A CLASSIFICATION-FREE CONSTRUCTION OF RANK-ONE SYMMETRIC SPACES

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We survey some recent work by the two authors, as well as less recent joint work by M. Cowling and A. Dooley and the two authors, concerning a unified presentation of rank-one symmetric spaces.

There are three inter-related algebraic structures that are involved: Clifford modules, Lie algebras of Heisenberg type (in short, H-type algebras), and compositions of quadratic forms (which we prefer to call *C-modules*). Symmetric spaces show up when either of these structures enjoys a special property, that we generically refer to as the "J^2-condition". This type of condition showed up already in the work of E. Heintze [Hein] on non-compact spaces and of [Chi] on compact ones.

The non-compact case was treated in [CDKR2], on the basis of the results of [CDKR1], and the language of H-type algebras was privileged. When passing to the compact case (which is the object of the more recent study), the rôle of nilpotent Lie algebras is less significant, and the language of composition of quadratic forms, or C-modules, appears to be much more appropriate.

In this note we rephrase the main results of [CDKR2] in this language and state some of the new results for compact spaces. A wider and more complete exposition with all the proofs will appear elsewhere.

For the non-compact case, [CDKR2] contains a classification-free proof that all rank-one symmetric spaces arise from the construction given there. Since our arguments show that each of the non-compact spaces are embedded into compact dual spaces, it follows that our construction gives locally all compact symmetric spaces of rank one. That this is even globally true is contained in the arguments of [Chi]. From a practical point of view, instead of this classification-free method it is easier to classify all the spaces that our construction gives and compare with the well-known list obtained from Lie theory.

Our purpose here to furnish a framework in which (after knowing from one proof or another that all rank one spaces are included) it is easy to make unified computations. our approach does not require any *a priori* classification. If we include lists of specific cases at various places, we do so in order to provide the reader with concrete examples.

The referee pointed out the similarity of the content of Section 4 with Chevalley's construction of the normed division algebras from Clifford algebras [Chev].

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1. C-modules

Let C, V be finite-dimensional vector spaces over $\mathbb{R}$, each equipped with an inner product[1]. A *C-module structure* on V is a bilinear map

$$J : C \times V \longrightarrow V \ ,$$

satisfying the following properties:

(i) there is an element $e \neq 0$ in C such that $J(e,v) = v$ for every $v \in V$;
(ii) for every $\zeta \in C$ and $v \in V$,

$$|J(\zeta,v)| = |\zeta||v| \ . \tag{1}$$

This structure is referred to in the literature as *composition of quadratic forms*, or *orthogonal multiplication*.

The degenerate cases, where either $V = \{0\}$, or C is reduced to $\mathbb{R}e$ (in which case J is simply scalar multiplication), are not excluded.

The element $J(\zeta,v) \in V$ will also be denoted as $J_\zeta v$, or as ζv. For $v \in V$, we set

$$Cv = \{\zeta v : \zeta \in C\} \subseteq V \ .$$

If $v \neq 0$, the map $\zeta \mapsto \zeta v$ is, up to the scalar factor $|v|$, an isometry of C onto Cv. Polarizing (1) in both v and ζ, we obtain the identity

$$\langle \zeta v, \eta w\rangle + \langle \eta v, \zeta w\rangle = 2\langle \zeta,\eta\rangle\langle v,w\rangle \ , \tag{2}$$

for $\zeta,\eta \in C$ and $v,w \in V$.

