by integration by parts. The last one follows from the fact that $\overline{\Phi^\lambda_{j,k}(-z,-t)}=\Phi^\lambda_{k,j}(z,t)$. □

Proposition 2.7. *For each* $N\in\mathbb{N}$*, there is a Schwartz norm* $\|\ \|_N$ *such that, if* $f\in\mathcal{S}(H_1)$*,*

$$\big|\hat f(\lambda,j,k)\big|\le\frac{\|f\|_N}{\big(1+|\lambda|(1+j+k)\big)^N}\ .$$

Moreover $\hat f(\lambda,j,k)$ *is smooth in* λ *for* $\lambda\neq0$*.*

Proof. If $f\in\mathcal{S}(H_1)$, we have in the first place

$$\big|\hat f(\lambda,j,k)\big|=\left|\int_{H_1}f(z,t)\Phi^\lambda_{j,k}(z,t)\,dz\,dt\right|\le\|f\|_1\|\Phi^\lambda_{j,k}\|_\infty\le\|f\|_1\ .$$

By Lemma 2.6,

$$\begin{aligned}|\lambda|^N\big|\hat f(\lambda,j,k)\big|&=\big|\widehat{T^Nf}(\lambda,j,k)\big|\le\|T^Nf\|_1\ ,\\ |\lambda|^N(2k+1)^N\big|\hat f(\lambda,j,k)\big|&=\big|\widehat{L^Nf}(\lambda,j,k)\big|\le\|L^Nf\|_1\ ,\\ |\lambda|^N(2j+1)^N\big|\hat f(\lambda,j,k)\big|&=|\lambda|^N(2j+1)^N\big|\widehat{f^*}(\lambda,k,j)\big|\le\|L^Nf^*\|_1\ ,\end{aligned}$$

Putting all these estimates together,

$$\big(1+|\lambda|(1+j+k)\big)^N|\hat f(\lambda,j,k)|\le C\big(\|f\|_1+\|Tf\|_1+\|Lf\|_1+\|L^*f\|_1\big)\ ,$$

and this last expression is controlled by a Schwartz norm.

The smoothness of $\hat f(\lambda,j,k)$ for $\lambda\neq0$ is obvious. □

Theorem 2.8. *For* $f\in\mathcal{S}(H_1)$ *the following* inversion formula *holds:*

(2.19)
$$\begin{aligned}f(z,t)&=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\mathrm{tr}\,\big(\pi^\lambda(f)\pi^\lambda(z,t)^*\big)\,|\lambda|\,d\lambda\\&=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\sum_{j,k\in\mathbb{N}}\hat f(\lambda,j,k)\overline{\Phi^\lambda_{j,k}(z,t)}\,|\lambda|\,d\lambda\ .\end{aligned}$$

Proof. Since

$$\hat f(\lambda,j,k)=\langle\mathcal{F}_tf(\cdot,-\lambda)\,|\,\overline{\varphi^\lambda_{j,k}}\rangle\ ,$$

for every $\lambda \neq 0$, and $\mathcal{F}_t f(\cdot, -\lambda) \in L^2(\mathbb{C})$,

$$\mathcal{F}_t f(z, -\lambda) = \frac{|\lambda|}{2\pi} \sum_{j,k\in\mathbb{N}} \hat{f}(\lambda, j, k)\overline{\varphi^\lambda_{j,k}(z)} ,$$

with convergence in $L^2(\mathbb{C})$. The convergence is also pointwise. In fact, by (2.18), $|\varphi_{j,k}(z)| \leq 1$, so that, by Proposition 2.7,

$$\sum_{j,k\in\mathbb{N}} |\hat{f}(\lambda, j, k)||\varphi^\lambda_{j,k}(z)| \leq \sum_{j,k\in\mathbb{N}} \frac{\|f\|_N}{\big(1 + |\lambda|(1 + j + k)\big)^N} < \infty ,$$

taking $N > 2$.

Therefore,

$$\mathcal{F}_t f(z, -\lambda)e^{-i\lambda t} = \frac{|\lambda|}{2\pi} \sum_{j,k\in\mathbb{N}} \hat{f}(\lambda, j, k)\overline{\Phi^\lambda_{j,k}(z, t)} ,$$

and the Fourier inversion formula on $\mathbb{R}$ does the rest. □

We give a first application of this formula, which will be used in the sequel.

Corollary 2.9. *Let $\mathcal{S}_0(H_1) \subset \mathcal{S}(H_1)$ be the space of the functions f such that $\hat{f}(\lambda, j, k)$ is non-zero only for j, k varying in a finite set, and, for these values of j, k, $\hat{f}(\lambda, j, k)$ is C^∞ in λ and supported on a compact subset of $\mathbb{R} \setminus \{0\}$. Then $\mathcal{S}_0(H_1)$ is dense in $L^2(H_1)$.*

Given any finite family of functions $v_{j,k} \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, with $(j, k) \in B$, the function

$$f(z, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{(j,k)\in B} v_{j,k}(\lambda)\overline{\Phi^\lambda_{j,k}(z, t)}\, |\lambda|\, d\lambda$$

is in $\mathcal{S}_0(H_1)$ and $\hat{f}(\lambda, j, k)$ equals $v_{j,k}(\lambda)$ if $(j, k) \in B$ and zero otherwise.

Proof. It is sufficient to prove that any $g \in \mathcal{S}(H_1)$ can be approximated in the L^2-norm by functions in $\mathcal{S}_0(H_1)$. Given $\varepsilon > 0$, take $K \subset \mathbb{R} \setminus \{0\}$ and $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus K} \sum_{j+k>N} |\hat{g}(\lambda, j, k)|^2\, |\lambda|\, d\lambda < \varepsilon^2 ,$$

and fix $\eta \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ equal to 1 on K, with $0 \leq \eta(\lambda) \leq 1$ for every λ. Let $K' = \operatorname{supp} \eta$ and define

$$\begin{aligned} f_\varepsilon(z, t) &= \frac{1}{(2\pi)^2} \int_{K'} \sum_{j+k\leq N} \hat{g}(\lambda, j, k)\overline{\Phi^\lambda_{j,k}(z, t)}\, \eta(\lambda)|\lambda|\, d\lambda \\ &= \frac{1}{(2\pi)^2} \sum_{j+k\leq N} \int_{K'} \hat{g}(\lambda, j, k)\overline{\varphi^\lambda_{j,k}(z)}\eta(\lambda)|\lambda|e^{-i\lambda t}\, d\lambda \\ &= \frac{1}{2\pi} \sum_{j+k\leq N} \mathcal{F}_t^{-1}\big(\hat{g}(\lambda, j, k)\overline{\varphi^\lambda_{j,k}(z)}\eta(\lambda)|\lambda|\big)(-t) . \end{aligned}$$

It follows from Proposition 2.7 that $\hat{g}(\lambda, j, k)\overline{\varphi^\lambda_{j,k}(z)}\eta(\lambda)|\lambda| \in \mathcal{S}(\mathbb{C}\times\mathbb{R})$. Therefore $f_\varepsilon \in \mathcal{S}(H_1)$.

Moreover, $\widehat{f_\varepsilon}(\lambda, j, k) = \eta(\lambda)\hat{g}(\lambda, j, k)$ if $j + k \le N$, and 0 otherwise. Hence $f_\varepsilon \in \mathcal{S}_0(H_1)$. Finally, by the Plancherel formula, $\|g - f_\varepsilon\|_2 < C\varepsilon$.

The proof of the last statement is now obvious. □

We conclude this section by discussing the possibility of defining the Fourier transform for a general distribution $u \in \mathcal{S}'(H_1)$. In analogy with (2.6), we are tempted to define $\hat{u}(\lambda, j, k) = \langle u, \Phi^\lambda_{j,k}\rangle$. But $\Phi^\lambda_{j,k}$ is not in $\mathcal{S}(H_1)$ because it does not have any decay in t, even though it satisfies the required decay conditions in z. We can however apply u to a "packet" of $\Phi^\lambda_{j,k}$.

Given $\psi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, define

$$\tilde{\psi}_{j,k}(z, t) = \int_{\mathbb{R}} \Phi^\lambda_{j,k}(z, t)\psi(\lambda)\, d\lambda = 2\pi \mathcal{F}_t^{-1}\big(\varphi^\lambda_{j,k}(z)\psi(\lambda)\big)(t) \ .$$

Since $\varphi^\lambda_{j,k}(z)\psi(\lambda) \in \mathcal{S}(\mathbb{C} \times \mathbb{R})$ as a function of z and λ, it follows that $\tilde{\psi}_{j,k} \in \mathcal{S}(H_1)$. We can then defined distributions $u_{j,k} \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$ by setting

$$\langle \hat{u}_{j,k}, \psi\rangle = \langle u, \tilde{\psi}_{j,k}\rangle \ . \tag{2.20}$$

It is a simple verification that, if $u \in L^1(H_1)$, then $\hat{u}_{j,k}$ coincides with $\hat{f}(\lambda, j, k)$ as a function of λ.

The convolution formula in (2.13) has the following extension.

Lemma 2.10. *If $u \in \mathcal{S}'(H_1)$, and $f \in \mathcal{S}_0(H_1)$,*

$$\widehat{(f * u)}_{j,k} = \sum_{\ell \in \mathbb{N}} \hat{f}(\cdot, j, \ell)\hat{u}_{\ell,k} \ .$$

Proof. By (2.20), if $\psi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ and $\check{f}(z, t) = f(-z, -t)$,

$$\begin{aligned}\langle \widehat{(f * u)}_{j,k}, \psi\rangle &= \langle f * u, \tilde{\psi}_{j,k}\rangle \\ &= \langle u, \check{f} * \tilde{\psi}_{j,k}\rangle \ .\end{aligned}$$

By (2.9),

$$\begin{aligned}\check{f} * \tilde{\psi}_{j,k}(z, t) &= \int_{H_1} f(w, u)\tilde{\psi}_{j,k}\big((w, u)(z, t)\big)\, dw\, du \\ &= \int_{H_1}\int_{\mathbb{R}} f(w, u)\Phi^\lambda_{j,k}\big((w, u)(z, t)\big)\psi(\lambda)\, d\lambda\, dw\, du \\ &= \sum_{\ell \in \mathbb{N}} \int_{\mathbb{R}} \hat{f}(\lambda, j, \ell)\Phi^\lambda_{\ell,k}(z, t)\psi(\lambda)\, d\lambda \\ &= \sum_{\ell \in \mathbb{N}} \big(\psi\hat{f}(\cdot, j, \ell)\big)\tilde{}_{\ell,k} \ .\end{aligned}$$

But, according to (2.20),

$$\big\langle u, \big(\psi\hat{f}(\cdot, j, \ell)\big)\tilde{}_{\ell,k}\big\rangle = \langle \hat{u}_{\ell,k}, \psi\hat{f}(\cdot, j, \ell)\rangle = \langle \hat{f}(\cdot, j, \ell)\hat{u}_{\ell,k}, \psi\rangle \ ,$$

and this concludes the proof. □

A complete description of the image of $\mathcal{S}'(H_1)$ under Fourier transform is possible[18], but we do not go into it because it will not be needed.

[18]See D. Geller, ...

3. Fourier multipliers

It follows from Lemma 2.6 that the matrices $\hat{f}(\lambda) = \{\hat{f}(\lambda, j, k)\}_{j,k}$ and $\widehat{Lf}(\lambda)$ are related by the identity

$$\widehat{Lf}(\lambda) = \hat{f}(\lambda) \begin{pmatrix} |\lambda| & 0 & \cdots & 0 & \cdots \\ 0 & 3|\lambda| & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & & (2k+1)|\lambda| & \\ \vdots & \vdots & & & \ddots \end{pmatrix} . \tag{3.1}$$

Similarly,

$$\widehat{Tf}(\lambda) = -i\lambda \hat{f}(\lambda) = \hat{f}(\lambda)(-i\lambda I) . \tag{3.2}$$

Similar formulas can be obtained for the other left-invariant vector fields. In analogy with (3.2) in Chapter I, define the complex vector fields

$$Z = \frac{1}{2}(X - iY) = \partial_z - \frac{i}{4}\bar{z}\partial_t , \qquad \bar{Z} = \frac{1}{2}(X + iY) = \partial_{\bar{z}} + \frac{i}{4}z\partial_t ,$$

on H_1. Observe that, by (2.8), for $\lambda > 0$,

$$\begin{aligned} Z\Phi^{\lambda}_{j,k}(z,t) &= \Big(\partial_z - \frac{i}{4}\bar{z}\partial_t\Big) e^{i\lambda t}\varphi_{j,k}(\lambda^{\frac{1}{2}} z) \\ &= e^{i\lambda t}\Big(\lambda^{\frac{1}{2}}(\partial_z \varphi_{j,k})(\lambda^{\frac{1}{2}} z) + \frac{1}{4}\lambda\bar{z}\varphi_{j,k}(\lambda^{\frac{1}{2}} z)\Big) \\ &= e^{i\lambda t}\Big(\lambda^{\frac{1}{2}}\mathcal{Z}\varphi_{j,k}(\lambda^{\frac{1}{2}} z) - \lambda\frac{\bar{z}}{4}\varphi_{j,k}(\lambda^{\frac{1}{2}} z) + \frac{1}{4}\lambda\bar{z}\varphi_{j,k}(\lambda^{\frac{1}{2}} z)\Big) \\ &= -\sqrt{\frac{k\lambda}{2}}\Phi^{\lambda}_{j,k-1}(z,t) . \end{aligned}$$

Integrating by parts, we find that

$$\widehat{Zf}(\lambda, j, k) = \sqrt{\frac{k\lambda}{2}}\hat{f}(\lambda, j, k-1) .$$

Similarly,

$$\widehat{\bar{Z}f}(\lambda, j, k) = -\sqrt{\frac{(k+1)\lambda}{2}}\hat{f}(\lambda, j, k+1) .$$

For $\lambda < 0$, the two expressions are interchanged:

$$\widehat{Zf}(\lambda, j, k) = -\sqrt{\frac{(k+1)|\lambda|}{2}}\hat{f}(\lambda, j, k+1) , \qquad \widehat{\bar{Z}f}(\lambda, j, k) = \sqrt{\frac{k|\lambda|}{2}}\hat{f}(\lambda, j, k-1) .$$

Define the matrices

$$U_\lambda = \begin{pmatrix} 0 & \sqrt{|\lambda|/2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{|\lambda|} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & & 0 & \sqrt{k|\lambda|/2} & \\ \vdots & \vdots & & & \ddots & \ddots \end{pmatrix} ,$$

$$L_\lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots \\ -\sqrt{|\lambda|/2} & 0 & \cdots & 0 & \cdots \\ 0 & -\sqrt{|\lambda|} & \ddots & & \\ \vdots & \vdots & \ddots & 0 & \\ 0 & 0 & & -\sqrt{k\lambda/2} & \ddots \\ \vdots & \vdots & & & \ddots \end{pmatrix} .$$

Then, for $\lambda > 0$,

$$\widehat{Zf}(\lambda) = \hat{f}(\lambda)U_\lambda \, , \qquad \widehat{\bar{Z}f}(\lambda) = \hat{f}(\lambda)L_\lambda \, , \tag{3.3}$$

and, for $\lambda < 0$,

$$\widehat{Zf}(\lambda) = \hat{f}(\lambda)L_\lambda \, , \qquad \widehat{\bar{Z}f}(\lambda) = \hat{f}(\lambda)U_\lambda \, , \tag{3.4}$$

In all these cases, we see that the action of a left-invariant differential operator on f is reflected on the Fourier transform side by *right multiplication* by a matrix depending on λ. The same holds for the integral operators $Tf = f * k$ in (1.1). By (2.13) we know in fact that

$$\widehat{Tf}(\lambda) = \hat{f}(\lambda)\hat{k}(\lambda) \, ,$$

at least for $k \in L^1(H_1)$. We prove now that this is a general fact for bounded operators on $L^2(H_1)$.

Definition. *We say that a matrix-valued funtion* $M(\lambda) = \big(m(\lambda, j, k)\big)_{j,k\in\mathbb{N}}$ *is a* bounded Fourier multiplier *for* H_1 *if* $m(\cdot, j, k) \in L^\infty(\mathbb{R})$ *for every* $j, k \in \mathbb{N}$, *and if*

$$\|M\|_\infty = \operatorname{ess\,sup} \|M(\lambda)\|_{\mathcal{L}(\ell^2)} < \infty \, .$$

Theorem 3.1. *If M is a bounded Fourier multiplier for H_1, the requirement that*

$$\widehat{T_M f}(\lambda) = \hat{f}(\lambda)M(\lambda) \tag{3.5}$$

defines a bounded left-invariant operator T_M on $L^2(H_1)$, with $\|T_M\|_{\mathcal{L}\left(L^2(H_1)\right)} = \|M\|_\infty$.

Conversely, for any bounded left-invariant operator T on $L^2(H_1)$, there is a bounded Fourier multiplier M such that $T = T_M$.

