HARDY SPACES IN ONE COMPLEX VARIABLE

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Contents

CHAPTER I
HARDY SPACES ON THE UNIT DISC

1. Definition and basic properties

We begin by presenting the main properties of Hardy spaces on the unit disc

$$D = \{z \in \mathbb{C} : |z| < 1\} \ .$$

We shall usually prefer to denote by $\mathbb{T}$, rather than ∂D or similar, the boundary of D. So

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{it} : t \in \mathbb{R}/2\pi\mathbb{Z}\} \ . \tag{1.1}$$

The reason is that emphasis will be put on the group structure of $\mathbb{T}$. The natural identification between $\mathbb{T}$ and $\mathbb{R}/2\pi\mathbb{Z}$ (both algebraic and topological) will be always assumed. Hence functions defined on $\mathbb{T}$ will be identified with functions on $\mathbb{R}/2\pi\mathbb{Z}$, i.e. with functions on the line, periodic of period 2π.

Integrals on $\mathbb{T}$ will be understood with respect to the *normalized* Lebesgue measure $\frac{1}{2\pi}\,dt$. We shall use as alternative notation for an integral on $\mathbb{T}$ any of the following[1]:

$$\int_{\mathbb{T}} f(e^{it})\,dt \ , \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\,dt \ , \qquad \int_{\mathbb{T}} f(t)\,dt \ , \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

The spaces $L^p(\mathbb{T})$ must be intended w.r. to the normalized Lebesgue measure.

The 2-dimensional Lebesgue measure on $\mathbb{C}$ will be denoted by dz. Hence, in the polar coordinates $z = re^{it}$,

$$dz = r\,dr\,dt \ .$$

This may cause some confusion in the occasions where we shall use line integrals in $\mathbb{C}$,

$$\int_\gamma f(z)\,dz \qquad \text{or} \qquad \oint_\gamma f(z)\,dz \ ,$$

in which case the symbol dz denotes a linear differential form. However, the meaning of the symbol dz will be revealed in each case by the domain of integration.

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[1] To be precise, in the first two integrals $\mathbb{T}$ is identified with the unit circle, in the last two with $\mathbb{R}/2\pi\mathbb{Z}$. We may switch from one notation to the other, omitting explicit reference to composition with the map $t \mapsto e^{it}$.

Let $f(z)$ be a holomorphic function on D. Given $r \in [0,1)$ and $p \geq 1$, define

$$M_p(f,r) = \left(\int_{\mathbb{T}} |f(re^{it})|^p \, dt \right)^{1/p} , \tag{1.2}$$

and also

$$M_\infty(f,r) = \max_{e^{it} \in \mathbb{T}} |f(re^{it})| \ . \tag{1.3}$$

If we set

$$f_r(e^{it}) = f(re^{it})$$

for $0 \leq r < 1$, we can then say that

$$M_p(f,r) = \|f_r\|_p \ .$$

Definition[2]**.** *Let $1 \leq p \leq \infty$. We denote by $H^p(D)$ the space of holomorphic functions f on D such that*

$$\sup_{0 \leq r < 1} M_p(f,r) = \|f\|_{H^p} < \infty \ . \tag{1.4}$$

It is easy to check that (1.4) defines a norm. For the implication $\|f\|_{H^p} = 0 \Rightarrow f = 0$, one has to observe that any $f \in H^p(D)$ is continuous on D, so that $\|f_r\|_p = 0 \Rightarrow f_r = 0$, if $r < 1$.

It is quite obvious that $H^\infty(D)$ consists of the bounded holomorphic functions on D.

Proposition 1.1. *If $1 \leq p < q \leq \infty$, then $H^q(D) \subset H^p(D)$, and, for $f \in H^q(D)$,*

$$\|f\|_{H^p} \leq \|f\|_{H^q} \ .$$

Proof. The inequality $\|f_r\|_p \leq \|f_r\|_q$ follows easily from Hölder's inequality if $q < \infty$, and from the trivial majorization if $q = \infty$. Taking suprema in r, the inequality is preserved. $\square$

All these inclusions are proper. Interesting examples in this respect are the functions

$$f_\alpha(z) = \frac{1}{(1-z)^\alpha} ,$$

with[3] $\alpha > 0$. Given p, we want to determine the values of α for which f_α is in $H^p(D)$. It is obvious that $f_\alpha \notin H^\infty(D)$ for $\alpha > 0$, so we take $p < \infty$.