Denote by C' the orthogonal complement of e in C. Taking $\eta = 1$ and $\zeta = z \in C'$ in (2), we find that J_z is skew-symmetric. More generally, if $\zeta = ae + z$, with $a \in \mathbb{R}$ and $z \in C'$, we set[2] $\bar\zeta = ae - z$. Then

$${}^tJ_\zeta = J_{\bar\zeta} \ . \tag{3}$$

It follows that

$$\langle J_{\bar\zeta} J_\zeta v, w\rangle = \langle \zeta v, \zeta w\rangle = |\zeta|^2\langle v,w\rangle \ ,$$

i.e.

$$J_{\bar\zeta} J_\zeta = |\zeta|^2 I \ . \tag{4}$$

If we define $J^* : V \times V \to C$ by

$$\langle J^*(v,v'),\zeta\rangle = \langle J(\zeta,v'),v\rangle \ ,$$

then J^* is "C-hermitean":

$$J^*(v',v) = \overline{J^*(v,v')} \ ,$$

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[1] As long as no confusion arises, scalar products will be denoted by $\langle\ ,\ \rangle$, and the induced norms by $|\ |$. A different notation will be introduced when necessary.

[2] Later on, we shall also write $a = \Re\mathrm{e}\,\zeta$, $z = \Im\mathrm{m}\,\zeta$.

$J^*(v,v') = 0$ if $v' \perp Cv$, and $J^*(\zeta v, v) = |v|^2\zeta$. Therefore, if $|v| = 1$, $J^*(v,v')v$ is the orthogonal projection of v' on Cv.

We say that two C-module structures (C,V,J) and $(\tilde{C},\tilde{V},\tilde{J})$ are isomorphic if there are orthogonal transformations $\varphi \in O(C)$ and $\psi \in O(V)$ such that the diagram

$$
\begin{array}{ccc}
C \times V & \xrightarrow{J} & V \\
\varphi \downarrow \quad \downarrow \psi & & \downarrow \psi \\
\tilde{C} \times \tilde{V} & \xrightarrow{\tilde{J}} & \tilde{V}
\end{array}
\tag{5}
$$

is commutative.

2. C-modules and Clifford modules

If we set $\zeta = z \in C'$ in (4), we find that

$$J_z^2 = -|z|^2 I\ .$$

Assume that both V and C' are non-trivial. Then V inherits the structure of a module over the Clifford algebra Cliff$\,(C')$ generated by an orthonormal basis $\{z_1,\dots,z_k\}$ of C', with the relations $z_iz_j + z_jz_i = -2\delta_{ij}$ for every i,j. Condition (1) implies that the scalar product in V is invariant under the action of Pin$\,(C')$, the multiplicative group generated by the unit elements of C' inside Cliff$\,(C')$.

This construction can be reversed: let Z be any finite dimensional vector space with a scalar product, Cliff$\,(Z)$ the Clifford algebra generated by Z, and V a module over Cliff$\,(Z)$, endowed with a Pin$\,(Z)$-invariant scalar product (when this holds, we say that V is an *orthogonal Clifford module*). If $C = \mathbb{R}e \oplus Z$, there is a unique C-module structure (C,V,J) inducing the given Clifford module structure on V [Ka].

Notice that the natural notion of (orthogonal) equivalence for orthogonal Clifford modules is (5) with the added conditions $C = \tilde{C}$ and $\varphi = I$. This is the same as requiring that ψ is "unitary" w.r. to the C-valued hermitean form J^*. Non-equivalent Clifford modules may induce isomorphic C-modules.

The main facts about orthogonal Clifford modules can be derived from [Hu]. If the dimension d of C is not divisible by 4, Cliff$\,(C')$ has, up to orthogonal equivalence, only one irreducible module V_0, and every other module can be realized as $V_k \sim V_0 \otimes_{\mathbb{F}} \mathbb{F}^k$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, depending on the congruence class of d mod. 8.

If d is divisible by 4, then Cliff$\,(C')$ has two inequivalent irreducible modules, V_1, V_2, and any module V can be realized as $V_{k,h} = (V_1 \otimes_{\mathbb{F}} \mathbb{F}^k) \oplus (V_2 \otimes_{\mathbb{F}} \mathbb{F}^h)$, with $\mathbb{F} = \mathbb{R}$ if d is divisible by 8, and $\mathbb{F} = \mathbb{H}$ otherwise. If $k \neq h$, $V_{k,h}$ and $V_{h,k}$ are non-equivalent as Clifford modules, but the induced C-modules are isomorphic.