Proof. Assume that M is a bounded Fourier multiplier. Since, for any pair of operators A, B on a Hilbert space H,

$$\|AB\|_{HS} \le \|A\|_{HS} \|B\|_{\mathcal{L}(H)} \, ,$$

taking $f \in L^2(H_1)$,

$$\int_{\mathbb{R}} \|\pi^\lambda(f)M(\lambda)\|_{HS}^2 \, |\lambda| \, d\lambda \le \|M\|_\infty^2 \int_{\mathbb{R}} \|\pi^\lambda(f)\|_{HS}^2 \, |\lambda| \, d\lambda \, .$$

By Plancherel's formula, T_M is bounded on $L^2(H_1)$, and $\|T_M\|_{\mathcal{L}\left(L^2(H_1)\right)} \leq \|M\|_\infty$. The opposite inequality will follow from the second part of the proof.

By (2.11),

$$\widehat{L_{(w,u)}f}(\lambda) = \Phi^\lambda(w,u)\hat{f}(\lambda) \ .$$

Therefore

$$(\widehat{L_{(w,u)}Tf})(\lambda) = \Phi^\lambda(w,u)\widehat{Tf}(\lambda) = (\widehat{L_{(w,u)}f})(\lambda)\hat{k}(\lambda) \ ,$$

which implies that T commutes with $L_{(w,u)}$ for every $(w,u) \in H_1$.

Suppose now that T commutes with left translations and is bounded on $L^2(H_1)$. As a consequence of the Schwartz kernel theorem, as it has already been mentioned, there is a distribution $u \in \mathcal{S}'(H_1)$ such that $Tf = f * u$ for $f \in \mathcal{S}(H_1)$. We want to show that the distributions $\hat{u}_{j,k}$ defined in (2.20) are in fact bounded functions, and that $M(\lambda) = \{u_{j,k}(\lambda)\}_{j,k}$ is a bounded Fourier multiplier.

Take $f \in \mathcal{S}_0(H_1)$ such that $\hat{f}(\lambda, j, k) = \psi(\lambda)\delta_{j,0}\delta_{k,p}$, where $\psi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ and $p \in \mathbb{N}$. Then, by Lemma 2.9,

$$\widehat{(f * u)}_{j,k} = \delta_{j,0}\psi\hat{u}_{p,k} \ .$$

If $f * u \in L^2(H_1)$, as we are assuming, a necessary condition is that for every ψ as above and every $p, k \in \mathbb{N}$, $\psi\hat{u}_{p,k}$ be square integrable in λ. This implies that each $\hat{u}_{j,k}$ is locally integrable on $\mathbb{R} \setminus \{0\}$. We can then define $M(\lambda)$ for a.e. λ.

Take now an infinite matrix A with only finite many entries different from 0, and $\psi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Let $f \in \mathcal{S}_0(H_1)$ be such that $\hat{f}(\lambda) = \psi(\lambda)A$. We have

$$\begin{aligned}
\int_{\mathbb{R}} \|\hat{f}(\lambda)M(\lambda)\|_{HS}^2 \,|\lambda|\, d\lambda &= \int_{\mathbb{R}} |\psi(\lambda)|^2 \|AM(\lambda)\|_{HS}^2 \,|\lambda|\, d\lambda \\
&\leq \|T\|^2_{\mathcal{L}(L^2(H_1))} \int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}^2 \,|\lambda|\, d\lambda \\
&= \|T\|^2_{\mathcal{L}(L^2(H_1))} \int_{\mathbb{R}} |\psi(\lambda)|^2 \|A\|_{HS}^2 \,|\lambda|\, d\lambda \ .
\end{aligned}$$

Since this must hold for every ψ, it follows that for a.e. λ,

$$\|AM(\lambda)\|_{HS} \leq \|T\|_{\mathcal{L}(L^2(H_1))} \|A\|_{HS} \ .$$

Given $v \in \ell^2$ with only finitely many components different from 0, take A_v as the matrix having the components of v on the top row, and 0 on the others. Then

$$\|A_v M(\lambda)\|_{HS} = \|M(\lambda)^* v\|_{\ell^2} \leq \|T\|_{\mathcal{L}(L^2(H_1))} \|A_v\|_{HS} = \|T\|_{\mathcal{L}(L^2(H_1))} \|v\|_{\ell^2} \ .$$

Therefore $\|M(\lambda)\|_{\mathcal{L}(\ell^2)} = \|M(\lambda)^*\|_{\mathcal{L}(\ell^2)} \leq \|T\|$. That each $\hat{u}_{j,k} \in L^\infty(\mathbb{R})$ follows easily from the fact that it is measurable, and

$$\hat{u}_{j,k} = \langle M(\lambda)e_k | e_j \rangle_{\ell^2} \ .$$

Hence M is a bounded Fourier multiplier and $\|M\|_\infty \leq \|T\|_{\mathcal{L}(L^2(H_1))}$. $\square$

How to transfer all this discussion to right-invariant operators is rather clear. The right-invariant analogues of the $Z_j, \bar{Z}_j$ are

$$Z^{(r)} = \frac{1}{2}(X^{(r)} - iY^{(r)}) = \partial_z + \frac{i}{4}\bar{z}\partial_t \ , \qquad \bar{Z}^{(r)} = \frac{1}{2}(X^{(r)} + iY^{(r)}) = \partial_{\bar{z}} - \frac{i}{4}z\partial_t \ .$$

Then $\widehat{Z^{(r)}f}(\lambda)$ can be expressed as in (3.3) and (3.4), only with the order of the two factors on the right-hand side reversed. This fact goes together with identities like

$$Z^{(r)}\Phi^\lambda_{j,k}(z,t) = -\sqrt{\frac{(j+1)\lambda}{2}}\Phi^\lambda_{j+1,k}(z,t) \ , \tag{3.6}$$

valid for $\lambda > 0$. Precisely, we have

$$\widehat{Z^{(r)}f}(\lambda) = U_\lambda \hat{f}(\lambda) \ , \qquad \widehat{\bar{Z}^{(r)}f}(\lambda) = L_\lambda \hat{f}(\lambda) \ , \tag{3.7}$$

and, for $\lambda < 0$,

$$\widehat{Z^{(r)}f}(\lambda) = L_\lambda \hat{f}(\lambda) \ , \qquad \widehat{\bar{Z}^{(r)}f}(\lambda) = U_\lambda \hat{f}(\lambda) \ , \tag{3.8}$$

Theorem 3.1 has the same formulation, except for the order of the two factors in (3.5), for right-invariant convolution operators, $Tf = k * f$.

4. Radial functions and diagonal multipliers

In Section 1 we presented the *rotations* $(z,t) \longmapsto (Uz,t)$ of H_n, with U a unitary transformation of H_n. For a function f defined on H_n, we set

$$f_U(z,t) = f(Uz,t) \ .$$

The fact that rotations are automorphisms of H_n implies that

$$(f * g)_U = f_U * g_U \ . \tag{4.1}$$

If $n = 1$, rotations are just scalar multiplications by complex numbers of modulus 1. We then write

$$f_\theta(z,t) = f(e^{i\theta}z,t) \ .$$

We say that a function on H_1 is *radial* if it depends only on $|z|$ and t, or, equivalently, if $f_\theta = f$ for every θ. More generally, we say that f is *of type* $m \in \mathbb{Z}$ if

$$f_\theta = e^{im\theta} f$$

for every θ, or, equivalently, if $e^{-im \arg z} f(z,t)$ is radial.

These notions can be adapted to distributions as follows: a distribution u is of type m if

$$\langle u, f_\theta \rangle = e^{-im\theta} \langle u, f \rangle \ ,$$

for every test function f.

Clearly, by expansion in Fourier serie in the angular variable, every function (or distribution) decomposes as a sum of functions of the different types (with convergence in a sense that depends on the function itself).

The function $\Phi^\lambda_{j,k}$ is of type $k - j$ if $\lambda > 0$ and of type $j - k$ if $\lambda < 0$. This gives the following result.

Lemma 4.1. *A function u is of type m if and only if $\hat{f}(\lambda, j, k) = 0$, unless $\lambda > 0$ and $j - k = m$, or $\lambda < 0$ and $k - j = m$. In particular, f is radial if and only if $\hat{f}(\lambda)$ is diagonal for every $\lambda \neq 0$.*

If u is a radial tempered distributions, then $\hat{u}_{j,k} = 0$ for $j \neq k$.

Proof. Suppose that f is of type m, and take $\lambda > 0$. Then

$$
\begin{aligned}
e^{im\theta}\hat{f}(\lambda, j, k) &= \int_{H_1} f(e^{i\theta}z, t)\Phi^{\lambda}_{j,k}(z, t)\, dz\, dt \\
&= \int_{H_1} f(z, t)\Phi^{\lambda}_{j,k}(e^{-i\theta}z, t)\, dz\, dt \\
&= e^{-i(k-j)\theta}\hat{f}(\lambda, j, k) \ ,
\end{aligned}
$$

for every θ. So, if $m \neq j - k$, necessarily $\hat{f}(\lambda, j, k) = 0$. The rest of the proof follows in the same way. $\square$

Consider now a convolution operator $Tf = f * u$, with u. If u is of type m,

$$(Tf)_\theta = f_\theta * u_\theta = e^{im\theta} f_\theta * u = e^{im\theta} T(f_\theta) \ .$$

Conversely, if $(Tf)_\theta = e^{im\theta}T(f_\theta)$ for every θ and every test function f, then

$$u * f = e^{im\theta}(u * f_\theta)_{-\theta} = e^{im\theta} u_{-\theta} * f \ .$$

Hence $u = e^{im\theta}u_{-\theta}$, i.e. u is of type m.

The special case $m = 0$ concerns the left-invariant operators that also commute with rotations.

Theorem 4.2. *Let $Tf = f * u$ be a bounded operator on $L^2(H_1)$. The following conditions are equivalent:*

(i) *T commutes with rotations;*
(ii) *u is radial;*
(iii) *the Fourier multiplier $M(\lambda)$ is diagonal for a.e. λ;*
(iv) *$T = \mu(iT, L)$, for some bounded spectral multiplier $\mu(\lambda, \xi)$ on the Heisenberg fan F_1.*

If these conditions are satisfied, then $\mu(\lambda, \xi)$ and the diagonal entries $m(\lambda, k, k)$ of $M(\lambda)$ are related by the identity

$$m(\lambda, k, k) = \mu\big(\lambda, |\lambda|(2k + 1)\big) \ .$$

Proof. The equivalence between (i) and (ii) follows from the previous remarks. The implication (ii)⇒(iii) follows from Lemma 4.1, since $\hat{u}_{j,k} = m(\cdot, j, k)$.

Given a Borel subset ω in $\mathbb{R}^2$, define the bounded Fourier multiplier $M_\omega = \big(M_\omega(\lambda, j, k)\big)_{j,k}$ as

$$M_\omega(\lambda, j, k) = \begin{cases} 1 & \text{if } \big(\lambda, |\lambda|(2k+1)\big) \in \omega \text{ and } j = k \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

and let $E(\omega)$ be the corresponding orthogonal projection on $L^2(H_1)$. This define a resolution of the identity and its support is the Heisenberg fan F_1.

For $f, g \in \mathcal{S}(H_1)$, it follows from the Plancherel formula that

$$\nu_{f,g}(\omega) = \langle E(\omega) f | g \rangle = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{\{(j,k):(\lambda,|\lambda|(2k+1))\in\omega\}} \hat{f}(\lambda, j, k) \overline{\hat{g}(\lambda, j, k)}\, |\lambda|\, d\lambda\ .$$

This implies that

$$\begin{aligned}\int_{F_1} \varphi(\lambda, \xi)\, d\nu_{f,g}(\lambda, \xi) = \\ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}} \varphi\big(\lambda, |\lambda|(2k+1)\big) \hat{f}(\lambda, j, k) \overline{\hat{g}(\lambda, j, k)}\, |\lambda|\, d\lambda\ .\end{aligned} \tag{4.2}$$

Let

$$A = \int_{F_1} \lambda\, dE(\lambda, \xi)\ , \qquad B = \int_{F_1} \xi\, dE(\lambda, \xi)\ .$$

The domain $D(A)$ of A consists of the functions f such that

$$\int_{F_1} \lambda^2\, d\nu_{f,f}(\lambda, \xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}} \lambda^2 |\hat{f}(\lambda, j, k)|^2\, |\lambda|\, d\lambda < \infty\ .$$

Let V be the space of finite families $v = \{v_{j,k}\}_{(j,k)\in B_v}$ of functions $v_{j,k} \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. By Corollary 2.9, V consists of the finite families $\{\hat{g}(\cdot, j, k)\}$ with $g \in \mathcal{S}_0(H_1)$. Then

$$\begin{aligned}\Big(\int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}} \lambda^2 |\hat{f}(\lambda, j, k)|^2\, |\lambda|\, d\lambda \Big)^{\frac{1}{2}} &= \sup_{v\in V} \frac{\int_{\mathbb{R}} \sum_{(j,k)\in B_v} \lambda \hat{f}(\lambda, j, k) \overline{v_{j,k}(\lambda)}\, |\lambda|\, d\lambda}{\Big(\int_{\mathbb{R}} \sum_{(j,k)\in B_v} |v_{j,k}(\lambda)|^2\, |\lambda|\, d\lambda \Big)^{\frac{1}{2}}} \\ &= 2\pi \sup_{g\in\mathcal{S}_0(H_1)} \frac{\langle f | -iTg \rangle}{\|g\|_2} \\ &= 2\pi \sup_{g\in\mathcal{S}_0(H_1)} \frac{\langle iTf | g \rangle}{\|g\|_2}\end{aligned}$$

Since $\mathcal{S}_0(H_1)$ is dense in $L^2(H_1)$, $f \in D(A)$ if and only if iTf (defined as a distribution) is in L^2, i.e. $D(A) = D(iT)$. Moreover, for $f, g \in \mathcal{S}(H_1)$,

$$\begin{aligned}\langle Af | g \rangle &= \int_{F_1} \lambda\, d\nu_{f,g}(\lambda, \xi) \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}} \lambda \hat{f}(\lambda, j, k) \overline{\hat{g}(\lambda, j, k)}\, |\lambda|\, d\lambda \\ &= \langle iTf, g \rangle\ .\end{aligned}$$

Hence $A = iT$. A similar argument shows that $B = L$.

Applying (4.2) with $\varphi = \mu$, we then have, by (2.16),

$$\begin{aligned}\langle \mu(iT, L) f | g \rangle &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{j,k\in\mathbb{N}} \mu\big(\lambda, |\lambda|(2k+1)\big) \hat{f}(\lambda, j, k) \overline{\hat{g}(\lambda, j, k)}\, |\lambda|\, d\lambda \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \operatorname{tr} \big(\hat{f}(\lambda) M(\lambda) \hat{g}(\lambda)^* \big)\, |\lambda|\, d\lambda \\ &= \langle T_M f | g \rangle\ .\end{aligned}$$

This proves the equivalence between (iii) and (iv).

Finally, assume that $T = T_M$, with M satisfying (iii). Given $f \in \mathcal{S}(H_n)$, let f_m, $m \in \mathbb{Z}$, be the components of f of type m. Then Tf_m is also of type m by Lemma 4.1. Therefore,

$$\begin{aligned} T(f_\theta) &= \sum_{m\in\mathbb{Z}} T\big((f_m)_\theta\big) \\ &= \sum_{m\in\mathbb{Z}} e^{im\theta} T(f_m) \\ &= \sum_{m\in\mathbb{Z}} T(f_m)_\theta \\ &= (Tf)_\theta \ . \end{aligned}$$

This shows that (iii)⇒(i). □

5. Radiality in H_n

We extend the Fourier analysis on H_1 presented in the last three sections to H_n. A large part of what we are going to say is a straightforward adaptation of what has been presented in detail for H_1 (only with more complicated notation), and we leave the verification to the reader. The new facts will arise when we will present the different notions of radiality.

Let $\mathbf{j} = (j_1, \dots, j_n), \mathbf{k} = (k_1, \dots, k_n)$ be two n-tuples in $\mathbb{N}^n$. For $\lambda \in \mathbb{R} \setminus \{0\}$, we set

$$\Phi^\lambda_{\mathbf{j},\mathbf{k}}(z,t) = e^{i\lambda t} \prod_{i=1}^n \varphi^\lambda_{j_i,k_i}(z_i) \ . \tag{5.1}$$

For $f \in L^1(H_n)$, define

$$\hat f(\lambda, \mathbf{j}, \mathbf{k}) = \int_{H_n} f(z,t) \Phi^\lambda_{\mathbf{j},\mathbf{k}}(z,t)\, dz\, dt \ , \tag{5.2}$$

and, with $\Phi^\lambda(z,t) = \big(\Phi^\lambda_{\mathbf{j},\mathbf{k}}(z,t)\big)_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n}$,

$$\hat f(\lambda) = \big(\hat f(\lambda, \mathbf{j}, \mathbf{k})\big)_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n} = \int_{H_n} f(z,t)\Phi^\lambda(z,t)\, dz\, dt \ . \tag{5.3}$$

Theorem 5.1.