The answer is based on the following lemma[4].

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[2] We define Hardy spaces only for $p \geq 1$. The definition makes perfect good sense also for $p < 1$, except that in this case (1.4) does not define a norm. H^p-spaces with $p < 1$ have many intereseting features, that we will not discuss in this course.

[3] It is possible to choose a determination of the α-power of $1-z$ on D for every $\alpha \in \mathbb{C}$, bacause $1-z$ does not vanish on D and D is simply connected. The "principal" determination is $(1-z)^\alpha = e^{\alpha \log(1-z)}$, with $\arg(1-z) \in (-\pi/2, \pi/2)$ (observe that $\Re\mathrm{e}(1-z) > 0$ for $z \in D$).

[4] We write $f \asymp g$ for $r \to 1$ to denote that the ratio $|f/g|$ is bounded from above and from below by positive constants on a neighborhood of 1.

Lemma 1.2. *For $s > 0$ fixed, consider the integral*

$$I_s(r) = \int_{-\pi}^{\pi} \frac{1}{|1 - re^{it}|^s}\, dt \ ,$$

as a function of $r \in [0,1)$. Then, for $r \to 1$,

(1) *if $s < 1$, $I_s(r) \asymp 1$;*
(2) *if $s = 1$, $I_1(r) \asymp |\log(1-r)|$;*
(3) *if $s > 1$, $I_s(r) \asymp (1-r)^{-(s-1)}$.*

Proof. Take $r > \frac{1}{2}$. Considering that the triangle with vertices 1, r and re^{it} is obtuse in r, and that $|\sin\theta| \geq \frac{2}{\pi}|\theta|$ for $\theta \in [-\pi/2, \pi/2]$, we have that, for $t \in [-\pi, \pi]$,

$$\begin{aligned} |1 - re^{it}| &> \max\left\{r|1 - e^{it}|, 1 - r\right\} \\ &\geq \frac{1}{2}\left(\frac{1}{2}|1 - e^{it}| + 1 - r\right) \\ &= \frac{1}{2}\left(\left|\sin\frac{t}{2}\right| + 1 - r\right) \\ &\geq \frac{1}{2}\left(\frac{1}{\pi}|t| + 1 - r\right) \\ &\geq \frac{1}{2\pi}(|t| + 1 - r) \ . \end{aligned} \tag{1.5}$$

We also have

$$|1 - re^{it}| \leq |1 - e^{it}| + |e^{it} - re^{it}| = |1 - e^{it}| + 1 - r \leq |t| + 1 - r \ ,$$

so that

$$I_s(r) \asymp \int_{-\pi}^{\pi} \frac{1}{(|t| + 1 - r)^s}\, dt \ ,$$

for every $s > 0$. Then,

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{(|t| + 1 - r)^s}\, dt &= 2\int_0^{\pi} \frac{1}{(t + 1 - r)^s}\, dt \\ &= \begin{cases} 2\big(\log(1 - r + \pi) - \log(1 - r)\big) & \text{if } s = 1 \ , \\ \frac{2}{1-s}\big((1 - r + \pi)^{1-s} - (1 - r)^{1-s}\big) & \text{if } s \neq 1 \ . \end{cases} \end{aligned}$$

The conclusion follows easily. □

Proposition 1.3. *$f_\alpha \in H^p(D)$ if and only if $\alpha p < 1$.*

Proof. Observe that

$$M_p(f_\alpha, r) = \left(\frac{1}{2\pi} I_{\alpha p}(r)\right)^{1/p} \ .$$

It follows from Lemma 1.2 that $M_p(f_\alpha, r)$ is bounded in r if $\alpha p < 1$, and

$$\lim_{r \to 1} M_p(f_\alpha, r) = +\infty$$

if $\alpha p \geq 1$. □

We show now that $H^p(D)$ is complete. A couple of preliminary facts have their own independent importance.

Lemma 1.4. *Let $f \in H^p(D)$. Then, for every $z \in D$,*

$$|f(z)| \le C_p \frac{\|f\|_{H_p}}{(1-|z|)^{1/p}} \ .$$

Proof. The statement is obvious for $p=\infty$, so we assume that p is finite.