3. The automorphism group of a C-module structure

If (C,V,J) is a C-module structure, we denote by M the group of all its automorphisms. These are the pairs $(\varphi,\psi) \in O(C) \times O(V)$ for which the diagram (5), with $(\tilde{C}, \tilde{V}, \tilde{J}) = (C, V, J)$, commutative. Since $\varphi(e) = e$ for any $(\varphi, \psi) \in M$, we can regard M as a subgroup of $O(C') \times O(V)$.

There are two notable subgroups of M. The notation $U_{\mathbb{F}}(k)$ stands for $O(k)$ if $\mathbb{F} = \mathbb{R}$, $U(k)$ if $\mathbb{F} = \mathbb{C}$, and $\mathrm{Sp}(k)$ if $\mathbb{F} = \mathbb{H}$.

(i) The group M_1 of pairs (I, ψ), where ψ is an automorphism of V as an orthogonal Clifford module. If d is not divisible by 4 and $V = V_k$, then $M_1 = U_{\mathbb{F}}(k)$, acting on the factor $\mathbb{F}^k$ of the tensor product. If d is divisible by 4 and $V = V_{k,h}$, then $M_1 = U_{\mathbb{F}}(k) \times U_{\mathbb{F}}(h)$.

(ii) The group M_2 generated by the pairs (φ_u, J_u), where u is a unit vector in C' and

$$\varphi_u(z) = 2\langle z, u\rangle u - z \ , \tag{6}$$

i.e. the reflection in C' w.r. to the line $\mathbb{R}u$. Clearly, M_2 is isomorphic to a quotient of $\mathrm{Pin}\,(C')$ modulo a finite subgroup.

Clearly, M_1 is normal in M. It is proved in [Ri] that $M_1 \cap M_2$ has at most 4 elements and that $M_1 M_2$ has index at most 2 in M. In particular, the Lie algebra of M is the semi-direct sum of the Lie algebras of M_1 and M_2.

4. The J^2-condition

Sometimes the composition of the J-action of certain pairs elements of C on V is equal to the action of another element of C. The following is a typical instance.

For any unit element $z_0 \in C'$, the maps J_{ae+bz_0} form a division algebra isomorphic to $\mathbb{C}$, hence the composition of two of them is of the same type, according to the arithmetic laws in $\mathbb{C}$. In particular, $\zeta^{-1} = |\zeta|^{-2}\bar{\zeta}$ for $\zeta \neq 0$. More generally, given $\zeta = ae + z \in C$ and a rational expression $R(\lambda, \bar{\lambda}) = \frac{P(\lambda,\bar{\lambda})}{Q(\lambda,\bar{\lambda})}$ with real coefficients such that $Q(\zeta, \bar{\zeta}) \neq 0$, then $R(\zeta, \bar{\zeta})$ is a well-defined element $\eta \in C$, which satisfies $J_\eta J_{Q(\zeta,\bar{\zeta})} = J_{P(\zeta,\bar{\zeta})}$.

However the general situation is different, and in most cases $C(Cv)$ is strictly larger than Cv. We say that a C-module V satisfies the J^2-*condition* if

$$C(Cv) = Cv$$

for every $v \in V$. In other words, we require that for every $\zeta, \eta \in C$ and $v \in V$ there is $\tau = \tau(\zeta, \eta, v)$ such that

$$J_\zeta J_\eta v = J_\tau v \ . \tag{7}$$

This condition has two implications (when V is non-trivial). The first is that, for every $v \neq 0$, Cv is a $\mathrm{Cliff}\,(C')$ submodule of V, necessarily irreducible. We shall call it *the C-line through v and the origin.* Moreover, one can inductively construct *orthonormal C-bases* of V, i.e. sets $\{v_1, \ldots, v_k\}$ such that $\langle v_i, v_j\rangle = \delta_{i,j}$ for every i, j and

$$V = Cv_1 \oplus \cdots \oplus Cv_k \ .$$

The fact that any non-zero element of V is contained in an irreducible submodule implies that V must be *isotypic*, i.e. all its irreducible sub-modules are equivalent (this is a restriction only when d is divisible by 4).