(1) $\Phi^\lambda(z,t)$ *is unitary for every* (z,t)*;*
(2) $\Phi^\lambda\big((z,t),(w,u)\big) = \Phi^\lambda(z,t)\Phi^\lambda(w,u)$, $\Phi^\lambda\big((z,t)^{-1}\big) = \Phi^\lambda(z,t)^*$*;*
(3) $L\Phi^\lambda_{\mathbf{j},\mathbf{k}} = |\lambda|\big(2|\mathbf{k}| + n\big)\Phi^\lambda_{\mathbf{j},\mathbf{k}}$ *and* $T\Phi^\lambda_{\mathbf{j},\mathbf{k}} = i\lambda\Phi^\lambda_{\mathbf{j},\mathbf{k}}$*;*
(4) $\widehat{f * g}(\lambda) = \hat f(\lambda)\hat g(\lambda)$ *for every* $f, g \in L^1(H_n)$ *and every* $\lambda \neq 0$*;*
(5) $\widehat{Lf}(\lambda, \mathbf{j}, \mathbf{k}) = |\lambda|\big(2|\mathbf{k}| + n\big)\hat f(\lambda, \mathbf{j}, \mathbf{k})$ *and* $\widehat{Tf}(\lambda, \mathbf{j}, \mathbf{k}) = -i\lambda \hat f(\lambda, \mathbf{j}, \mathbf{k})$*;*

(6) *if* $\lambda > 0$ *and* $\mathbf{e}_i = (0, 0, \cdots, 1, \ldots, 0)$, *with the* 1 *in the* i*-th position,*

$$\widehat{Z_i f}(\lambda, \mathbf{j}, \mathbf{k}) = \sqrt{\frac{k_i \lambda}{2}} \hat{f}(\lambda, \mathbf{j}, \mathbf{k} - \mathbf{e}_i) \ ,$$

$$\widehat{\bar{Z}_i f}(\lambda, \mathbf{j}, \mathbf{k}) = -\sqrt{\frac{(k_i + 1)\lambda}{2}} \hat{f}(\lambda, \mathbf{j}, \mathbf{k} + \mathbf{e}_i) \ ,$$

$$\widehat{Z_i^{(r)} f}(\lambda, \mathbf{j}, \mathbf{k}) = \sqrt{\frac{(j_i + 1)\lambda}{2}} \hat{f}(\lambda, \mathbf{j} + \mathbf{e}_i, \mathbf{k}) \ ,$$

$$\widehat{\bar{Z}_i^{(r)} f}(\lambda, \mathbf{j}, \mathbf{k}) = -\sqrt{\frac{j_i \lambda}{2}} \hat{f}(\lambda, \mathbf{j} - \mathbf{e}_i, \mathbf{k}) \ ,$$

(7) *if* $\lambda < 0$,

$$\widehat{Z_i f}(\lambda, \mathbf{j}, \mathbf{k}) = -\sqrt{\frac{(k_i + 1)|\lambda|}{2}} \hat{f}(\lambda, \mathbf{j}, \mathbf{k} + \mathbf{e}_i) \ ,$$

$$\widehat{\bar{Z}_i f}(\lambda, \mathbf{j}, \mathbf{k}) = \sqrt{\frac{k_i |\lambda|}{2}} \hat{f}(\lambda, \mathbf{j}, \mathbf{k} - \mathbf{e}_i) \ ,$$

$$\widehat{Z_i^{(r)} f}(\lambda, \mathbf{j}, \mathbf{k}) = -\sqrt{\frac{j_i |\lambda|}{2}} \hat{f}(\lambda, \mathbf{j} - \mathbf{e}_i, \mathbf{k}) \ ,$$

$$\widehat{\bar{Z}_i^{(r)} f}(\lambda, \mathbf{j}, \mathbf{k}) = \sqrt{\frac{(j_i + 1)|\lambda|}{2}} \hat{f}(\lambda, \mathbf{j} + \mathbf{e}_i, \mathbf{k}) \ ,$$

(8) *the following Plancherel formula holds, for* $f \in L^2(H_n)$*:*

$$\int_{H_n} |f(z,t)|^2 \, dz \, dt = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}^2 \, |\lambda|^n \, d\lambda \ .$$

(9) *the following inversion formula holds, for* $f \in \mathcal{S}(H_n)$*:*

$$f(z,t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \operatorname{tr}\left(\hat{f}(\lambda)\Phi^\lambda(z,t)^*\right) |\lambda|^n \, d\lambda \ .$$

Moreover, Theorem 3.1 has the same formulation on H_n.

The formulas at points (6) and (7) can be expressed in analogy with (3.3) and (3.4). Define the matrices $U_{\lambda,i}, L_{\lambda,i}$ with indices $(\mathbf{j}, \mathbf{k}) \in (\mathbb{N}^n)^2$ by

$$(U_{\lambda,i})_{\mathbf{j},\mathbf{k}} = \sqrt{\frac{k_i |\lambda|}{2}} \delta_{\mathbf{j},\mathbf{k}-\mathbf{e}_i} \ , \qquad (L_{\lambda,i})_{\mathbf{j},\mathbf{k}} = -\sqrt{\frac{(k_i + 1)|\lambda|}{2}} \delta_{\mathbf{j},\mathbf{k}+\mathbf{e}_i} \ . \tag{5.4}$$

Then, for $\lambda > 0$,

$$\widehat{Z_i f}(\lambda) = \hat{f}(\lambda) U_{\lambda,i} \ , \qquad \widehat{\bar{Z}_i f}(\lambda) = \hat{f}(\lambda) L_{\lambda,i} \ , \tag{5.5}$$

and, for $\lambda < 0$,

$$\widehat{Z_i f}(\lambda) = \hat{f}(\lambda) L_{\lambda,i} \ , \qquad \widehat{\bar{Z}_i f}(\lambda) = \hat{f}(\lambda) U_{\lambda,i} \ , \tag{5.6}$$

The representation-theoretic formulation of (1) and (2) can be given in terms of the unitary representations π^λ (for $\lambda \neq 0$) of H_n on $\ell^2(\mathbb{N}^n)$, such that $\pi^\lambda(z,t)$ is the operator defined by the matrix $\Phi^\lambda(z,t)$ in the canonical basis.

If $n > 1$, we must distinguish between two notions of radiality.

We say that a function $f(z,t)$ is *radial* if it only depends on $|z|$ and t. This is equivalent to saying that $f(Uz,t) = f(z,t)$ for every unitary $n \times n$ matrix U and every (z,t).

We say that a function $f(z,t)$ is *poliradial* if it depends on $|z_1|, \dots, |z_n|$ and t. This is equivalent to saying that $f(U_\theta z,t) = f(z,t)$ for every unitary diagonal $n\times n$ matrix,

$$U_\theta = \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_n} \end{pmatrix} ,$$

and every (z,t). We shall write $f_\theta(z,t)$ for $f(U_\theta z,t)$.

The natural extensions of Lemma 4.1 and Theorem 4.2 concerns poliradial functions and joint spectral multipliers of T and the *partial sub-Laplacians* $L_i = -X_i^2 - Y_i^2$. We group them in one statement, disregarding the first part of Lemma 4.1[19].

Theorem 5.2. *A function $f \in L^1(H_n)$ is poliradial if and only if $\hat f(\lambda)$ is diagonal for every $\lambda \neq 0$. For a bounded operator $Tf = f * u$ on $L^2(H_1)$, the following conditions are equivalent:*

(i) *T commutes with the rotations U_θ, i.e. $T(f_\theta) = (Tf)_\theta$ for every θ;*
(ii) *u is poliradial;*
(iii) *the Fourier multiplier $M(\lambda)$ is diagonal for a.e. λ;*
(iv) *$T = \mu(iT, L_1, \dots, L_n)$, for some bounded spectral multiplier $\mu(\lambda, \xi_1, \dots, \xi_n)$.*

If these conditions are satisfied, then $\mu(\lambda, \xi_1, \dots, \xi_n)$ and the diagonal entries $m(\lambda, \mathbf{k}, \mathbf{k})$ of $M(\lambda)$ are related by the identity

$$m(\lambda, \mathbf{k}, \mathbf{k}) = \mu\big(\lambda, |\lambda|(2k_1+1), \dots, |\lambda|(2k_n+1)\big) .$$

The condition on $\hat f(\lambda)$ characterizing radial functions must be more restrictive. We shall show that it consists in the fact that $\hat f(\lambda, \mathbf{k}, \mathbf{k}) = \hat f(\lambda, \mathbf{k}', \mathbf{k}')$ if $|\mathbf{k}| = |\mathbf{k}'|$.

Lemma 5.3. *A polyradial function $f \in \mathcal{S}(H_n)$ is radial if and only if for every i, i',*

$$Z_i Z_{i'}^{(r)} f = Z_{i'} Z_i^{(r)} f .$$

Proof. Write $f(z,t) = g(r_1, \dots, r_n, t)$, with $r_i = |z_i|^2$. Then f is radial if and only if g only depends on $r_1 + \cdots + r_n$ and t, i.e. if and only if $\partial_{r_i} g = \partial_{r_{i'}} g$ for every i, i'. Since

$$\bar z_i \partial_{z_{i'}} f - \bar z_{i'} \partial_{z_i} f = \bar z_i \bar z_{i'} \partial_{r_{i'}} g - \bar z_{i'} \bar z_i \partial_{r_i} g = \bar z_i \bar z_{i'} (\partial_{r_{i'}} - \partial_{r_i}) g ,$$

it follows that f is radial if and only if

$$(\bar z_i \partial_{z_{i'}} - \bar z_{i'} \partial_{z_i}) f = 0$$

[19] One can introduce the notion of type $\mathbf{m}$, for $\mathbf{m} \in \mathbb{Z}^n$, and extend that part too.

for every i, i'. Now,

$$(Z_i Z_{i'}^{(r)} - Z_{i'} Z_i^{(r)})f = \Big(\partial_{z_i} - \frac{i}{4}\bar z_i \partial_t\Big)\Big(\partial_{z_{i'}} + \frac{i}{4}\bar z_{i'}\partial_t\Big) f$$
$$- \Big(\partial_{z_{i'}} - \frac{i}{4}\bar z_{i'} \partial_t\Big)\Big(\partial_{z_i} + \frac{i}{4}\bar z_i\partial_t\Big) f$$
$$= \frac{i}{2}(\bar z_{i'}\partial_{z_i} - \bar z_i \partial_{z_{i'}})\partial_t f \ .$$

Therefore, if f is radial, $(Z_i Z_{i'}^{(r)} - Z_{i'} Z_i^{(r)})f = 0$. Conversely, if $Z_i Z_{i'}^{(r)} f = Z_{i'} Z_i^{(r)} f$, we obtain that $\partial_t f$ is radial. Since $\partial_t f \in \mathcal{S}(H_n)$,

$$f(z,t) = \int_{-\infty}^t \partial_t f(z,u)\, du$$

is also radial. □

Theorem 5.4. *A function $f \in \mathcal{S}(H_n)$ is radial if and only if $\hat f(\lambda)$ is diagonal and $\hat f(\lambda, \mathbf{k}, \mathbf{k})$ only depends on $|\mathbf{k}|$.*

Proof. Assume that f is radial. In particular, it is poliradial, hence $\hat f(\lambda)$ is diagonal. Moreover, by Lemma 5.3, $Z_i Z_{i'}^{(r)} f = Z_{i'} Z_i^{(r)} f$ for every i, i'. By Theorem 5.1, (6) and (7), we obtain that, for $\lambda > 0$,

$$(\widehat{Z_i Z_{i'}^{(r)}} f)(\lambda, \mathbf{k} - \mathbf{e}_{i'}, \mathbf{k} + \mathbf{e}_i) = \frac{\sqrt{(k_i+1)k_{i'}\lambda}}{2} \hat f(\lambda, \mathbf{k}, \mathbf{k}) \ ,$$

and

$$(\widehat{Z_{i'} Z_i^{(r)}} f)(\lambda, \mathbf{k}' - \mathbf{e}_i, \mathbf{k}' + \mathbf{e}_{i'}) = \frac{\sqrt{k_i'(k_{i'}'+1)\lambda}}{2} \hat f(\lambda, \mathbf{k}', \mathbf{k}') \ .$$

Fix two different indices $i, i'$and take $\mathbf{k}, \mathbf{k}'$ such that $\mathbf{k} + \mathbf{e}_i = \mathbf{k}' + \mathbf{e}_{i'}$. Then the two left-hand sides coincide, and so do the expressions under square root. We must then have

$$\hat f(\lambda, \mathbf{k}, \mathbf{k}) = \hat f(\lambda, \mathbf{k}', \mathbf{k}') \ .$$

This means that, moving one unit from one entry of $\mathbf{k}$ to another entry, the value fo the Fourier coefficient does not change. Repeating this operation, we can pass from any $\mathbf{k}$ to any other $\mathbf{k}'$ with the length.

For $\lambda < 0$ the argument is the same, and this proves the first part of the statement.

Suppose conversely that $f \in \mathcal{S}(H_n)$ and $\hat f(\lambda, \mathbf{j}, \mathbf{k}) = \delta_{\mathbf{j},\mathbf{k}} \nu(\lambda, |\mathbf{k}|)$. By the inversion formula,

$$f(z,t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{k\in\mathbb{N}} \nu(\lambda, k) \overline{\Psi_k^\lambda(z,t)}\, |\lambda|^n \, d\lambda \ ,$$

with

$$\Psi_k^\lambda(z,t) = \sum_{|\mathbf{k}|=k} \Phi_{\mathbf{k},\mathbf{k}}^\lambda(z,t) \ .$$

Since each $\Phi^\lambda_{\mathbf{k},\mathbf{k}}$ is poliradial, Ψ^λ_k is poliradial too. As in Section 3, for $\lambda > 0$,

$$Z_i \Phi^\lambda_{\mathbf{k},\mathbf{k}} = -\sqrt{\frac{k_i \lambda}{2}} \Phi^\lambda_{\mathbf{k},\mathbf{k}-\mathbf{e}_i}$$

and

$$Z_i^{(r)} \Phi^\lambda_{\mathbf{k},\mathbf{k}} = -\sqrt{\frac{(k_i+1)\lambda}{2}} \Phi^\lambda_{\mathbf{k}+\mathbf{e}_i,\mathbf{k}} \ .$$

Therefore

$$Z_i Z_{i'}^{(r)} \Phi^\lambda_{\mathbf{k},\mathbf{k}} = \frac{\sqrt{k_i(k_{i'}+1)}\lambda}{2} \Phi^\lambda_{\mathbf{k}+\mathbf{e}_{i'},\mathbf{k}-\mathbf{e}_i} \ .$$

Summing over $\mathbf{k}$ and setting $\mathbf{k}' = \mathbf{k}+\mathbf{e}_{i'}-\mathbf{e}_i$, we find that $Z_i Z_{i'}^{(r)} \Psi^\lambda_k = Z_{i'} Z_i^{(r)} \Phi^\lambda_k$. As in the proof of Lemma 5.3, this implies that $(\bar z_{i'} \partial_{z_i} - \bar z_i \partial_{z_{i'}}) \partial_t \Psi^\lambda_k = 0$. But, since $\Psi^\lambda_k(z,t) = \psi^\lambda_k(z) e^{i\lambda t}$, we conclude that ψ^λ_k only depends on $|z|$.

Finally, repeating the same argument for $\lambda < 0$, we conclude that Ψ^λ_k is radial for every $\lambda \neq 0$, and hence f is radial. □

We need at this point to describe the orthogonal projection P from $L^2(H_n)$ onto the subspace of radial functions. This requires some notions concerning the group $U(n)$ of unitary transformations of $\mathbb{C}^n$. With the natural topology, induced from $\mathbb{C}^{n^2}$, $U(n)$ is compact and the proup operations are continuous. The basic fact is the existence of a unique Borel probability measure m (called the *Haar measure*) which is invariant under left and right translations[20], i.e. such that

$$m(E) = m(gE) = m(Eg) = m(E^{-1}) \ ,$$

for every Borel set E and every $g \in U(n)$. As a consequence,

$$\begin{aligned} \int_{U(n)} f(x)\, dm(x) &= \int_{U(n)} f(gx)\, dm(x) \\ &= \int_{U(n)} f(xg)\, dm(x) = \int_{U(n)} f(x^{-1})\, dm(x) \ , \end{aligned}$$

for every integrable function f and every $g \in U(n)$.