Take r with $|z|<r<1$, and let γ_r be the circle centered at the origin and radius r, oriented counterclockwise. Then

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(w)}{w-z}\, dw \ .$$

Setting $w = re^{it}$, we have $dw = ire^{it}\, dt$, so that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(re^{it})}{1-\frac{z}{r}e^{-it}}\, dt \ .$$

If $z = |z|e^{i\theta}$, this becomes

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(re^{it})}{1-\frac{|z|}{r}e^{i(\theta-t)}}\, dt \ .$$

By Hölder's inequality,

$$|f(z)| \le M_p(f,r) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left|1-\frac{|z|}{r}e^{i(\theta-t)}\right|^{p'}}\, dt \right)^{1/p'} ,$$

where p' is the dual exponent of p.

Making the change of variable $\theta - t = u$ and using the periodicity of the integrand, we find that

$$\int_{-\pi}^{\pi} \frac{1}{\left|1-\frac{|z|}{r}e^{i(\theta-t)}\right|^{p'}}\, dt = \int_{-\pi}^{\pi} \frac{1}{\left|1-\frac{|z|}{r}e^{iu}\right|^{p'}}\, du = I_{p'}\left(\frac{|z|}{r}\right) .$$

By Lemma 1.2, since $p' > 1$,

$$\begin{aligned} |f(z)| &\le \|f\|_{H^p} \left(\frac{1}{2\pi} I_{p'}\left(\frac{|z|}{r}\right)\right)^{1/p'} \\ &\le C_p \|f\|_{H^p} \left(1-\frac{|z|}{r}\right)^{-\frac{p'-1}{p'}} \\ &= C_p \|f\|_{H^p} \left(1-\frac{|z|}{r}\right)^{-1/p} . \end{aligned}$$

Letting $r \to 1$, we conclude the proof. □

Corollary 1.5. *Convergence in $H^p(D)$ implies uniform convergence on compact subsets of D.*

Proof. Let $K \subset D$ be compact. Then there is $r < 1$ such that $K \subset \overline{D_r}$, the closed disc of radius r centered at the origin. By Lemma 1.4, if $f_n \to f$ in $H^p(D)$,

$$\|f_n - f\|_{\infty,K} = \max_{z\in K} |f_n(z) - f(z)| \le C_p \frac{\|f_n - f\|_{H^p}}{(1-r)^{1/p}} ,$$

also tends to zero. □

Theorem 1.6. *$H^p(D)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $H^p(D)$. By Lemma 1.4, for every $r<1$ $\{f_n\}$ is a Cauchy sequence in $C(\overline{D_r})$, with the uniform norm. By the completeness of $C(\overline{D_r})$, there is $g_r \in C(\overline{D_r})$ such that $f_n \to g_r$ uniformly on $\overline{D_r}$.

Obviously, if $r_1 < r_2 < 1$, g_{r_1} and g_{r_2} coincide on $\overline{D_{r_1}}$. It follows that the various g_r, with $r<1$, are all restrictions of a unique function g continuous on D.

We now prove that g is holomorphic in D. By Morera's theorem, this is true if and only if for every closed arc γ in D,

$$\oint_\gamma g(z)\,dz = 0 \ .$$

Let γ be such an arc. Since γ is contained in $\overline{D_r}$ for some $r<1$, $f_n \to g$ uniformly on γ. Therefore

$$\oint_\gamma g(z)\,dz = \lim_{n\to\infty} \oint_\gamma f_n(z)\,dz \ ,$$

and each of these integrals is zero because the f_n are holomorphic.

We finally prove that $f_n \to g$ in $H^p(D)$. Given $\varepsilon > 0$, let $\bar{n}$ be such that $\|f_n - f_m\|_{H^p} < \varepsilon$ for $n, m \geq \bar{n}$. Take $r<1$. Since $f_m \to g$ uniformly on the circle $|z| = r$, if $n \geq \bar{n}$,

$$M_p(f_n - g, r) = \lim_{m\to\infty} M_p(f_n - f_m, r) \leq \lim_{m\to\infty} \|f_n - f_m\|_{H^p} \leq \varepsilon \ .$$