The second implication is that every non-zero element of V induces a composition law on C, given by

$$\zeta \cdot_v \eta = \tau \iff J_\zeta J_\eta v = J_\tau v \ .$$

Under any such product, C is a division algebra, not necessarily associative. It is proved in [CDKR1] that associativity holds if and only if $\cdot_v$ does not depend on v. If this is not the case, then V must necessarily be irreducible. All these facts give the following list, up to isomorphism, of C-module structures satisfying the J^2-condition:

(i) $C = \mathbb{R}$, $V = \mathbb{R}^k$, $k \geq 1$, $J =$ scalar multiplication;
(ii) $C = \mathbb{C}$, $V = \mathbb{C}^k$, $k \geq 1$, $J =$ scalar multiplication;
(iii) $C = \mathbb{H}$, $V = \mathbb{H}^k$, $k \geq 1$, $J =$ scalar multiplication from the left;
(iv) $C = \mathbb{O}$, $V = \mathbb{O}$, $J =$ multiplication from the left;
(v) C arbitrary, $V = \{0\}$.

The product on C fails to be associative only in case (iv)[3]. We shall refer to this distinction as the "associative" and the "non-associative" case. In the associative case, we simply write $\zeta \cdot \eta$ for the unique product in C.

5. The space $W = C \oplus V$ and the non-compact symmetric spaces

Assume that (C, V, J) satisfies the J^2-condition.

In order to describe the rank-one symmetric spaces, we want to make one further step, by giving a reasonable structure to the space W, defined as the orthogonal sum of C and V, leaving the original scalar product on each summand. We keep the notation $|\ |$ for the induced norm on W.

What we really need is to introduce a reasonable notion of C-line through two points in W. In the associative case, we can define a new C-module structure $(C, W, \tilde{J})$, setting

$$\tilde{J}(\zeta, v + \eta) = J(\zeta, v) + \zeta \cdot \eta \ .$$

This however does not work in the non-associative case, and any attempt of making W a C-module would destroy the J^2-condition.

What we can do instead is to introduce an equivalence relation $\sim$ on $W \setminus \{0\}$ as follows. Let $w = (\zeta, v)$, $w' = (\zeta', v')$ in $W \setminus \{0\}$.

(i) If ζ and ζ' are non-zero, we say that $w \sim w'$ if $\zeta^{-1} v = {\zeta'}^{-1} v'$.
(ii) If $\zeta = \zeta' = 0$, then $w \sim w'$ if $v \in C v'$.

We then say that the *C-line through w and the origin*, denoted as Cw, is the union of the equivalence class of w with the origin.

One can distinguish between two kinds of C-lines: those included in V, coinciding with the irreducible Clifford submodules of V, and those not contained in V. Each of the latter ones contains one and only one element of the form (e, v), so that they can be parametrized by V. Notice that $C(e, v) = C(\zeta, \zeta v)$ for every $\zeta \in C \setminus \{0\}$. Clearly, C itself is the C-line $C(e, 0)$.

On the unit ball B_W in W we introduce the Riemannian metric that assigns to a pair of tangent vectors $X, Y \in T_w B_W$ (identified with W itself) at the point

[3] There is no product in C in case (v), but it is convenient to include it in the associative case.

$w \in B_W$ the scalar product $\langle X, Y\rangle_{w-}$ such that

$$\|X\|^2_{w-} = \begin{cases} \frac{|X|^2}{(1-|w|^2)^2} & \text{if } X \in Cw \ , \\ \frac{|X|^2}{1-|w|^2} & \text{if } X \in (Cw)^\perp \ , \end{cases}$$
$$\langle X, Y\rangle_{w-} = 0 \qquad \text{if } X \in Cw \text{ and } Y \in (Cw)^\perp \ . \tag{8}$$

Notice that, for $w = 0$, Cw is not defined, and (8) must be read as $\|X\|_{0-} = |X|$.
One of the results of [CDKR2] can be restated as follows.