Lemma 5.5. *The orthogonal projection P from $L^2(H_n)$ onto the subspace of radial functions $L^2_{\rm rad}(H_n)$ is given by*

$$Pf(z,t) = \int_{U(n)} f(Uz,t)\, dm(U) \ , \tag{5.7}$$

and

$$\widehat{Pf}(\lambda,\mathbf{j},\mathbf{k}) = \delta_{\mathbf{j},\mathbf{k}} \binom{|\mathbf{k}|+n-1}{n-1}^{-1} \sum_{\mathbf{k}':|\mathbf{k}'|=|\mathbf{k}|} \hat f(\lambda,\mathbf{k}',\mathbf{k}') \ . \tag{5.8}$$

P is well-defined and continuous on the following spaces:

(i) *$L^p(H_n)$, for $1 \le p \le \infty$;*
(ii) *$\mathcal{S}(H_n)$ into itself continuously, and it can therefore be extended by duality to $\mathcal{S}'(H_n)$;*
(iii) *$\mathcal{S}_0(H_n)$, defined as in Corollary 2.9.*

[20]See, e.g., the notes of the course *Analisi di Fourier non commutativa.*

If X denotes each of these spaces, the image of P in X is the subspace $X_{\rm rad}$ of all radial functions (or distributions) in the space itself. In particular $\mathcal{S}_{\rm rad}(H_n)$ and $\mathcal{S}_{0,{\rm rad}}(H_n)$ are dense in $L^p_{\rm rad}(H_n)$ for $1 \le p < \infty$.

Proof. Since

$$Pf(Uz,t) = \int_{U(n)} f(U'Uz,t)\,dm(U') = \int_{U(n)} f(U'z,t)\,dm(U') = Pf(z,t)\ ,$$

the image of P consists of radial functions. If f is already radial, it is clear that $Pf = f$. Hence $P^2 = P$ and the image consists of all radial functions. Given $f, g \in L^2(H_n)$,

$$\begin{aligned} \langle Pf|g\rangle &= \int_{H_n}\int_{U(n)} f(Uz,t)\overline{g(z,t)}\,dm(U)\,dz\,dt \\ &= \int_{U(n)}\int_{H_n} f(Uz,t)\overline{g(z,t)}\,dz\,dt\,dm(U) \\ &= \int_{U(n)}\int_{H_n} f(z,t)\overline{g(U^{-1}z,t)}\,dz\,dt\,dm(U) \\ &= \langle f|Pg\rangle\ . \end{aligned}$$

Hence P is an orthogonal projection.

Formula (5.8) is a direct consequence of Theorem 5.4 and the Plancherel formula, once we have observed that the binomial coefficient in front of the sum gives the number of $\mathbf{k}'$ with the same length of $\mathbf{k}$..

If $f \in L^p(H_n)$, the Minkowski integral inequality gives

$$\|Pf\|_p = \left\| \int_{U(n)} f_U\,dm(U)\right\|_p \le \int_{U(n)} \|f_U\|_p\,dm(U) = \|f\|_p\ .$$

Boundedness of P on $\mathcal{S}(H_n)$ follows from (5.7) by differentiation under the integral sign, and on $\mathcal{S}_0(H_n)$ from (5.8).

The last part is then obvious. □

Theorem 5.6. *For a bounded operator $Tf = f * u$ on $L^2(H_1)$, the following conditions are equivalent:*

(i) *T commutes with all the rotations of H_n, i.e. $T(f_U) = (Tf)_U$ for every $U \in U(n)$;*
(ii) *T maps radial functions into radial functions;*
(iii) *u is radial;*
(iv) *the Fourier multiplier $M(\lambda)$ has the form*

$$M(\lambda) = \Big(\delta_{\mathbf{j},\mathbf{k}}\nu\big(\lambda, |\mathbf{k}|\big)\Big)_{\mathbf{j},\mathbf{k}}\ ;$$

(v) *$T = \mu(iT, L)$, for some bounded spectral multiplier $\mu(\lambda,\xi)$ on F_n.*

If these conditions are satisfied, then the spectral multiplier $\mu(\lambda,\xi)$ and the diagonal entries $\nu(\lambda,k)$ of $M(\lambda)$ are related by the identity

$$\nu(\lambda,k) = \mu\big(\lambda, |\lambda|(2k+n)\big)\ .$$

Proof. The implication (i)⇒(ii) is trivial, because if f is radial, $(Tf)_U = T(f_U) = Tf$ for every U, hence Tf is radial. To prove that (ii)⇒(iii), take a radial function $\varphi \in \mathcal{S}(H_n)$ with $\int_{H_n} \varphi = 1$. Then the functions

$$\varphi_\varepsilon(z,t) = \frac{1}{\varepsilon^{2n+2}}\varphi\Big(\frac{z}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)$$

form an approximate identity as $\varepsilon \to 0$. In particular,

$$\lim_{\varepsilon\to 0} T(\varphi_\varepsilon) = \lim_{\varepsilon \to 0} u * \varphi_\varepsilon = u$$

in $\mathcal{S}'(H_n)$. But $T(\varphi_\varepsilon)$ is radial by assumption, hence u is also radial.

Assume now that u is radial. Then, for any $U \in U(n)$,

$$(Tf)_U = (u * f)_U = u_U * f_U = u * f_U = T(f_U) \ ,$$

which give the implication (iii)⇒(i).

The implication (iv)⇒(ii) is obvious by Theorem 5.4. To prove that (i)⇒(iv), we fix $\varphi \in \mathcal{D}(\mathbb{R}\setminus\{0\})$ and $k \in \mathbb{N}$, and define

$$\tilde\varphi_k(z,t) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}} \varphi(\lambda) \sum_{|\mathbf{k}|=k} \overline{\Phi^\lambda_{\mathbf{k},\mathbf{k}}(z,t)}\, |\lambda|^n \, d\lambda \ .$$

Then $\tilde\varphi_k \in \mathcal{S}_0(H_n)$ and, by Theorem 5.4, it is radial. Hence, since we know that (i) implies (ii), $T\tilde\varphi_k$ is also radial.

Since T commutes with diagonal rotations, we know from Theorem 5.2 that the Fourier multiplier of T is diagonal. Let $m(\lambda,\mathbf{k},\mathbf{k})$ be its entries on the diagonal. Then

$$\widehat{T\tilde\varphi_k}(\lambda,\mathbf{k},\mathbf{k}) = \widehat{\tilde\varphi_k}(\lambda,\mathbf{k},\mathbf{k})m(\lambda,\mathbf{k},\mathbf{k}) = \delta_{|\mathbf{k}|,k}\varphi(\lambda)m(\lambda,\mathbf{k},\mathbf{k}) \ ,$$

must depend only on λ and $|\mathbf{k}|$. By the arbitrarity of φ and k, $m(\lambda,\mathbf{k},\mathbf{k}) = \nu(\lambda,|\mathbf{k}|)$.

Finally the equivalence between (iv) and (v) is proved as in Theorem 4.2. □

Define

$$\Psi^\lambda_k(z,t) = \binom{k+n-1}{n-1}^{-1} \sum_{|\mathbf{k}|=k} \Phi^\lambda_{\mathbf{k},\mathbf{k}}(z,t) \ , \tag{5.9}$$

for $k \in \mathbb{N}$. It follows from this definition that

$$\Psi^\lambda_k(z,t) = e^{i\lambda t}\psi_k\big(|\lambda||z|^2\big) \ , \tag{5.10}$$

where ψ_k is a Schwartz function.

The relevance of the functions Ψ^λ_k in our context is clarified by the next statement.

Proposition 5.7. *Up to scalar multiples, Ψ_k^λ is the unique radial function in* $\mathrm{span}\,\{\Phi^\lambda_{\mathbf{k},\mathbf{k}} : |\mathbf{k}| = k\}$*. If $|\mathbf{k}| = k$, then $P\Phi^\lambda_{\mathbf{k},\mathbf{k}} = \Psi_k^\lambda$, and, if $f \in L^1(H_n)$,*

$$\widehat{Pf}(\lambda, \mathbf{k}, \mathbf{k}) = \int_{H_n} f(z,t)\Psi_k^\lambda(z,t)\,dz\,dt\ . \tag{5.11}$$

Proof. We start by proving (5.10). If $f \in L^1 \cap L^2(H_n)$, it follows directly from (5.8), and it extends to any integrable f by continuity, since $\Psi_k^\lambda \in L^\infty(H_n)$.

Take now
$$u = \sum_{|\mathbf{k}|=k} c_{\mathbf{k}}\Phi^\lambda_{\mathbf{k},\mathbf{k}} \in L^\infty(H_n)\ .$$

Then, for $f \in L^1(H_n)$,

$$\begin{aligned}\langle Pu, f\rangle &= \langle u, Pf\rangle\\ &= \sum_{|\mathbf{k}|=k} c_{\mathbf{k}}\widehat{Pf}(\lambda, \mathbf{k}, \mathbf{k})\\ &= \sum_{|\mathbf{k}|=k} c_{\mathbf{k}}\langle \Psi_k^\lambda, f\rangle\ ,\end{aligned}$$

by (5.11). Hence Pu is a scalar multiple of Ψ_k^λ, and the rest of the statement follows easily. □

6. Applications

Fourier analysis on H_n can be used to derive the regularity properties of the sub-Laplacian. We shall prove estimates showing that if f and $L^k f$ are in $L^2(H_n)$, then all the derivatives of f up to a certain order are (at least locally) in L^2. The general name for this type of results is *sub-elliptic estimates*, and they are typical of hypoelliptic operators.

In the second part of this Section, we characterize Fourier transforms of radial Schwartz functions.

The most efficient way to state hypoelliptic estimates for L is in terms of left-invariant vector fields, because they can be stated in global form.

Consider "higher-order left-invariant derivatives" of f, meant as expressions like $T^2X_1Y_2Y_1X_2^3 f$, or like $\bar Z_2^2 Z_1\bar Z_1 Tf$. The order of the factors in these expressions is relevant; we call them *non-commutative monomials*[21]. Clearly one can switch from non-commutative polynomials in the X_j, Y_j, T to non-commutative polynomials in the $Z_j, \bar Z_j, T$ by simple formal manipulations. The use of the $Z_j, \bar Z_j$ is preferable for us, because formulas for the Fourier transform are simpler. Observe also that, due to the relations
$$[X_j, Y_j] = T\ , \qquad [Z_j, \bar Z_j] = \frac{i}{2}T\ ,$$

[21] We mention that it follows from the Poincaré-Birckhoff-Witt theorem (see, e.g., the notes *Sub-Laplacians on nilpotent Lie groups*). that every left-invariant differential operator on H_n can be written as a non-commutative polynomial.

different non-commutative polynomials can give the same differential operator[22]. In particular, one can always replace a non-commutative polynomial by another one not containing T, without altering the differential operator.

Let $P(Z,\bar{Z})$ be a non-commutative polynomial in the $Z_j,\bar{Z}_j$ only. The degree of P is defined in the usual way. If T also appears in P, it must be counted as a factor of degree 2. With this convention, the *non-isotropic order* of a left-invariant diffeerential operator is well defined, as the degree of any non-commutative polynomial representing it.

Theorem 6.1. *Let $N\in\mathbb{N}$, and assume that f and $L^{N/2}f$ are in $L^2(H_n)$. Then also $P(Z,\bar{Z},T)f\in L^2(H_n)$ for every P of non-isotropic degree at most N, and*

$$\|Pf\|_2\le C_N\big(\|f\|_2+\|L^{N/2}f\|_2\big)\ .$$

Moreover, all partial derivatives of f up to the order $[N/2]$ are locally in L^2.

Proof. It is sufficient to take a monomial $P=P(Z,\bar{Z})$ in the $Z_j,\bar{Z}_j$ of degree $d\le 2N$. For each $\lambda\ne 0$, $\widehat{Pf}(\lambda)$ is given by

$$\widehat{Pf}(\lambda)=\hat{f}(\lambda)P(U_\lambda,L_\lambda)\ ,\qquad\text{or}\qquad \widehat{Pf}(\lambda)=\hat{f}(\lambda)P(L_\lambda,U_\lambda)\ ,$$

depending on the signum of λ, where U_λ,L_λ stand for the matrices $U_{\lambda,i},L_{\lambda,i}$ in (5.4). The diagonal matrix D_λ with

$$(D_\lambda)_{\mathbf{k},\mathbf{k}}=1+\big(|\lambda|(2|\mathbf{k}|+n)\big)^{N/2}$$

is such that

$$\big((I+\widehat{L^{N/2})f}\big)(\lambda)=\hat{f}(\lambda)D_\lambda\ .$$

Therefore, if

$$M_\lambda=\begin{cases}P(U_\lambda,L_\lambda)D_\lambda^{-1} & \text{if }\lambda>0\ ,\\ P(L_\lambda,U_\lambda)D_\lambda^{-1} & \text{if }\lambda<0\ ,\end{cases}$$

we have that

$$\widehat{Pf}(\lambda)=\big((I+\widehat{L^{N/2})f}\big)(\lambda)M_\lambda\ .$$

Observe that both $P(U_\lambda,L_\lambda)$ and $P(L_\lambda,U_\lambda)$ have non-zero entries only on one single diagonal, and the $\mathbf{k}$-th entry along this diagonal is dominated by $\big(|\lambda|(|\mathbf{k}|+1)\big)^{d/2}$. Therefore the norms $\|M_\lambda\|_{\mathcal{L}(\ell^2)}$ are uniformly bounded in λ. It follows from Theorem 3.1 (which holds also on H_n, as already mentioned in Section 5) that

$$\|Pf\|_2\le C\|f+L^{N/2}f\|_2\le C\big(\|f\|_2+\|L^{N/2}f\|_2\big)\ .$$

The last part of the statement follows from the fact that, by the explicit expression of the vector fields, L^2-estimates for $Z_ig,\bar{Z}_ig,Tg$ imply local L^2-estimates for $\partial_{z_i}g,\partial_{\bar{z}_i}g,\partial_tg$. An induction argument shows that, in our hypotheses, we can control locally all partial derivatives of f up to order $[N/2]$. □

A similar argument, made simpler by the fact that all the matrices involved are diagonal, gives the following result.

[22]In more correct terms, a non-commutative polynomial is an element of the tensor algebra $\mathcal{T}$ over $\mathbb{C}$ generated by the $Z_j,\bar{Z}_j,T$. The map assigning to each element of $\mathcal{T}$ the corresponding composition of left-invariant vector fields on H_n has a kernel, equal to the ideal $\mathcal{I}$ generated by the elements $Z_j\otimes\bar{Z}_k-\bar{Z}_k\otimes Z_j-\frac{i}{2}\delta_{j,k}T$. The quotient $\mathcal{T}/\mathcal{I}$ identifies left-invariant differential operators on H_n and is called the *universal enveloping algebra* of the Lie algebra $\mathfrak{h}_n$.

Theorem 6.2. . *Assume that f and $L^s f$ are in $L^2(H_n)$ for some $s \in \mathbb{R}^+$. Then $L^{s_1}(i^{-1}T)^{s_2-s_1} f \in L^2(H_n)$, for $s_1, s_2 \geq 0$, $s_2 \leq s$, and*

$$\|L^{s_1}(i^{-1}T)^{s_2-s_1} f\|_2 \leq C_s\big(\|f\|_2 + \|L^s f\|_2\big) \ .$$

A further refinement gives global Sobolev estimates for poliradial functions.

Corollary 6.3. *Suppose that f is poliradial and that $f, L^{N/2} f \in L^2(H_n)$. Then $\partial_z^\alpha \partial_{\bar z}^\beta \partial_t^m f \in L^2(H_n)$ for all multi-indices α, β and all $m \in \mathbb{N}$ such that $|\alpha| + |\beta| + 2m \leq N$, and*

$$\|\partial_z^\alpha \partial_{\bar z}^\beta \partial_t^m f\|_2 \leq C_N\big(\|f\|_2 + \|L^{N/2} f\|_2\big) \ .$$

Proof. Consider the *right-invariant sub-Laplacian*

$$L^{(r)} = -\sum_{i=1}^n \big((X_i^{(r)})^2 + (Y_i^{(r)})^2\big) \ .$$

If $\widehat{Lf}(\lambda) = \hat f(\lambda) M(\lambda)$, then $\widehat{L^{(r)} f}(\lambda) = M(\lambda) \hat f(\lambda)$. Since $M(\lambda)$ is diagonal, it commutes with $\hat f(\lambda)$ if f is poliradial. Hence $L^{(r)} f = Lf$. Since the analogue of Theorem 6.1 also holds for right-invariant operators, we have that $P(Z_i^{(r)}, \bar Z_i^{(r)}) f \in L^2(H_n)$ for every non-commutative monomial P of degree at most $2N$.

Starting from the identities

$$Z_i + Z_i^{(r)} = 2\partial_{z_i} \ , \qquad \bar Z_i + \bar Z_i^{(r)} = 2\partial_{\bar z_i} \ ,$$

and expressing ∂_t as a commutator, we reduce the problem of estimating $\partial_z^\alpha \partial_{\bar z}^\beta \partial_t^m f$ to proving that

$$\|Q(Z, \bar Z, Z^{(r)}, \bar Z^{(r)}) f\|_2 \leq C\big(\|f\|_2 + \|L^{N/2} f\|_2\big) \ ,$$

if $Q = Q(Z, \bar Z, Z^{(r)}, \bar Z^{(r)})$ is a non-commutative monomial of degree at most N.

It is a general fact (and it can be easily verified from the explicit formulas or from the Fourier transforms) that right-invariant vector fields commute with left-invariant ones. Hence they also commute with L, and, by Theorem 3.1 extended to H_n, with its spectral projection, and ultimately with its fractional powers. The same can be said interchanging left and right.