Since this holds for every $r<1$, $\|f_n - g\|_{H^p} \leq \varepsilon$. □

2. Harmonic versus holomorphic functions

A C^2-function u defined on an open set $\Omega \subseteq \mathbb{R}^n$ is called *harmonic* on Ω if its *Laplacian* Δu, defined as

$$\Delta u = \sum_{j=1}^n \partial^2_{x_j} u$$

is identically zero on Ω. A holomorphic function f is harmonic on its domain in $\mathbb{R}^2$. This follows from the fact that holomorphic functions are C^2 (in fact analytic) and from the Cauchy-Riemann equation

$$\bar{\partial}_z f = \frac{1}{2}\big(\partial_x f + i\partial_y f\big) = 0 \ .$$

Differentiating in x, we find that

$$\partial^2_x f = -i\partial_x\partial_y f = -\partial^2_y f \ .$$

Since Δ is real, if u is harmonic, so are $\bar{u}$, $\Re e u$ and $\Im m u$. In particular, anti-holomorphic functions are also harmonic.

Harmonic functions are characterized by the *mean value property.* Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, and $d\sigma$ the surface measure on it.

Definition. *A continuous function u satisfies the* mean value property *on Ω if, for any $x \in \Omega$ and any $r > 0$ such that the closed ball $\overline{B(x,r)}$ is contained in Ω,*

$$u(x) = \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} u(x+ry)\, d\sigma(y)\ . \tag{2.1}$$

In two dimensions, (2.1) becomes

$$u(z) = \int_{\mathbb{T}} u(z + re^{it})\, dt\ .$$

Theorem 2.1. *Let u be a continuous function on Ω. Then u is harmonic if and only if the mean value property holds.*

Proof. Suppose u is C^2 in Ω. Given $x \in \Omega$, consider the function

$$\varphi(r) = \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} u(x+ry)\, d\sigma(y)\ ,$$

defined for $r \in [0, r_0)$, where r_0 is the radius of the largest ball centered at x and contained in Ω. It is easy to verify that φ is continuous, $\varphi(0) = u(x)$ and that

$$\varphi'(r) = \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \frac{\partial}{\partial r} u(x+ry)\, d\sigma(y)\ .$$

For any r, the last integral can be written as an integral on the boundary of the ball $B(x,r)$, in terms of the surface measure $d\sigma_r$ on it:

$$\int_{S^{n-1}} \frac{\partial}{\partial r} u(x+ry)\, d\sigma(y) = \frac{1}{r^{n-1}} \int_{\partial B(x,r)} \nu \cdot \nabla u(y)\, d\sigma_r(y)\ ,$$

where ν denotes the outer normal to $\partial B(x,r)$.

By Green's formula, we obtain that

$$\varphi'(r) = \frac{1}{\sigma(S^{n-1}) r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy\ .$$

It follows that, if u is harmonic, then $\varphi' = 0$, hence φ is identically equal to $u(x)$, which proves the mean value property.

Conversely, if u is C^2 and satisfies the mean value property, the same argument shows that $\int_{B(x,r)} \Delta u(y)\, dy = 0$ for any ball $B(x,r) \subset \Omega$. This implies that u is harmonic.

It remains to prove that if u is only continuous on Ω and satisfies the mean value property, then it is C^2. Take a ball $B(x_0, r) \subset \Omega$, and choose a C^2-function $\psi(x)$ on $\mathbb{R}^n$, which is radial, non-negative, non-identically zero, and supported on $B(0, r/2)$. Define

$$v(x) = \int_{B(0,r/2)} u(x+y)\psi(y)\, dy\ , \tag{2.2}$$

which is well-defined for $x \in B(x_0, r/2)$. Since ψ is radial, we can consider its "profile" ψ_0, defined on the positive half-line so that $\psi(x) = \psi_0(|x|)$. We then have, integrating in polar coordinates:

$$\begin{aligned} v(x) &= \int_0^{\frac{r}{2}} \int_{S^{n-1}} u(x+\rho w)\psi_0(\rho)\rho^{n-1}\, d\sigma(w)\, d\rho \\ &= \int_0^{\frac{r}{2}} \psi_0(\rho)\rho^{n-1}\Big(\int_{S^{n-1}} u(x+\rho w)\, d\sigma(w)\Big)\, d\rho \\ &= \Big(\sigma(S^{n-1})\int_0^{\frac{r}{2}} \psi_0(\rho)\rho^{n-1}\, d\rho\Big) u(x)\ . \end{aligned}$$