Theorem 1. *Under the metric* (8)*, B_W is a rank-one symmetric space of the non-compact type. All rank-one symmetric spaces can be constructed in this way. Precisely, with respect to the list in Section 4, (i) and (v) give the real hyperbolic spaces, respectively in the Cayley-Klein model and in the Poincaré model, (ii) gives the complex hyperbolic spaces, (iii) the quaternionic ones, and (iv) the exceptional space $F_{4,-20}/\mathrm{Spin}\,(9)$.*

For every $w \neq 0$, let $B_w = B_W \cap Cw$. With the induced metric, B_w is isometric to the d-dimensional real hyperbolic space in the Poincaré model.

We denote by G the isometry group of B_W, and by K the stabilizer of the origin in G. Since the diameters $B_W \cap \mathbb{R}w$ are the geodesics through the origin, and the distance from w to the origin depends only on $|w|$, it is rather clear that K consists of linear transformations. Combining the results of the previous sections with [CDKR2], we have the following realizations of the groups M and M_1 as the subgoups of K fixing the elements of the geodesic $B_W \cap \mathbb{R}e$ and of $B_{(e,0)}$ respectively.

Theorem 2. *The group K consists of the linear transformations in $O(W)$ that map C-lines into C-lines. In particular, it contains $-I$, the geodesic inversion around the origin. It also contains the group M of automorphisms of the C-module structure, as the subgroup of those elements that stabilize each point in the real line $\mathbb{R}e \subset C$. The subgroup of K stabilizing each point of C is the group $M_1 \subset M$ of automorphisms of V as an orthogonal Clifford module.*

Since rank-one symmetric spaces are also characterized as the two-point homogeneous spaces (i.e. given any two pairs (x, y), (x', y') of points in the space, with $d(x, y) = d(x', y')$, there is an isometry mapping x to x' and y to y', see [Hel]), it is interesting to see how this fact follows from our description of B_W and of its isometries.

The first step is to prove that K acts transitively on geodesic spheres centered at the origin, i.e. on the spheres $|w| = r < 1$. The second step is to show transitivity of G on B_W. Transitivity of K on geodesic spheres follows from the following two facts. The first of them is Kostant's "double transitivity theorem" [CDKR2].

Theorem 3. *The group M acts transitively on the product $S_{C'} \times S_V$ of the unit spheres in C' and V respectively. For a fixed unit vector $v_0 \in V$, decompose $v \in V$ as $v = \zeta v_0 + v'$, with $v' \in (Cv_0)^\perp$, and denote by T the torus in K consisting of the transformations*

$$\tau_\theta(\zeta, \eta v_0 + v') = \big(\cos\theta\zeta - \sin\theta\eta, (\cos\theta\eta + \sin\theta\zeta)v_0 + v'\big) \ .$$

Then K is generated by T and M and it acts transitively on the unit sphere S_W in W.

The structure of G itself is better described on the unbounded model of B_W.

Following [CDKR2], we introduce the *Cayley transform*

$$c(\zeta,v)=\big((1-\zeta)^{-1}(1+\zeta),(1-\zeta)^{-1}v\big) \tag{9}$$

to establish a bijection between B_W and the domain

$$D_W=\Big\{(\zeta,v):\Re\mathrm{e}\,\zeta>|v|^2\Big\}\ .$$

The group $\tilde G=cGc^{-1}$ is the group of isometries of D_W with respect to the metric transported from B_W via the Cayley transform. Clearly, $\tilde K=cKc^{-1}$ is the stabilizer in $\tilde G$ of the point $c(0)=(1,0)$. We also have the following notable subgroups of $\tilde G$:

(i) the one-parameter dilation group $\tilde A=\{\tilde a_t\}_{t\in\mathbb{R}}$, where

$$\tilde a_t(\zeta,v)=(e^{2t}\zeta,e^tv)\ ;$$

(ii) the translation group $\tilde N=\{\tilde n_{(z,u)}:z\in C'\ ,\ u\in V\}$, with

$$\tilde n_{(z,u)}(\zeta,v)=\Big(\zeta+z+2J^*(v,u)+|u|^2,v+u\Big)\ , \tag{10}$$

acting simply transitively on the level sets of the height function $h(\zeta,v)=\Re\mathrm{e}\,\zeta-|v|^2$.