We can therefore write

$$Q = Q_1(Z, \bar Z) Q_2(Z^{(r)} \bar Z^{(r)}) \ ,$$

where Q_1 and Q_2 are monomials of degrees d_1 and d_2 respectively, with $d_1 + d_2 \leq N$. By Theorems 6.1 and 6.2,

$$\begin{aligned}
\|Qf\|_2 &\leq C\big(\|Q_2(Z^{(r)} \bar Z^{(r)}) f\|_2 + \|L^{d_1/2} Q_2(Z^{(r)} \bar Z^{(r)}) f\|_2\big) \\
&\leq C\big(\|f\|_2 + \|(L^{(r)})^{d_2/2} f\|_2 + \|Q_2(Z^{(r)} \bar Z^{(r)}) L^{d_1/2} f\|_2\big) \\
&\leq C\big(\|f\|_2 + \|(L^{(r)})^{d_2/2} f\|_2 + \|L^{d_1/2} f\|_2 + \|(L^{(r)})^{d_2/2} L^{d_1/2} f\|_2\big) \\
&= C\big(\|f\|_2 + \|L^{d_2/2} f\|_2 + \|L^{d_1/2} f\|_2 + \|L^{(d_1+d_2)/2} f\|_2\big) \\
&\leq C\big(\|f\|_2 + \|L^{N/2} f\|_2\big) \ . \qquad \square
\end{aligned}$$

We conclude this section with the proof of some identities providing conditions on the Fourier transform side that correspond to decay at infinity of the function. In $\mathbb{R}^n$ this is provided by formulas like $\mathcal{F}(x_j f)(\xi) = i\partial_{\xi_j}\hat{f}(\xi)$. In H_n, staying within radial functions, we look for formulas relating $\hat{f}(\lambda)$ with $\widehat{|z|^2 f}(\lambda)$ and with $\widehat{tf}(\lambda)$.

If f is radial, we set

$$\tilde{f}(\lambda, k) = \hat{f}(\lambda, \mathbf{k})$$

if $|\mathbf{k}| = k$.

Lemma 6.4. *Assume that $f \in \mathcal{S}(H_n)$ is radial. Then*

$$\widetilde{(|z|^2 f)}(\lambda, k) = \frac{2}{|\lambda|}\big((2k+n)\tilde{f}(\lambda, k) - (k+n)\tilde{f}(\lambda, k+1) - k\tilde{f}(\lambda, k-1)\big) \ ,$$

and

$$\widetilde{(tf)}(\lambda, k) = -i\partial_\lambda \tilde{f}(\lambda, k) - \frac{i}{2\lambda}\big(n\tilde{f}(\lambda, k) - (k+n)\tilde{f}(\lambda, k+1) + k\tilde{f}(\lambda, k-1)\big) \ .$$

Proof. Take $f \in \mathcal{S}(H_n)$, not necessarily radial. For each $i = 1, \dots, n$,

$$Z_i - Z_i^{(r)} = -\frac{i}{2}\bar{z}_i\partial_t \ , \qquad \bar{Z}_i - \bar{Z}_i^{(r)} = \frac{i}{2}z_i\partial_t \ .$$

Therefore

$$\widehat{(\bar{z}_i\partial_t f)}(\lambda, \mathbf{j}, \mathbf{k}) = \begin{cases} i\sqrt{2k_i\lambda}\hat{f}(\lambda, \mathbf{j}, \mathbf{k}-\mathbf{e}_i) - i\sqrt{2(j_i+1)\lambda}\hat{f}(\lambda, \mathbf{j}+\mathbf{e}_i, \mathbf{k}) & \text{if } \lambda > 0 \ , \\ -i\sqrt{2(k_i+1)|\lambda|}\hat{f}(\lambda, \mathbf{j}, \mathbf{k}+\mathbf{e}_i) + i\sqrt{2j_i|\lambda|}\hat{f}(\lambda, \mathbf{j}-\mathbf{e}_i, \mathbf{k}) & \text{if } \lambda < 0 \ , \end{cases}$$

and

$$\widehat{(z_i\partial_t f)}(\lambda, \mathbf{j}, \mathbf{k}) = \begin{cases} i\sqrt{2(k_i+1)\lambda}\hat{f}(\lambda, \mathbf{j}, \mathbf{k}+\mathbf{e}_i) - i\sqrt{2j_i\lambda}\hat{f}(\lambda, \mathbf{j}-\mathbf{e}_i, \mathbf{k}) & \text{if } \lambda > 0 \ , \\ -i\sqrt{2k_i|\lambda|}\hat{f}(\lambda, \mathbf{j}, \mathbf{k}-\mathbf{e}_i) + i\sqrt{2(j_i+1)|\lambda|}\hat{f}(\lambda, \mathbf{j}+\mathbf{e}_i, \mathbf{k}) & \text{if } \lambda < 0 \ . \end{cases}$$

Therefore, for every $\lambda \neq 0$,

$$\begin{aligned}\widehat{(|z_i|^2\partial_t^2 f)}(\lambda, \mathbf{j}, \mathbf{k}) = &-2(j_i+k_i+1)|\lambda|\hat{f}(\lambda, \mathbf{j}, \mathbf{k}) \\ &+ 2|\lambda|\sqrt{(j_i+1)(k_i+1)}\hat{f}(\lambda, \mathbf{j}+\mathbf{e}_i, \mathbf{k}+\mathbf{e}_i) \\ &+ 2|\lambda|\sqrt{j_ik_i}\hat{f}(\lambda, \mathbf{j}-\mathbf{e}_i, \mathbf{k}-\mathbf{e}_i) \ .\end{aligned}$$

Summing over i and restricting to f radial and $\mathbf{j} = \mathbf{k}$, we obtain that

$$\begin{aligned}\widetilde{(|z|^2\partial_t^2 f)}(\lambda, k) = &-2(2k+n)|\lambda|\tilde{f}(\lambda, k) + 2|\lambda|(k+n)\tilde{f}(\lambda, k+1) \\ &+ 2|\lambda|k\tilde{f}(\lambda, k-1) \ .\end{aligned}$$

But $(\widetilde{|z|^2\partial_t^2 f}) = (\widetilde{\partial_t^2 |z|^2 f}) = -\lambda^2(\widetilde{|z|^2 f})$, and this proves the first formula.

In order to prove the second formula, we start from the derivative of $\tilde{f}(\lambda, k)$ in λ. By (5.10) and (5.11), if f is radial,

$$\tilde{f}(\lambda, k) = \int_{H_n} f(z,t) e^{i\lambda t} \psi_k\big(|\lambda||z|^2\big)\, dz\, dt \ ,$$

so that

$$\begin{aligned}\partial_\lambda \tilde{f}(\lambda, k) = i &\int_{H_n} t f(z,t) e^{i\lambda t}\psi_k\big(|\lambda||z|^2\big)\, dz\, dt \\ &+ \operatorname{sgn}\lambda \int_{H_n} |z|^2 f(z,t) e^{i\lambda t}\psi_k'\big(|\lambda||z|^2\big)\, dz\, dt \\ &= i\widetilde{(tf)}(\lambda, k) + \operatorname{sgn}\lambda \int_{H_n} |z|^2 f(z,t) e^{i\lambda t}\psi_k'\big(|\lambda||z|^2\big)\, dz\, dt \ .\end{aligned}$$

Observing that

$$\sum_{i=1}^n z_i \partial_{z_i} \psi_k\big(|\lambda||z|^2\big) = |\lambda||z|^2 \psi_k'\big(|\lambda||z|^2\big) \ ,$$

we obtain

$$\begin{aligned}\partial_\lambda \tilde{f}(\lambda, k) &= i\widetilde{(tf)}(\lambda, k) \\ &\qquad - \frac{1}{\lambda}\int_{H_n}\sum_{i=1}^n \partial_{z_i}\big(z_i f(z,t)\big)\Psi_k^\lambda(z,t)\, dz\, dt \\ &= i\widetilde{(tf)}(\lambda, k) - \frac{1}{\lambda}\Big(\widetilde{\sum_{i=1}^n \partial_{z_i}(z_i f)}\Big)(\lambda, k)\end{aligned} \tag{6.1}$$

(observe that $\sum \partial_{z_i}\big(z_i f\big)$ is also radial).

Using again the fact that $\bar{Z}_i - \bar{Z}_i^{(r)} = \frac{i}{2} z_i \partial_t$, and that $Z_i + Z_i^{(r)} = 2\partial_{z_i}$, we have

$$i\partial_t \partial_{z_i}(z_i f) = (Z_i + Z_i^{(r)})(\bar{Z}_i - \bar{Z}_i^{(r)}) f \ .$$

With computations similar to the previous ones, we find that, for $\lambda \neq 0$,

$$\lambda \widehat{\partial_{z_i}(z_i f)}(\lambda, \mathbf{k}, \mathbf{k}) = \frac{\lambda}{2}\big(\hat{f}(\lambda, \mathbf{k}, \mathbf{k}) + k_i \hat{f}(\lambda, \mathbf{k}-\mathbf{e}_i, \mathbf{k}-\mathbf{e}_i) - (k_i+1)\hat{f}(\lambda, \mathbf{k}+\mathbf{e}_i, \mathbf{k}+\mathbf{e}_i)\big) \ .$$

Summing over i,

$$\Big(\widetilde{\sum_{i=1}^n \partial_{z_i}(z_i f)}\Big)(\lambda, k) = \frac{1}{2}\big(n\tilde{f}(\lambda, k) + k\tilde{f}(\lambda, k-1) - (k+n)\tilde{f}(\lambda, k+1)\big) \ .$$

Inserting this identity in (6.1), we find the stated formula. □

The formula indicated in Lemma 6.4 are rather complicated, since they involve strange second-order differences in k. A considerable simplification occurs if we combine the two formulas to express the Fourier transform of $\big(\frac{|z|^2}{4} \pm it\big)f$. Setting $w_\pm(z,t) = \frac{|z|^2}{4} \pm it$, we have the following identities:

$$\widetilde{(w_+ f)}(\lambda,k) = \begin{cases} \partial_\lambda \tilde f(\lambda,k) - \dfrac{k+n}{\lambda}\big(\tilde f(\lambda,k+1) - \tilde f(\lambda,k)\big) & \text{if } \lambda > 0\ , \\ \partial_\lambda \tilde f(\lambda,k) - \dfrac{k}{\lambda}\big(\tilde f(\lambda,k) - \tilde f(\lambda,k-1)\big) & \text{if } \lambda < 0\ , \end{cases} \tag{6.2}$$

and

$$\widetilde{(w_- f)}(\lambda,k) = \begin{cases} -\partial_\lambda \tilde f(\lambda,k) + \dfrac{k}{\lambda}\big(\tilde f(\lambda,k) - \tilde f(\lambda,k-1)\big) & \text{if } \lambda > 0\ , \\ -\partial_\lambda \tilde f(\lambda,k) + \dfrac{k+n}{\lambda}\big(\tilde f(\lambda,k+1) - \tilde f(\lambda,k)\big) & \text{if } \lambda < 0\ . \end{cases} \tag{6.3}$$

CHAPTER V
SPECTRAL MULTIPLIERS OF THE SUB-LAPLACIAN

1. The heat kernel on H_n

Our discussion of spectral L^p-multipliers of L on H_n begins with the study of the heat kernel. This is defined as the kernel $p_s(z,t)$, for $s>0$, such that

$$f * p_s = e^{-sL} f \ .$$

We recall some general facts about semigroups and evolution equations on Lie groups.

The semigroup property $e^{-(s_1+s_2)L} = e^{-s_1 L} e^{-s_2 L}$ implies that

$$p_{s_1+s_2} = p_{s_1} * p_{s_2}$$

for every $s_1, s_2 > 0$. This identity extends to $s=0$ if we set $p_0 = \delta_0$, consistently with the fact that $e^{0L} = I$. Moreover, the map $s \longmapsto e^{-sL}$ is continuous from $[0,\infty)$ to $\mathcal{L}(L^2)$ with the strong topology.

The identity $\frac{d}{ds} e^{-sL} f = L e^{-sL} f$ holds for every $s>0$ and $f \in L^2(H_n)$. Therefore, the function $u(s,z,t) = e^{-sL} f(z,t)$ satisfies the homogeneous *heat equation*

$$(\partial_s + L) u = 0 \ ,$$

which implies that

$$(\partial_s + L) p_s = 0$$

in the sense of distributions.

By the already cited Hörmander's theorem[23], the operator $\partial_s + L$ is hypoelliptic, which implies that $p_s(z,t)$ is smooth in (s,z,t).

Finally, it follows from Hunt's theorem[24] that the p_s define probability measures, i.e.

$$p_s(z,t) \geq 0 \ , \qquad \int_{H_n} p_s(z,t)\, dz\, dt = 1 \ . \tag{1.1}$$

We shall prove now further properties of the p_s, using the Fourier analysis on H_n.

[23] See L. Hörmander, *Hypoelliptic second-order differential equations*, Acta Math. vol.119 (1967), p.147-171.

[24] See A. Hunt, *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. vol.81 (1956), p.264-293.

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By Theorem 5.6 in Chapter IV,

$$\widetilde{p_s}(\lambda,k) = e^{-s|\lambda|(2k+n)} \ ,$$

so that, by the inversion formula in Theorem 5.1 of Chapter IV,

$$p_s(z,t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{k\in\mathbb{N}} \binom{k+n-1}{n-1} e^{-s|\lambda|(2k+n)} \Psi_k^\lambda(z,t)\, |\lambda|^n \, d\lambda \ . \tag{1.2}$$

We show that p_s is a Schwartz function, and, more precisely that it decays exponentially at infinity.

Proposition 1.1. *For* $\lambda \neq 0$*,*

$$\sum_{k\in\mathbb{N}} \binom{k+n-1}{n-1} e^{-s|\lambda|(2k+n)} \Psi_k^\lambda(z,t) = \frac{2^n e^{i\lambda t}}{\sinh^n(|\lambda|s)} e^{-\frac{|\lambda||z|^2}{\tanh(|\lambda|s)}} \ .$$

The proof is based on the following lemma.

Lemma 1.2. *For* $0 \le r < 1$ *and* $z \in \mathbb{C}$*,*

$$\sum_{k\in\mathbb{N}} r^k \varphi_{k,k}(z) = \frac{1}{1-r} e^{-\frac{1+r}{4(1-r)}|z|^2} \ .$$

Proof. By (3.3) and (3.4) in Chapter I,

$$h_{j,k}(z) = (-2)^j e^{\frac{|z|^2}{4}} \partial_{\bar z}^k \partial_z^j e^{-\frac{|z|^2}{2}} \ ,$$

and, by (2.2) in Chapter IV,

$$\varphi_{k,k}(z) = \frac{(-2)^k}{k!} e^{\frac{|z|^2}{4}} \partial_{\bar z}^k \partial_z^k e^{-\frac{|z|^2}{2}} = \frac{1}{2^k k!} e^{\frac{|z|^2}{4}} \Delta^k e^{-\frac{|z|^2}{2}} \ .$$

Then

$$\sum_{k\in\mathbb{N}} r^k \varphi_{k,k}(z) = e^{\frac{|z|^2}{4}} \sum_{k\in\mathbb{N}} \frac{r^k}{2^k k!} \Delta^k e^{-\frac{|z|^2}{2}} = e^{\frac{|z|^2}{4}} F(r,z) \ ,$$

where the series converges for $r<1$ because $|\varphi_{j,k}(z)| \le 1$. Recalling that, for $s>0$,

$$\mathcal{F}\big(\frac{1}{2\pi s} e^{-\frac{|z|^2}{4s}}\big) = e^{-s|\zeta|^2} \ , \tag{1.3}$$

we have

$$\mathcal{F}(\Delta^k e^{-\frac{|z|^2}{2}}) = \pi |\zeta|^{2k} e^{-\frac{|\zeta|^2}{2}} \ .$$

Therefore

$$\begin{aligned} F(r,z) &= \pi \mathcal{F}^{-1}\Big(\sum_{k\in\mathbb{N}} \frac{r^k}{2^k k!} |\zeta|^{2k} e^{-\frac{|\zeta|^2}{2}} \Big) \\ &= \pi \mathcal{F}^{-1}(e^{-\frac{1-r}{2}|\zeta|^2}) \\ &= \frac{1}{1-r} e^{-\frac{|z|^2}{2(1-r)}} \ . \end{aligned}$$

Multiplying by $e^{\frac{|z|^2}{4}}$, the proof is completed. □

Proof of Proposition 1.1. Assume first that $\lambda = 1$. By (5.9) and (5.10) in Chapter IV, and by Lemma 1.2,

$$\begin{aligned}\sum_{k\in\mathbb{N}}\binom{k+n-1}{n-1}e^{-s(2k+n)}\Psi_k^1(z,t) &= e^{it}\sum_{\mathbf{k}\in\mathbb{N}^n}e^{-s(2|\mathbf{k}|+n)}\varphi_{\mathbf{k},\mathbf{k}}(z)\\ &= e^{it}\prod_{i=1}^n\Big(\sum_{k\in\mathbb{N}}e^{-s(2k+1)}\varphi_{k,k}(z_i)\Big)\\ &= e^{it}\prod_{i=1}^n\frac{e^{-s}}{1-e^{-2s}}e^{-\frac{1+e^{-2s}}{1-e^{-2s}}|z_i|^2}\\ &= \frac{e^{it}}{(e^s-e^{-s})^n}e^{-\frac{e^s+e^{-s}}{e^s-e^{-s}}|z|^2}\\ &= \frac{2^n e^{it}}{\sinh^n s}e^{-\frac{|z|^2}{\tanh s}}\ .\end{aligned}$$

For a generic λ, it is sufficient to replace t by λt, s by $|\lambda|s$, and z by $\lambda^{\frac12}z$ or $|\lambda|^{\frac12}\bar z$, depending on the signum of λ. □

By (1.2), we then have

$$\begin{aligned}p_s(z,t) &= \frac{1}{2\pi^{n+1}}\int_{\mathbb{R}}\frac{e^{i\lambda t}}{\sinh^n(|\lambda|s)}e^{-\frac{|\lambda||z|^2}{\tanh(|\lambda|s)}}\,|\lambda|^n\,d\lambda\\ &= \frac{1}{2\pi^{n+1}}\int_{\mathbb{R}}\frac{e^{i\lambda t}}{\sinh^n(\lambda s)}e^{-\frac{\lambda|z|^2}{\tanh(\lambda s)}}\,\lambda^n\,d\lambda\ .\end{aligned} \tag{1.4}$$

Observe that

$$p_s(z,t) = s^{-(n+1)}p_1\big(s^{-\frac12}z, s^{-1}t\big)\ , \tag{1.5}$$

which is the analogue of the scaling property for the Gauss-Weierstrass heat kernel for the Laplacian in $\mathbb{R}^n$.