Since the expression in parentheses is positive, we conclude that v is a non-zero constant multiple of u on $B(x_0, r/2)$. On the other hand, changing variable of integration in (2.2), we find that, for $x \in B(x_0, r/2)$,

$$v(x) = \int_{B(x,r/2)} u(y)\psi(y-x)\, dy = \int_{B(x_0,r)} u(y)\psi(y-x)\, dy$$

(the last equality takes into account the fact that $\psi = 0$ in the extra part of the domain of integration). This last expression shows, by differentiating under integral sign, that v is C^2 on $B(x_0, r/2)$. The same is then true for u, and, since x_0 is arbitrary, u is C^2 on Ω. $\square$

Corollary 2.2. *Harmonic functions satisfy the maximum modulus principle: if u is harmonic on a connected open set Ω, and it attains its maximum modulus at a point in Ω, then it is constant.*

Proof. Suppose that $x_0 \in \Omega$ is such that $|u(x_0)| = \max_{x\in\Omega}|u(x)|$. By replacing u by $e^{i\theta}u$ for an appropriate θ, we can assume that $u(x_0)$ is real and non-negative. Let $\overline{B(x_0, r)} \subset \Omega$. Since $\Re e u$ is harmonic,

$$\begin{aligned} u(x_0) &= \Re e u(x_0) \\ &= \frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}} \Re e u(x_0 + ry)\, d\sigma(y)\ . \end{aligned}$$

Therefore

$$\frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}} \big(u(x_0) - \Re e u(x_0 + ry)\big)\, d\sigma(y) = 0\ ,$$

with a continuous integrand $u(x_0) - \Re e u(x_0 + ry) \geq 0$. Therefore $u(x_0) = \Re e u(x_0 + ry)$ for every $y \in S^{n-1}$. Since $|u(x_0 + ry)| \leq u(x_0)$, we conclude that $u(x_0 + ry) = u(x_0)$.

This proves that u is constant on a neighborhood of x_0. It follows that the set of points $x \in \Omega$ where $u(x) = u(x_0)$ is both closed and open. Since Ω is connected, this set is all of Ω. $\square$

We shall now restrict ourselves to $n = 2$, and focus our attention on the relations between harmonic and holomorphic functions. We shall need the following lemma of Fourier analysis.

Lemma 2.3. *Let f be a C^2-function on $\mathbb{T}$, and denote by $\hat{f}(n)$, with $n \in \mathbb{Z}$ its Fourier coefficients. Then*

$$n^2|\hat{f}(n)| \leq \|f\|_{C^2} \ ,$$

and the Fourier series of f converges to f uniformly on $\mathbb{T}$.

Proof. Integrating by parts twice,

$$n^2\hat{f}(n) = \int_{\mathbb{T}} n^2 f(t)e^{-int}\,dt = -\int_{\mathbb{T}} f''(t)e^{-int}\,dt \ .$$

Therefore,

$$n^2|\hat{f}(n)| \leq \|f''\|_\infty \leq \|f\|_{C^2} \ .$$

By the Weierstrass test[5], it follows that the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$$

is uniformly convergent on $\mathbb{T}$. If g is its sum, integration term by term shows that $\hat{g}(n) = \hat{f}(n)$ for every n. This implies that $g = f$. □

Theorem 2.4. *Let u be harmonic on a neighborhood of the closed disc $\overline{B(z_0, R)} \subset \mathbb{C}$. Then u admits a power series expansion*

$$u(z) = a_0 + \sum_{n=1}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(\overline{z - z_0})^n \ , \tag{2.3}$$

uniformly convergent on $\overline{B(z_0, R)}$. In particular, u is real-analytic, and it is the sum of a holomorphic and an anti-holomorphic function.