We call $A=c^{-1}\tilde Ac$, $N=c^{-1}\tilde Nc$. In particular,

$$\begin{aligned} a_t(\zeta,v)&\overset{\mathrm{def}}{=}C^{-1}\tilde a_tC(\zeta,v)\\ &=\big((\sinh t\,\zeta+\cosh t)^{-1}(\cosh t\,\zeta+\sinh t),(\sinh t\,\zeta+\cosh t)^{-1}v\big)\ . \end{aligned} \tag{11}$$

Transitivity of G on B_W follows from the transitivity of K on spheres and the fact that $a_t(0,0)=(\tanh t,0)$.

Theorem 4. *B_W is two-point homogeneous. The group G admits the Iwasawa decomposition $G=NAK$ and the Cartan decomposition $G=KA_+K$, where $A_+=\{a_t\in A:t\geq 0\}$.*

The composition law on N (or $\tilde N$) corresponds to the product

$$(z,v)(z',v')=\big(z+z'+2\Im\mathrm{m}\,J^*(v,v'),v+v'\big) \tag{12}$$

on $C'\times V$. So N is a group of *Heisenberg type*, in the sense of [Ka], when the C-module structure is non-degenerate, and abelian otherwise. The correspondence between C-modules and groups of Heisenberg type was shown in [Ka]. In [CDKR1] one finds the proof of the correspondence between C-modules with the J^2-condition on one side, and groups of Heisenberg type appearing in rank-one Iwasawa decompositions on the other side.

6. The compact symmetric spaces

Starting from a C-module V satisfying the J^2-condition, we construct a compact symmetric space as a compactification of W.

The construction is modeled on the idea of obtaining a kind of "C-projective space" of the same dimension as W. In fact, such an object can be correctly defined in the associative case, but it does not make any sense in the non-associative case.

We define a set P_W as $W \cup W_\infty$, where W_∞ is the manifold of C-lines in W,

$$W_\infty = \big(W \setminus \{0\}\big)/\sim\ ,$$

with the quotient differentiable structure. We denote by $[w] \in W_\infty$ the equivalence class of $w \in W$. If $w = (\zeta, v)$, we also write $[\zeta, v]$ instead of $[w]$.

The topology of P_W is defined by postulating that

(i) W is open in P_W, with its own topology;

(ii) if $\{U_n\}$ is a fundamental system of neighborhoods of $[w_0] \in W_\infty$ in the topology of W_∞, a fundamental system $\{U'_n\}$ of neighborhoods of $\dot w$ in P_W is defined as

$$U'_n = U_n \cup \{w \in W : |w| > n\,,\ [w] \in U_n\}\ .$$

One easily verifies that P_W is compact and W is dense in it.

Keeping the same notation as in (8), we define the following Riemannian metric on W:

$$\begin{aligned} \|X\|^2_{w+} &= \begin{cases} \frac{|X|^2}{(1+|w|^2)^2} & \text{if } X \in Cw\ ,\\ \frac{|X|^2}{1+|w|^2} & \text{if } X \in (Cw)^\perp\ ,\end{cases} \\ \langle X, Y\rangle_{w+} &= 0 \qquad \text{if } X \in Cw \text{ and } Y \in (Cw)^\perp\ . \end{aligned} \tag{13}$$

In order to give P_W a differentiable structure and, at the same time a Riemannian metric, we introduce a finite family of isometries on W, which admit a continuous extension to P_W, and impose that these extensions are smooth and isometric.