Corollary 1.3. *For $s>0$, $p_s\in\mathcal{S}(H_n)$.*

Proof. We can take $s=1$ and prove that $\mathcal{F}_t p_1$ is a Schwartz function. By (1.4),

$$\mathcal{F}_t p_1(z,\lambda) = \frac{\lambda^n}{\pi^n\sinh^n\lambda}e^{-\frac{\lambda|z|^2}{\tanh\lambda}}\ .$$

This function is analytic in (z,λ) and all of its derivatives decay rapidly at infinity. □

We introduce on H_n the homogeneous norm

$$\big|(z,t)\big| = \big(|z|^4+16t^2\big)^{\frac14} = \big||z|^2+4it\big|^{\frac12} = 2|w_+|^{\frac12}\ . \tag{1.6}$$

Proposition 1.4. *The homogeneous norm* (1.6) *satisfies the inequality*

$$\big|(z,t)(w,u)\big| \le \big|(z,t)\big| + \big|(w,u)\big| \ .$$

Proof. We have

$$\begin{aligned}\big|(z,t)(w,u)\big|^2 &= \Big||z+w|^2 + 4i\big(t+u-\frac{1}{2}\Im\mathrm{m}\langle z|w\rangle\big)\Big| \\ &= \big||z|^2+|w|^2+2\langle w|z\rangle + 4i(t+u)\big| \\ &\le \big||z|^2+4it\big| + \big||w|^2+4iu\big| + 2|z||w| \\ &\le \big|(z,t)\big|^2 + \big|(w,u)\big|^2 + 2\big|(z,t)\big|\big|(w,u)\big| \\ &= \Big(\big|(z,t)\big| + \big|(w,u)\big|\Big)^2 \ . \qquad \square\end{aligned}$$

The following estimate can be seen as a Gaussian estimate w.r. to the homogeneous norm (1.6).

Corollary 1.5. *There is* $a>0$ *such that*

$$p_1(z,t) \le Ce^{-a\left|(z,t)\right|^2} \ .$$

Proof. From (1.4) we obtain that

$$p_1(z,t) \le \frac{1}{\pi^{n+1}} \int_0^{+\infty} \frac{\lambda^n}{\sinh^n\lambda} e^{-\frac{\lambda|z|^2}{\tanh\lambda}}\, d\lambda \ .$$

Since the function $\lambda/\tanh\lambda$ is bounded from below by 1, we have

$$p_1(z,t) \le \frac{e^{-b|z|^2}}{\pi^{n+1}} \int_0^{+\infty} \frac{\lambda^n}{\sinh^n\lambda}\, d\lambda = Ce^{-b|z|^2} \ . \tag{1.7}$$

Consider now

$$\mathcal{F}p_1(\zeta,\lambda) = \frac{1}{\cosh^n\lambda} e^{-\frac{\tanh\lambda}{4\lambda}|\zeta|^2} \ .$$

It can be extended analytically in λ to the strip $\big\{\lambda+i\tau : |\tau| < \frac{\pi}{2}\big\}$. For $|\tau| \le \frac{\pi}{2}-\delta$, $\mathcal{F}p_1(\zeta,\lambda+i\tau)$ is integrable in (ζ,λ) and rapidly decreasing in λ, uniformly in ζ and τ. Therefore, a change of contour integration in the plane $\lambda+i\tau$ gives

$$\begin{aligned}p_1(z,t) &= \frac{1}{(2\pi)^{2n+1}} \int_{\mathbb{C}^n\times\mathbb{R}} \mathcal{F}p_1(\zeta,\lambda+i\tau) e^{i\left(t(\lambda+i\tau)+\Re\mathrm{e}\langle z|\zeta\rangle\right)}\, d\zeta\, d\lambda \\ &= \frac{e^{-\tau t}}{(2\pi)^{2n+1}} \int_{\mathbb{C}^n\times\mathbb{R}} \mathcal{F}p_1(\zeta,\lambda+i\tau) e^{i\left(t\lambda+\Re\mathrm{e}\langle z|\zeta\rangle\right)}\, d\zeta\, d\lambda \\ &\le Ce^{-\tau t} \int_{\mathbb{C}^n\times\mathbb{R}} \big|\mathcal{F}p_1(\zeta,\lambda+i\tau)\big|\, d\zeta\, d\lambda \\ &\le C_\tau e^{-\tau t} \ .\end{aligned}$$

Combining this with (1.7), we obtain that, for some $\alpha>0$,

$$p_1(z,t) \le Ce^{-\alpha\left(|z|^2+|t|\right)} \ .$$

This gives the conclusion, since $\big|(z,t)\big|^2 \sim |z|^2+|t|$. $\square$

2. Smooth multipliers and Schwartz kernels

In his Section, we use the estimates obtained for the heat kernel to prove that certain spectral multipliers of L correspond to convolution with a Schwartz kernel.

Theorem 2.1. *Let m be a smooth function supported on the interval $\left[\frac{1}{4}, 4\right]$. Then $m(L)f = f * k$, where $k \in \mathcal{S}(H_n)$.*

This theorem will be proved in a few steps. The multiplier must be first decomposed appropriately, in order to take advantage of the heat kernel estimates.

Let $\mu(\tau) = m(-\log \tau)$. Then μ is supported on $[e^{-4}, e^{-\frac{1}{4}}] \subset [-\pi, \pi]$. Extending μ as a periodic function of period 2π, it can be expanded into a Fourier series

$$\mu(\tau) = \sum_{j \in \mathbb{Z}} a_j e^{ij\tau} \ ,$$

with rapidly decreasing coefficients. Since $\sum_j a_j = \mu(0) = 0$, we can write

$$\mu(\tau) = \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j (e^{ij\tau} - 1) \ ,$$

hence, for $\xi > 0$,

$$m(\xi) = \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j (e^{ije^{-\xi}} - 1) \ . \tag{2.1}$$

We set

$$m_j(\xi) = e^{ije^{-\xi}} - 1 = \sum_{\ell=1}^{\infty} \frac{(ij)^\ell}{\ell!} e^{-\ell\xi} \ ,$$

so that

$$m_j(L) = \sum_{\ell=1}^{\infty} \frac{(ij)^\ell}{\ell!} e^{-\ell L} \ .$$

This makes sense, because the series converges in the operator norm in $\mathcal{L}(L^2)$. Therefore, the convolution kernel k_j of $m_j(L)$ equals

$$k_j(z,t) = \sum_{\ell=1}^{\infty} \frac{(ij)^\ell}{\ell!} p_\ell(z,t) \ . \tag{2.2}$$

Lemma 2.2. *We have*

$$\|k_j\|_2 \leq C|j| \ .$$

Proof. Let

$$\tilde{m}_j(\xi) = \frac{e^{ije^{-\xi}} - 1}{e^{-\xi}} \ .$$

Observe that $\tilde{m}_j$ is bounded on $\mathbb{R}^+$, and

$$\|\tilde{m}_j\|_\infty \leq |j| \ .$$

Then, if $\tilde{k}_j$ is the convolution kernel of $\tilde{m}_j(L)$ and $f \in L^2(H_n)$,

$$f * k_j = m_j(L)f = \tilde{m}_j(L)e^{-L}f = f * p_1 * \tilde{k}_j \ .$$

Therefore,

$$k_j = p_1 * \tilde{k}_j = \tilde{m}_j(L)p_1 \ .$$

It follows from Proposition 1.4 in Chapter I that

$$\|k_j\|_2 \leq \|\tilde{m}_j(L)\|_{\mathcal{L}(L^2)}\|p_1\|_2 = \|\tilde{m}_j\|_\infty \|p_1\|_2 \leq C|j| \ . \qquad \square$$

Lemma 2.3. *There is a constant $\nu > 0$ such that*

$$\int_{H_n} e^{|(z,t)|}|k_j(z,t)|\,dz\,dt \leq e^{\nu|j|} \ . \tag{2.3}$$

Moreover, for every $N \in \mathbb{N}$, there is $C_N > 0$ such that

$$\int_{H_n} \big|(z,t)\big|^N |k_j(z,t)|\,dz\,dt \leq C_N|j|^{N+n+2} \ . \tag{2.4}$$

Proof. By (2.2),

$$\int_{H_n} e^{|(z,t)|}|k_j(z,t)|\,dz\,dt \leq \sum_{\ell=1}^{\infty} \frac{|j|^\ell}{\ell!} \int_{H_n} e^{|(z,t)|}p_\ell(z,t)\,dz\,dt \ . \tag{2.5}$$

By Proposition 1.4,

$$\begin{aligned}
\int_{H_n} & e^{|(z,t)|}\big|f * g(z,t)\big|\,dz\,dt \leq \\
&\leq \int_{H_n}\int_{H_n} e^{|(z,t)|}\big|f\big((z,t)(w,u)^{-1}\big)\big|\big|g(w,u)\big|\,dw\,du\,dz\,dt \\
&= \int_{H_n}\int_{H_n} e^{|(z',t')(w,u)|}\big|f(z',t')\big|\big|g(w,u)\big|\,dw\,du\,dz'\,dt' \\
&\leq \Big(\int_{H_n} e^{|(z',t')|}\big|f(z',t')\big|\,dz'\,dt'\Big)\Big(\int_{H_n} e^{|(w,u)|}\big|g(w,u)\big|\,dw\,du\Big) \ .
\end{aligned}$$

Therefore, if

$$\nu = \int_{H_n} e^{|(z,t)|}p_1(z,t)\,dz\,dt \ ,$$

we deduce from the fact that $p_\ell = p_1 * \cdots p_1$ (ℓ times) that

$$\int_{H_n} e^{|(z,t)|}p_\ell(z,t)\,dz\,dt \leq \nu^\ell \ .$$

From (2.5) we obtain the first claimed estimate. We then pass to the second.

Given $r > 0$, we split the integral into the sum of the two integrals, extended to the set where $\big|(z,t)\big| > r$ and $\big|(z,t)\big| < r$ respectively.

We use (2.3) to estimate the first integral:

$$\int_{|(z,t)|>r} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le \big(\sup_{\rho>r} \rho^N e^{-\rho}\big) \int_{|(z,t)|>r} e^{|(z,t)|} |k_j(z,t)|\, dz\, dt$$
$$\le e^{\nu|j|} \big(\sup_{\rho>r} \rho^N e^{-\rho}\big) \ .$$

If we impose that $r > N$, then $\rho^N e^{-\rho}$ is a decreasing function of ρ, so that

$$\int_{|(z,t)|>r} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le e^{\nu|j|} r^N e^{-r} \ . \tag{2.6}$$

To estimate the second integral, we use instead Lemma 2.2 to obtain that

$$\int_{|(z,t)|<r} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le$$
$$\le \Big(\int_{|(z,t)|<r} \big|(z,t)\big|^{2N}\, dz\, dt\Big)^{\frac12} \Big(\int_{|(z,t)|<r} |k_j(z,t)|^2\, dz\, dt\Big)^{\frac12}$$
$$\le C|j| \Big(\int_{|(z,t)|<r} \big|(z,t)\big|^{2N}\, dz\, dt\Big)^{\frac12} \ .$$

Since

$$\int_{|(z,t)|<r} \big|(z,t)\big|^{2N}\, dz\, dt = \sum_{k=0}^{\infty} \int_{2^{-(k+1)}r<|(z,t)|<2^{-k}r} \big|(z,t)\big|^{2N}\, dz\, dt$$
$$\le \sum_{k=0}^{\infty} 2^{-2Nk}\, r^{2N} \int_{2^{-(k+1)}r<|(z,t)|<2^{-k}r} dz\, dt$$
$$\le \sum_{k=0}^{\infty} 2^{-2Nk}\, r^{2N} 2^{-(2n+2)k}\, r^{2n+2}$$
$$= C r^{2(N+n+1)} \ ,$$

we have

$$\int_{|(z,t)|<r} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le C|j| r^{N+n+1} \ . \tag{2.7}$$

Putting together (2.6) and (2.7), we have that, for $r > N$,

$$\int_{H_n} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le e^{\nu|j|} r^N e^{-r} + C|j| r^{N+n+1} \ .$$

If $\nu|j| > N$, taking $r = \nu|j|$, we have

$$\int_{H_n} \big|(z,t)\big|^N |k_j(z,t)|\, dz\, dt \le \nu^N |j|^N + C\nu^{N+n+1} |j|^{N+n+2}$$
$$\le C_N |j|^{N+n+2} \ .$$

There are only finitely many remaining values of j, so that (2.4) is proved. □

Lemma 2.4. *Let m and k be as in Theorem 2.1. Then, for every $N \in \mathbb{N}$,*

$$\int_{H_n} \big(1+|(z,t)|\big)^N |k(z,t)|\,dz\,dt \le C_N \|m\|_{C^{N+n+4}} \ .$$

Proof. By (2.1),

$$k(z,t) = \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j k_j(z,t) \ ,$$

so that, by (2.4),

$$\int_{H_n} \big|(z,t)\big|^N |k(z,t)|\,dz\,dt \le C_N \sum_{j\in\mathbb{Z}\setminus\{0\}} |a_j||j|^{N+n+2} \ .$$

The conclusion follows from the inequality

$$|a_j| = |\hat\mu(j)| \le |j|^{-M}\|\mu\|_{C^M} \le |j|^{-M}\|m\|_{C^M} \ ,$$

valid for every M. □

The control by the C^{N+n+4}-norm is not optimal, and it will be improved later. At this stage we do not need to be more precise than this.

End of the proof of Theorem 2.1. Let $\tilde m(\xi) = e^\xi m(\xi)$. Then $\tilde m$ satisfies the same assumptions of m, so that the convolution kernel $\tilde k$ of $\tilde m(L)$ satisfies, by Lemma 2.4,

$$\int_{H_n} \big|(z,t)\big|^N |\tilde k(z,t)|\,dz\,dt \le C_N \|\tilde m\|_{C^{N+n+4}} \le C'_N \|m\|_{C^{N+n+4}} \ .$$

From the identity $m(L) = e^{-L}\tilde m(L) = \tilde m(L)e^{-L}$, it follows that

$$k = \tilde k * p_1 = p_1 * \tilde k \ .$$

In particular, k is smooth. To prove that $k \in \mathcal{S}(H_n)$ is equivalent to proving that for every non-commutative polynomial $P(Z,\bar Z)$ in the left-invariant vector fields $Z_j, \bar Z_j$, $P(Z,\bar Z)k$ is rapidly decreasing at infinity. We then have

$$P(Z,\bar Z)k = \tilde k * P(Z,\bar Z)p_1 \ ,$$

where $P(Z,\bar Z)p_1 = g$ is rapidly decreasing at infinity.

Fix the polynomial P and $N \in \mathbb{N}$. For $\big|(z,t)\big| > 1$, we split the convolution integral in two parts. If $\big|(w,u)| < \frac12\big|(z,t)\big|$, then

$$\big|(w,u)^{-1}(z,t)\big| \ge \big|(z,t)\big| - \big|(w,u)\big| > \frac12\big|(z,t)\big| \ ,$$

so that

$$\begin{aligned}
&\int_{|(w,u)|<\frac12|(z,t)|} \big|\tilde k(w,u)\big|\big|g\big((w,u)^{-1}(z,t)\big)\big|\,dw\,du \le \\
&\qquad \le C \int_{|(w,u)|<\frac12|(z,t)|} \big|\tilde k(w,u)\big|\big|(w,u)^{-1}(z,t)\big|^{-N}\,dw\,du \\
&\qquad \le C'\big|(z,t)\big|^{-N}\|\tilde k\|_1 \\
&\qquad \le C''\|m\|_{C^{n+4}}\big|(z,t)\big|^{-N} \ .
\end{aligned} \tag{2.8}$$

Moreover,

$$\begin{aligned}&\int_{|(w,u)|>\frac12|(z,t)|}\big|\tilde k(w,u)\big|\big|g\big((w,u)^{-1}(z,t)\big)\big|\,dw\,du\le\\ \text{(2.9)}\qquad&\le 2^N\int_{|(w,u)|>\frac12|(z,t)|}\frac{|(w,u)|^N}{|(z,t)|^N}\big|\tilde k(w,u)\big|\big|g\big((w,u)^{-1}(z,t)\big)\big|\,dw\,du\\ &\le C\|m\|_{C^{N+n+4}}\big|(z,t)\big|^{-N}\ .\end{aligned}$$

Putting together (2.8) and (2.9), we find that

$$\big|P(Z,\bar Z)k(z,t)\big|\le C_{P,N}\frac{\|m\|_{C^{N+n+4}}}{\big(1+|(z,t)|\big)^N}\ ,\tag{2.10}$$

and this concludes the proof. □

3. Mihlin-Hörmander multipliers of L

We have the tools now to prove the sharp Mihlin-Hörmander theorem for multipliers of L.

We need however to make some preliminary digression on the realization of H_n as a space of homogeneous type, and on the corresponding Calderón-Zygmund theory.

There are two natural homogeneous-type structures on H_n, a left-invariant and a right-invariant one, that we denote here as (H_n, m, d_ℓ) and (H_n, m, d_r) respectively. In both cases the measure m is the Lebesgue measure, and they differ in the choice of the distance, which in one case is the *left-invariant distance*

$$d_\ell\big((z,t),(w,u)\big)=\big|(w,u)^{-1}(z,t)\big|\ ,$$

and in the other case the *right-invariant distance*

$$d_r\big((z,t),(w,u)\big)=\big|(z,t)(w,u)^{-1}\big|\ .$$

The terminology corresponds to the different invariance properties of d_ℓ and d_r: while

$$d_\ell\big((\zeta,\tau)(z,t),(\zeta,\tau)(w,u)\big)=d_\ell\big((z,t),(w,u)\big)\ ,$$

for every $(\zeta,\tau)\in H_n$, we have instead

$$d_r\big((z,t)(\zeta,\tau),(w,u)(\zeta,\tau)\big)=d_r\big((z,t),(w,u)\big)\ .$$

We then have two different notions of Calderón-Zygmund kernel.

Definition. *A distribution $u\in\mathcal S'(H_n)$ is a* left Calderón-Zygmund kernel on H_n *if*

(i) *the operator $Tf=f*u$ extends to a bounded operator on $L^2(H_n)$;*
(ii) *away from the origin, u coincides with a locally integrable function $u(z,t)$, and there is a constant $C>0$ such that, for every $(w,u)\neq(0,0)$,*

$$\int_{|(z,t)|>4|(w,u)|}\big|u\big((w,u)(z,t)\big)-u(z,t)\big|\,dz\,dt\le C\ .\tag{3.1}$$

A distribution $u \in \mathcal{S}'(H_n)$ *is a* right Calderón-Zygmund kernel on H_n *if*

(i) *the operator* $Tf = u * f$ *extends to a bounded operator on* $L^2(H_n)$*;*

(ii) *away from the origin,* u *coincides with a locally integrable function* $u(z,t)$*, and there is a constant* $C > 0$ *such that, for every* $(w,u) \neq (0,0)$*,*

$$\int_{\left|(z,t)\right|>4\left|(w,u)\right|} \left|u\big((z,t)(w,u)\big) - u(z,t)\right| dz\,dt \leq C \ . \tag{3.2}$$

The following statement belongs to the general Calderón-Zygmund theory (see Corollary 3.4 in Chapter II).

Proposition 3.1. *If* u *is a left Calderón-Zygmund kernel on* H_n*, the operator* $Tf = f * u$ *is weak-type (1,1) and bounded on* $L^p(H_n)$ *for* $1 < p \leq 2$*.*

If u *is a right Calderón-Zygmund kernel on* H_n*, the operator* $T'f = u * f$ *is weak-type (1,1) and bounded on* $L^p(H_n)$ *for* $1 < p \leq 2$*.*

Some comments are in order concerning the non-equivalence of the two notions, and the extension of L^p-boundedness to values of $p > 2$.

In contrast with what we have seen in $\mathbb{R}^n$, the fact that a convolution operator is bounded on $L^p(H_n)$ for some $p \in (1,\infty)$, does not imply[25] that the same operator is also bounded on $L^{p'}(H_n)$. This is related to the non-commutative structure of H_n. The correct duality result is as follows.

Proposition 3.2. *Let* $u \in \mathcal{S}'(H_n)$ *and* $p \in (1,\infty)$*. The operator* $Tf = f * u$ *is bounded on* $L^p(H_n)$ *if and only if* $T'g = u * g$ *is bounded on* $L^{p'}(H_n)$*. In this case the two operator norms coincide.*

Proof. Take $f, g \in \mathcal{S}(H_n)$. If $\check f(z,t) = f(-z,-t)$, an explicit computation shows that

$$\begin{aligned}\langle f * u, \check g\rangle &= \int_{H_n} f * u(z,t)\check g(z,t)\,dz\,dt \\ &= \langle u, \check f * \check g\rangle \\ &= \langle u * g, \check f\rangle \ .\end{aligned}$$

Therefore

$$\begin{aligned}\|T\|_{\mathcal{L}(L^p)} &= \sup_{\|f\|_p \leq 1, \|\check g\|_{p'} \leq 1} \left|\langle f * u, \check g\rangle\right| \\ &= \sup_{\|\check f\|_p \leq 1, \|g\|_{p'} \leq 1} \left|\langle u * g, \check f\rangle\right| \\ &= \|T'\|_{\mathcal{L}(L^{p'})} \ . \qquad \square\end{aligned}$$

We shall use the following consequence of Propositions 3.1 and 3.2.

[25] In fact this is false in general. This phenomenon is known as "asymmetry" of convolution operators, and it occurs on many non-commutative groups. It is not known if all infinite locally compact non-commutative groups exhibit asymmetry. For a large class of l.c. groups, called *amenable* and including the Heisenberg group, it is true however that if a convolution operator is bounded on some L^p, then it is also bounded on L^2.

Corollary 3.3. *Let u be a* two-sided Calderón-Zygmund kernel *(i.e. it is both left and right C-Z kernel). Then $Tf = f * u$ is bounded on $L^p(H_n)$ for $1 < p < \infty$, and it is weak-type (1,1).*

Take now a Mihlin-Hörmander multiplier $m(\xi)$ of order $s > \frac{1}{2}$ on $\mathbb{R}^+$. Then m is continuous and bounded, by (6.1) in Chapter II, so that $m(L)$ is bounded on $L^2(H_n)$. Hence there exists $u \in \mathcal{S}'(H_n)$ such that $m(L)f = f * u$.

We shall prove the following result[26].

Theorem 3.4. *Assume that $m(\xi)$ is a Mihlin-Hörmander multiplier on $\mathbb{R}^+$ of order $s > \frac{2n+1}{2}$. If $m(L)f = f * u$, then u is a two-sided Calderón-Zygmund kernel. Hence $m(L)$ is weak-type (1,1) and bounded on $L^p(H_n)$ for $1 < p < \infty$. Moreover,*

$$\|m(L)\|_{\mathcal{L}(L^p)} \leq C_p \|m\|_{MH_s} \ .$$

As in Section 6 of Chapter II, we take a non-negative C^∞-function η supported in $\left[\frac{1}{2}, 2\right]$, such that

$$\sum_{j \in \mathbb{Z}} \eta(2^j \xi) = 1$$

for $\xi > 0$. We define

$$m_j(\xi) = m(2^{-j}\xi)\eta(\xi) \ , \tag{3.3}$$

so that each m_j is supported in $\left[\frac{1}{2}, 2\right]$, and

$$m(\xi) = \sum_{j \in \mathbb{Z}} m_j(2^j \xi) \ . \tag{3.4}$$

We call u_j the distribution on H_n such that $m_j(L)f = f * u_j$. Our aim is to prove that the u_j are in fact integrable unctions, and that they satisfy conditions analogous to those imposed on the φ_j of Theorem 5.3 in Chapter II.

To begin with, we prove that the u_j are integrable functions and that also $\big(1 + |(z,t)|\big)^\varepsilon u_j(z,t)$ is integrable for some $\varepsilon > 0$. The following statement is not good enough, but it is one of the starting points for an interpolation argument, which will lead to the desired estimates.

Lemma 3.5. *Assume that $m \in H^N(\mathbb{R})$, with $N \geq n+5$, and is supported in $\left[\frac{1}{4}, 4\right]$. Then $m(L)f = f * u$, where u is integrable on H_n and*

$$\int_{H_n} \big(1 + |(z,t)|\big)^{2(N-n-5)} |u(z,t)|^2 \, dz \, dt \leq C_N \|m\|_{H^N}^2 \ .$$

Proof. It follows from Lemma 2.4, by a limiting argument, that u is integrable and

$$\int_{H_n} \big(1 + |(z,t)|\big)^{N-n-5} |u(z,t)| \, dz \, dt \leq C_N \|m\|_{C^{N-1}} \ .$$

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[26] This result has been proved independently by D. Müller and E.M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. 73 (1994), 413-440, and by W. Hebisch, *Multiplier theorem on generalized Heisenberg groups*, Coll. Math. 65 (1993), 231-239. The two papers contain different extensions to other nilpotent groups.

On the other hand, (2.10) implies that

$$\big(1+|(z,t)|\big)^{N-n-5}|u(z,t)| \le C_N\|m\|_{C^{N-1}}\ .$$

By (6.1) in Chapter II, $\|m^{(j)}\|_\infty \le C\|m^{(j)}\|_{H^1}$, so that $\|m\|_{C^{N-1}} \le \|m\|_{H^N}$. Putting these inequalities together, we conclude the proof. □

The other starting point for the interpolation is a sharper estimate for $s=\frac{3}{2}$.

Lemma 3.6. *Assume that $m\in H^{\frac{3}{2}}(\mathbb{R})$ and is supported in $\big[\frac14,4\big]$. Then $m(L)f = f*u$, where u is square-integrable on H_n and*

$$\int_{H_n}\big(1+|(z,t)|\big)^4|u(z,t)|^2\,dz\,dt \le C\|m\|^2_{H^{\frac32}}\ .$$

Proof. The left-hand side is equivalent to the L^2-norm of u plus the L^2-norm of w_+u, by (1.6). We then use the Plancherel formula, recalling that $\tilde u(\lambda,k) = m\big(|\lambda|(2k+n)\big)$:

$$\begin{aligned}
\|u\|_2^2 &= \sum_{k=0}^\infty \binom{n+k-1}{n-1}\int_{-\infty}^{+\infty}\big|\tilde u(\lambda,k)\big|^2\,|\lambda|^n\,d\lambda\\
&\sim \sum_{k=0}^\infty (k+1)^{n-1}\int_{-\infty}^{+\infty}\big|m\big(|\lambda|(2k+n)\big)\big|^2\,|\lambda|^n\,d\lambda\\
&= \sum_{k=0}^\infty \frac{(k+1)^{n-1}}{(2k+n)^{n+1}}\|m\|_2^2\\
&\le C\|m\|^2_{H^{\frac32}}\ .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|w_+u\|_2^2 &= \sum_{k=0}^\infty \binom{n+k-1}{n-1}\int_{-\infty}^{+\infty}\big|\widetilde{(w_+u)}(\lambda,k)\big|^2\,|\lambda|^n\,d\lambda\\
&\sim \sum_{k=0}^\infty (k+1)^{n-1}\int_{-\infty}^{+\infty}\big|\widetilde{(w_+u)}(\lambda,k)\big|^2\,|\lambda|^n\,d\lambda\ .
\end{aligned}$$

For $\lambda>0$ we have

$$\begin{aligned}
\widetilde{(w_+u)}(\lambda,k) &= \partial_\lambda m\big(\lambda(2k+n)\big)\\
&\qquad -\frac{k+n}{\lambda}\Big(m\big(\lambda((2k+n+2)\big)-m\big(\lambda(2k+n)\big)\Big)\\
&= (2k+n)m'\big(\lambda(2k+n)\big)-(k+n)\int_0^2 m'\big(\lambda(2k+n+s)\big)\,ds\\
&= -n\,m'\big(\lambda(2k+n)\big)\\
&\qquad -(k+n)\int_0^2\Big(m'\big(\lambda(2k+n+s)\big)-m'\big(\lambda(2k+n)\big)\Big)\,ds\\
&= A(\lambda,k)+B(\lambda,k)\ .
\end{aligned}$$

We make separate estimates of the contributions of $A(\lambda,k)$ and $B(\lambda,k)$ to the L^2-integral. Clearly,

$$\begin{aligned}\int_0^{+\infty} \left|A(\lambda,k)\right|^2 \lambda^n \, d\lambda &= n^2 \int_0^{+\infty} \left|m'\big(\lambda(2k+n)\big)\right|^2 \lambda^n \, d\lambda \\ &= \frac{n^2}{(2k+n)^{n+1}} \int_{\frac14}^4 \left|m'(\lambda)\right|^2 \lambda^n \, d\lambda \ ,\end{aligned}$$

so that

$$\begin{aligned}&\sum_{k=0}^{\infty} (k+1)^{n-1} \int_0^{+\infty} \left|A(\lambda,k)\right|^2 \lambda^n \, d\lambda \\ &\qquad \le C \sum_{k=0}^{\infty} (k+1)^{-2} \int_{\frac14}^4 \left|m'(\lambda)\right|^2 \lambda^n \, d\lambda \\ &\qquad \le C\|m'\|_2^2 \\ &\qquad \le C\|m\|_{H^{\frac32}} \ .\end{aligned}$$

The estimate for the contribution of $B(\lambda,k)$ is more delicate. We have

$$\begin{aligned}&\int_0^{+\infty} \left|B(\lambda,k)\right|^2 \lambda^n \, d\lambda = \\ &\quad = (k+n)^2 \int_0^{+\infty} \left| \int_0^2 \Big(m'\big(\lambda(2k+n+s)\big) - m'\big(\lambda(2k+n)\big) \Big) \, ds \right|^2 \lambda^n \, d\lambda \\ &\quad \le 2(k+n)^2 \int_0^2 \int_0^{+\infty} \left| m'\big(\lambda(2k+n+s)\big) - m'\big(\lambda(2k+n)\big) \right|^2 \lambda^n \, d\lambda \, ds \\ &\quad = \frac{2(k+n)^2}{(2k+n)^{n+1}} \int_0^2 \int_0^4 \left| m'\Big(\lambda + \frac{\lambda s}{2k+n}\Big) - m'(\lambda) \right|^2 \lambda^n \, d\lambda \, ds \\ &\quad \le \frac{C}{(k+1)^{n-1}} \int_0^2 \int_0^4 \left| m'\Big(\lambda + \frac{\lambda s}{2k+n}\Big) - m'(\lambda) \right|^2 \lambda \, d\lambda \, ds \\ &\quad \le \frac{C}{(k+1)^{n-2}} \int_0^4 \int_0^{\frac{2\lambda}{2k+n}} \left| m'(\lambda+h) - m'(\lambda) \right|^2 dh \, d\lambda \\ &\quad \le \frac{C}{(k+1)^{n-2}} \int_0^4 \int_0^{\frac{8}{2k+n}} \left| m'(\lambda+h) - m'(\lambda) \right|^2 dh \, d\lambda \ .\end{aligned}$$

Summing over k and using the the Plancherel formula in $\mathbb{R}$, we obtain that

$$\begin{aligned}
&\sum_{k=0}^{\infty}(k+1)^{n-1}\int_0^{+\infty}\big|B(\lambda,k)\big|^2\,\lambda^n\,d\lambda\\
&\qquad\le C\int_0^4\sum_{k=0}^{\infty}(k+1)\int_0^{\frac{8}{2k+n}}\big|m'(\lambda+h)-m'(\lambda)\big|^2\,dh\,d\lambda\\
&\qquad= C\int_0^4\int_0^{\frac{8}{n}}\Big(\sum_{k<\frac{4}{h}}(k+1)\Big)\big|m'(\lambda+h)-m'(\lambda)\big|^2\,dh\,d\lambda\\
&\qquad\le C\int_0^4\int_0^{\frac{8}{n}}\big|m'(\lambda+h)-m'(\lambda)\big|^2\,\frac{dh}{h^2}\,d\lambda\\
&\qquad\le C\int_0^{+\infty}\int_{\mathbb{R}}\big|m'(\lambda+h)-m'(\lambda)\big|^2\,d\lambda\,\frac{dh}{h^2}\\
&\qquad= C\int_0^{+\infty}\int_{\mathbb{R}}\tau^2|\hat m(\tau)|^2|e^{ih\tau}-1|^2\,d\tau\,\frac{dh}{h^2}\\
&\qquad= C\int_{\mathbb{R}}\tau^2|\hat m(\tau)|^2\int_0^{+\infty}\frac{|e^{ih\tau}-1|^2}{h^2}\,dh\,d\tau\\
&\qquad= C\int_{\mathbb{R}}|\tau|^3|\hat m(\tau)|^2\int_0^{+\infty}\frac{|e^{ih}-1|^2}{h^2}\,dh\,d\tau\\
&\qquad= C\|m\|^2_{H^{\frac{3}{2}}}\ .
\end{aligned}$$

We have so proved that

$$\sum_{k=0}^{\infty}(k+1)^{n-1}\int_0^{+\infty}\big|\widetilde{(w_+u)}(\lambda,k)\big|^2\,|\lambda|^n\,d\lambda\le C\|m\|^2_{H^{\frac{3}{2}}}\ .$$

Similar computations allow to obtain the same estimate for the integral over negative values of λ. □

Corollary 3.7. *Assume that $m\in H^s(\mathbb{R})$, with $s>\frac{2n+1}{2}$, and is supported in $\big[\frac12,2\big]$. If $m(L)f=f*u$, then, for $0<\varepsilon<s-\frac{2n+1}{2}$,*

$$\int_{H_n}\big(1+|(z,t)|\big)^{\varepsilon}|u(z,t)|\,dz\,dt\le C_\varepsilon\|m\|_{H^s}\ .$$

Proof. Let $\psi(\xi)$ be a smooth function supported in $\big[\frac14,4\big]$ and identically equal to 1 on $\big[\frac12,2\big]$. Given a bounded function $m(\xi)$ on $\mathbb{R}$, let u be the convolution kernel of $(m\psi)(L)$, and let S be the linear operator given by $Sm=u$. If, in particular, m is supported in $\big[\frac12,2\big]$, then $m\psi=m$, so that Sm is the kernel u in the statement.