We define the following involutions:

(i) for $\zeta \neq 0$

$$\varphi_0(\zeta, v) = (\zeta^{-1}, \zeta^{-1}v)\ ; \tag{14}$$

(ii) for a fixed unit vector $u \in V$, decompose $v \in V$ as $v = \eta u + v'$, with $v' \perp u$, and set

$$\psi_u(\zeta, \eta u + v') = (\eta, \zeta u + v')\ ,$$

and finally $\varphi_u = \varphi_o \circ \psi_u \circ \varphi_0$, i.e., for $\eta \neq 0$,

$$\varphi_u(\zeta, \eta u + v') = (\eta^{-1} \cdot_u \zeta, \eta^{-1}u + \eta^{-1}v')\ . \tag{15}$$

Theorem 5. *Let g be the Riemannian metric* (13) *on W. Given an orthonormal C-basis $\{u_1, \dots, u_k\}$ of V, set $\varphi_j = \varphi_{u_j}$. Then the maps φ_j, for $j = 0, \dots, k$, are involutive isometries and admit a homeomorphic extension to all of P_W. The collection $\big\{\big(\varphi_j(W), \varphi_j\big)\big\}_{j=0,\dots,k}$ is an atlas for a differentiable structure on P_W. Moreover, the metrics $g_j = \varphi_j^* g$ extend g to a Riemannian metric on P_W. The spaces obtained in this way are all the compact rank-one symmetric spaces. Precisely, with respect to the list in Section 4, (i), (ii), and (iii) give the real, complex and quaternionic projective spaces respectively, (iv) the exceptional symmetric space $F_4/\mathrm{Spin}\,(9)$, and (v) the real spheres.*

The extension of the φ_j to all of P_W can be easily described. We have

$$\varphi_0(0, v) = [1, v] \ , \quad \varphi_0\big([1, v]\big) = (0, v) \ , \quad \varphi_0\big([0, v]\big) = [0, v] \ ,$$

and, for a general φ_u as in (15),

$$\varphi_u(\zeta, v') = \begin{cases} \big[1, \zeta^{-1}(u + v')\big] & \text{if } \zeta \neq 0 \ , \\ [0, u + v'] & \text{if } \zeta = 0 \ , \end{cases}$$

$$\varphi_u\big([1, \eta u + v']\big) = \begin{cases} (\eta^{-1}, \eta^{-1} v') & \text{if } \eta \neq 0 \ , \\ [1, v'] & \text{if } \eta = 0 \ , \end{cases}$$

$$\varphi_u\big([0, \eta u + v']\big) = \begin{cases} (0, \eta^{-1} v') & \text{if } \eta \neq 0 \ , \\ [0, v'] & \text{if } \eta = 0 \ . \end{cases}$$

Denote by U the isometry group og P_W. It clearly contains K as the stabilizer of the origin. Notice that, by definition, the elements of K extend naturally to W_∞.

In analogy with (11), U also contains the 1-parameter group $B = \{b_t\}_{t \in \mathbb{T}}$ with

$$b_t(\zeta, v) = \big((\sin t\, \zeta + \cos t)^{-1}(\cos t\, \zeta - \sin t), (\sin t\, \zeta + \cos t)^{-1} v\big) \ , \tag{16}$$

appropriately extended to all of P_W (notice that $b_{-\pi/2} = -\varphi_0$).

Theorem 6. *P_W is two-point homogeneous, and $U = KBK$.*

It can be shown that the elements of U have the property of mapping C-lines into C-lines[4]. Besides the group U of isometries of P_W, there is a larger group acting on P_W and preserving C-lines, the *collineation group.* Inside the collineation group, U is a maximal compact subgroup.

It can also be shown that U is a compact real form of $G^{\mathbb{C}}$. Therefore, the inclusion $B_W \subset P_W$ is an instance of the Borel embedding of a symmetric space of non-compact type into a compact dual.

All this will be discussed elsewhere.

[4]For a precise statement of this property one should refer to the extension of the elements of U to P_W and to their action on the closure of C-lines in P_W, as well as on the "C-lines at infinity", modelled on projective lines in projective spaces, and whose properties we cannot describe here.

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