Denote by $L^2_\alpha(H_n)$ the space of functions f on H_n such that $\big(1+|(z,t)|\big)^\alpha f(z,t)\in L^2(H_n)$. Then Lemma 3.6 states that

$$S:H^{\frac32}(\mathbb{R})\longrightarrow L^2_{2}(H_n)\ ,$$

and Lemma 3.5 states that if $N\ge n+5$,

$$S:H^N(\mathbb{R})\longrightarrow L^2_{N-n-5}(H_n)\ .$$

A simple modification to the proof of the interpolation Lemma 7.2 in Chapter II shows that if

$$s = \frac{3}{2}\theta + (1-\theta)N \ , \qquad \alpha = 2\theta + (1-\theta)(N-n-5) \ ,$$

with $0 < \theta < 1$, then

$$S : H^s(\mathbb{R}) \longrightarrow L^2_\alpha(H_n) \ ,$$

For fixed $s > \frac{3}{2}$ and $N > s$, we find $\theta = \frac{N-s}{N-\frac{3}{2}}$, hence

$$\alpha = \alpha_N = 2\frac{N-s}{N-\frac{3}{2}} + (s-\frac{3}{2})\frac{N-n-5}{N-\frac{3}{2}} \ .$$

As $N \to +\infty$, α_N tends monotonically to $s+\frac{1}{2}$. Therefore

$$\int_{H_n} \big(1+|(z,t)|\big)^{2(s+\frac{1}{2}-\delta)} |u(z,t)|^2 \, dz\, dt \le C_\delta \|m\|^2_{H^s} \ , \tag{3.5}$$

for every $\delta > 0$.

Assume now that $s > \frac{2n+1}{2}$ and $0 < \varepsilon < s - \frac{2n+1}{2}$. Then, if $\delta < s - \frac{2n+1}{2} - \varepsilon$,

$$\begin{aligned}\int_{H_n} \big(1+|(z,t)|\big)^{\varepsilon} |u(z,t)| \, dz\, dt \le \bigg(\int_{H_n} \big(1+|(z,t)|\big)^{2(s+\frac{1}{2}-\delta)} |u(z,t)|^2 \, dz\, dt \bigg)^{\frac{1}{2}} \\ \times \bigg(\int_{H_n} \big(1+|(z,t)|\big)^{-2(s+\frac{1}{2}-\delta-\varepsilon)} \, dz\, dt \bigg)^{\frac{1}{2}} \ .\end{aligned}$$

The last integral is convergent, because the exponent is strictly smaller than the negative of the homogeneous dimension $Q = 2n+2$ of H_n. The conclusion follows from (3.5). □

We look now for a substitute of condition (c) of Theorem 5.3 in Chapter II on the u_j. We want to replace the ordinary L^1-Lipschitz condition, which concern the "abelian" differences $u_j(z+w, t+u) - u_j(z,t)$, with similar conditions, involving "non-abelian" differences of the form $u_j\big((z,t)(w,u)\big) - u_j(z,t)$ and $u_j\big((w,u)(z,t)\big) - u_j(z,t)$.

Lemma 3.8. *Assume that $m \in H^s(\mathbb{R})$, with $s > \frac{2n+1}{2}$, and is supported in $\big[\frac{1}{2}, 2\big]$. If $m(L)f = f * u$, then*

$$\int_{H_n} \big|u\big((z,t)(w,u)\big) - u(z,t)\big| \, dz\, dt \le C \|m\|_{H^s} \big|(w,u)\big| \ , \tag{3.6}$$

and

$$\int_{H_n} \big|u\big((w,u)(z,t)\big) - u(z,t)\big| \, dz\, dt \le C \|m\|_{H^s} \big|(w,u)\big| \ . \tag{3.7}$$

Proof. The generic increment (w,u) can be written as

$$(w,u) = (w,0)(\sqrt{|u|}e_1, 0)(\pm i\sqrt{|u|}e_1, 0)(-\sqrt{|u|}e_1, 0)(\mp i\sqrt{|u|}e_1, 0) \ ,$$

where the $\pm$ sign depends on the signum of u. This implies that it is sufficient to prove (3.6) and (3.7) for "horizontal" increments (i.e. with a zero t-component). In fact, restricting our attention to (3.6) and assuming $u>0$ for simplicity, we can then write

$$\begin{aligned}
&\int_{H_n} \big|u\big((z,t)(w,u)\big)-u(z,t)\big|\,dz\,dt \le \\
&\quad\le \int_{H_n} \big|u\big((z,t)(w,u)\big)-u\big((z,t)(w,0)(\sqrt{u}e_1,0)(i\sqrt{u}e_1,0)(-\sqrt{u}e_1,0)\big)\big|\,dz\,dt \\
&\qquad + \cdots + \int_{H_n} \big|u\big((z,t)(w,0)\big)-u(z,t)\big|\,dz\,dt \\
&\quad= \int_{H_n} \big|u\big((z,t)(-i\sqrt{u}e_1,0)\big)-u(z,t)\big|\,dz\,dt \\
&\qquad + \cdots + \int_{H_n} \big|u\big((z,t)(w,0)\big)-u(z,t)\big|\,dz\,dt \\
&\quad\le C\|m\|_{H^s}\big(|w|+4\sqrt{u}\big) \\
&\quad\le C'\|m\|_{H^s}\big|(w,u)\big| \ .
\end{aligned}$$

Suppose therefore that $u=0$ in (3.6). By composing if necessary, u with a unitary transformation of $\mathbb{C}^n$, we can assume that $w=re_1$, with $r>0$. We claim that we can transform the difference into an integral by the fundamental theorem of calculus.

Consider in fact the multiplier $\tilde{m}(\xi)=e^{\xi}m(\xi)$ is also in $H^s(H_n)$ and is supported in $\big[\frac{1}{2},2\big]$. If $\tilde{u}$ is the corresponding convolution kernel, then $\tilde{u}\in L^1(H_n)$, by Corollary 3.7, and $u=\tilde{u}*p_1=p_1*\tilde{u}$. Hence u is C^∞. Therefore

$$\begin{aligned}
u\big((z,t)(re_1,0)\big)-u(z,t) &= \int_0^r \frac{d}{ds}u\big((z,t)(se_1,0)\big)\,ds \\
&= \int_0^r X_1u\big((z,t)(se_1,0)\big)\,ds \ .
\end{aligned} \tag{3.8}$$

Now, $X_1u=\tilde{u}*X_1p_1\in L^1(H_n)$, and

$$\|X_1u\|_1\le C\|\tilde{u}\|_1\le C'\|\tilde{m}\|_{H^s}\le C''\|m\|_{H^s} \ .$$

Hence

$$\begin{aligned}
\int_{H_n} \big|u\big((z,t)(re_1,0)\big)-u(z,t)\big|\,dz\,dt &= \int_{H_n}\left|\int_0^r X_1u\big((z,t)(se_1,0)\big)\,ds\right|dz\,dt \\
&\le \int_0^r\int_{H_n}\big|X_1u\big((z,t)(se_1,0)\big)\big|\,dz\,dt\,ds \\
&= r\|X_1u\|_1 \ .
\end{aligned}$$

In (3.7) the increments are on the left, hence (3.8) must be replaced by

$$\begin{aligned}
u\big((re_1,0)(z,t)\big)-u(z,t) &= \int_0^r \frac{d}{ds}u\big((se_1,0)(z,t)\big)\,ds \\
&= \int_0^r X_1^{(r)}u\big((z,t)(se_1,0)\big)\,ds \ ,
\end{aligned}$$

where the right-invariant vector field $X_1^{(r)}$ appears. The identity

$$X_1^{(r)} u = X_1^{(r)}(p_1 * \tilde{u}) = (X_1^{(r)} p_1) * \tilde{u}$$

then leads us to the conclusion. □

We consider next condition (b) of Theorem 5.3 in Chapter II.

Lemma 3.9. *Assume that $m \in H^s(\mathbb{R})$, with $s > \frac{2n+1}{2}$, and is supported in $\left[\frac{1}{2}, 2\right]$. If $m(L)f = f * u$, then $\int_{H_n} u(z,t)\, dz\, dt = 0$.*

Proof. Let k_j be the kernel in (2.2). By (1.1),

$$\int_{H_n} k_j(z,t)\, dz\, dt = \sum_{\ell=1}^{\infty} \frac{(ij)^\ell}{\ell!} \int_{H_n} p_\ell(z,t)\, dz\, dt = \sum_{\ell=1}^{\infty} \frac{(ij)^\ell}{\ell!} = e^{ij} - 1 \ .$$

Since $s > \frac{3}{2}$, m is C^1, and so is μ. It follows that the coefficients a_j in (2.1) are summable, because, by the Parseval formula,

$$\sum_{j \in \mathbb{Z}\setminus\{0\}} |a_j| \le \Big(\sum_{j \in \mathbb{Z}\setminus\{0\}} |j a_j|^2 \Big)^{\frac{1}{2}} = \|\mu'\|_2 \ .$$

Since $u = \sum_{j \in \mathbb{Z}\setminus\{0\}} a_j k_j$, we have

$$\int_{H_n} u(z,t)\, dz\, dt = \sum_{j \in \mathbb{Z}\setminus\{0\}} a_j (e^{ij} - 1) = \mu(1) = m(0) = 0 \ . \qquad \square$$

Finally, some further remarks are needed, concerning the decomposition (3.4) of m.

Let m_j, with $j \in \mathbb{Z}$, be the multiplier in (3.3), with u_j such that $m_j(L)f = f * u_j$. By Proposition 1.4 in Chapter I, if $f, g \in L^2(H_n)$, then

$$\begin{aligned} \langle m(L) f | g \rangle &= \int_0^\infty m(\lambda)\, d\nu_{f,g}(\lambda) \\ &= \sum_{j \in \mathbb{Z}} \int_0^\infty m_j(2^j \lambda)\, d\nu_{f,g}(\lambda) \\ &= \sum_{j \in \mathbb{Z}} \langle m_j(2^j L) f | g \rangle \ . \end{aligned} \tag{3.9}$$

Hence $m(L) = \sum_{j \in \mathbb{Z}} m_j(2^j L)$ in the weak topology.

Lemma 3.10. *Let*

$$u_j^{(j)}(z,t) = 2^{-(n+1)j} u_j(2^{-\frac{j}{2}} z, 2^{-j} t) \ .$$

Then $m_j(2^j L) f = f * u_j^{(j)}$, and

$$u = \sum_{j \in \mathbb{Z}} u_j^{(j)} \ ,$$

in the sense of distributions.

Proof. By Theorem 5.6 and (5.10) in Chapter IV,

$$\widetilde{u_j}(\lambda, k) = \int_{H_n} u_j(z,t) e^{i\lambda t} \psi_k\big(|\lambda||z|^2\big)\, dz\, dt = m_j\big(|\lambda|(2k+n)\big) \ .$$

Therefore,

$$\widetilde{u_j^{(j)}}(\lambda, k) = \int_{H_n} u_j(z,t) e^{i\lambda 2^j t} \psi_k\big(|\lambda| 2^j |z|^2\big)\, dz\, dt = m_j\big(2^j|\lambda|(2k+n)\big) \ .$$

Applying Theorem 5.6 in Chapter IV again, we conclude that $u_j^{(j)}$ is the convolution kernel of $m_j(2^j L)$. By (3.9), if $f, g \in L^2(H_n)$ and $g^*(z,t) = \overline{g(-z,-t)}$,

$$\begin{aligned}
\langle u | f * g\rangle &= \langle u * g^* | f\rangle \\
&= \langle m(L) g^* | f\rangle \\
&= \sum_{j\in\mathbb{Z}} \langle m_j(2^j L) g^* | f\rangle \\
&= \sum_{j\in\mathbb{Z}} \langle u_j^{(j)} | f * g\rangle \ .
\end{aligned}$$

Take now $\varphi \in \mathcal{S}(H_n)$. We want to prove that

$$\langle u | \varphi\rangle = \sum_{j\in\mathbb{Z}} \langle u_j^{(j)} | \varphi\rangle \ .$$

Since u is radial, $\langle u|\varphi\rangle = \langle u|P\varphi\rangle$ (and the same holds for $u_j^{(j)}$), where P is the orthogonal projection from $L^2(H_n)$ onto $L^2_{\mathrm{rad}}(H_n)$ (see (5.7) in Chapter IV). By Lemma 5.5 in Chapter IV, $P\varphi \in \mathcal{S}_{\mathrm{rad}}(H_n)$.

It is then sufficient to prove that every $\varphi \in \mathcal{S}_{\mathrm{rad}}(H_n)$ can be written as $\varphi = f * g$, with $f, g \in L^2(H_n)$.

By Proposition 2.7 in Chapter IV and its extension to H_n,

$$\tilde\varphi(\lambda, k) \le \frac{C_N}{\big(1 + |\lambda|(k+1)\big)^N} \ ,$$

for every $N \in \mathbb{N}$. Therefore, taking $N > n+1$,

$$\begin{aligned}
\int_{\mathbb{R}} \sum_{k\in\mathbb{N}} \binom{k+n-1}{n-1} \big|\tilde\varphi(\lambda,k)\big|\, |\lambda|^n\, d\lambda &\le C_N \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} \frac{(k+1)^{n-1}}{\big(1+|\lambda|(k+1)\big)^N} |\lambda|^n\, d\lambda \\
&< +\infty \ .
\end{aligned}$$

By the Plancherel formula, if we impose that

$$\tilde f(\lambda, k) = \big|\tilde\varphi(\lambda,k)\big|^{\frac12} \ , \qquad \tilde g(\lambda,k) = \big|\tilde\varphi(\lambda,k)\big|^{\frac12} \arg\big(\tilde\varphi(\lambda,k)\big) \ ,$$

then $f, g \in L^2(H_n)$ and $f * g = \varphi$. $\square$

We have now all the ingredients for the proof of Theorem 3.4.

Proof of Theorem 3.4. By Lemma 3.10, $u = \sum_{j\in\mathbb{Z}} u_j^{(j)}$. By Lemmas 3.7, 3.8, 3.9, the u_j satisfy the following conditions with the same constant C:

(i) For some $\varepsilon > 0$, $\int_{H_n} \big(1 + |(z,t)|\big)^\varepsilon |u_j(z,t)|\, dz\, dt \le C_\varepsilon$;

(ii) $\int_{H_n} u_j(z,t)\, dz\, dt = 0$;

(iii) for every $(w,u) \in H_n$,

$$\int_{H_n} \big|u_j\big((z,t)(w,u)\big) - u_j(z,t)\big|\, dz\, dt \le C\big|(w,u)\big| \ ,$$

$$\int_{H_n} \big|u_j\big((w,u)(z,t)\big) - u_j(z,t)\big|\, dz\, dt \le C\big|(w,u)\big| \ .$$

A straightforward adaptation of the proof of Theorem 5.3 in Chapter II shows that u is a two-sided Calderón-Zygmund kernel. Corollary 3.3 can then be applied. $\square$