Proof. We can assume for simplicity that $z_0 = 0$ and $R = 1$, so that $B(0, R) = D$. The restriction of u to $\mathbb{T}$ is C^∞, hence its Fourier coefficients a_n form an absolutely summable sequence. The function

$$v(re^{it}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n$$

is uniformly convergent on $\overline{D}$ and harmonic in the interior. Since $v = u$ on the boundary, the maximum principle implies that $v = u$ in $\overline{D}$. □

Corollary 2.5. *Let u be harmonic on a connected and simply connected domain Ω. Then there are holomorphic functions f and g on Ω, unique up to additive constants, such that $u = f + \bar{g}$.*

Proof. By Theorem 2.4, any $z \in \Omega$ has a spherical neighborhood U_z where $u = f_z + \overline{g_z}$, with f_z, g_z holomorphic. If $U_z \cap U_{z'} \neq \emptyset$, on the intersection $f_z - f_{z'} = \overline{g_{z'} - g_z}$. Since the left-hand side is holomorphic and the right-hand side anti-holomorphic, this implies that

$$f_z - f_{z'} = \overline{g_{z'} - g_z} = c_{z,z'}$$

[5]The Weierstrass test says that if the functions f_n satisfy inequalities $|f_n(x)| \leq M_n$ on a set E, with $M_n \geq 0$ and $\sum M_n < \infty$, then the series $\sum f_n$ converges uniformly on E.

is constant. Therefore $f'_z = f'_{z'}$ on $U_z \cap U_{z'}$, so that all these derivatives can be unified into a unique holomorphic function h on Ω. Since Ω is simply connected, h admits a primitive f, holomorphic on all of Ω. Then $f_z - f = b_z$ is constant on each U_z. On each U_z,

$$u - f = \overline{g_z} + f_z - f = \overline{g_z} + b_z \ ,$$

showing that $u - f$ is anti-holomorphic on Ω. Setting $g = \overline{u - f}$ we have that $u = f + \bar{g}$. If f_1 and g_1 are holomorphic and $u = f_1 + \overline{g_1}$ on Ω, we repeat the argument given before: $f - f_1 = \overline{g_1 - g}$ must be constant. □

Corollary 2.6. *Let u be a real-valued harmonic function on a connected and simply connected domain Ω. There is a unique, up to additive constants, real-valued harmonic function $\tilde{u}$ on Ω such that $f = u + i\tilde{u}$ is holomorphic.*

Proof. According to Corollary 2.5, we can write $u = \frac{1}{2}(f+\bar{g})$, with f, g holomorphic. Since u is real, $\Im\mathrm{m} f = \Im\mathrm{m} g$. Then $f-g$ is a real-valued holomorphic function, hence it is constant. Adjusting, if necessary, f and g by adding appropriate constants, we can assume that $f = g$. So $u = \Re\mathrm{e} f$, and we then take $\tilde{u} = \Im\mathrm{m} f$.

If v is another real-valued harmonic function such that $u + iv$ is holomorphic, $v - \tilde{u} = -i(u + iv) + i(u + i\tilde{u})$ is a real-valued holomorphic function. Then it is constant. □

Definition. *The function $\tilde{u}$ is called a* harmonic conjugate *of u on Ω. If $u = u_1 + iu_2$ is harmonic and complex-valued, the harmonic conjugates of u are the functions $\tilde{u} = \widetilde{u_1} + i\widetilde{u_2}$, with $\widetilde{u_1}$ and $\widetilde{u_2}$ harmonic conjugates of u_1 and u_2 respectively.*

If $\Omega = D$, one usually normalizes "the" conjugate harmonic function of u by imposing that $\tilde{u}(0) = 0$. Consider the power series expansion (2.3) of u,

$$u(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z^n}$$

The fact that u is real is reflected by the conditions

$$a_0 \in \mathbb{R} \ , \qquad a_{-n} = \overline{a_n} \ .$$

Then

$$u(z) = a_0 + 2\Re\mathrm{e}\left(\sum_{n=1}^{\infty} a_n z^n \right) \ .$$

We then take

$$\tilde{u}(z) = 2\Im\mathrm{m}\left(\sum_{n=1}^{\infty} a_n z^n \right) \ ,$$

in order to have

$$u + i\tilde{u} = a_0 + 2\sum_{n=1}^{\infty} a_n z^n \ . \tag{2.6}$$

The power series expansion of $\tilde{u}$ is therefore

$$\tilde{u}(z) = -i\sum_{n=1}^{\infty} a_n z^n + i\sum_{n=1}^{\infty} a_{-n} \overline{z^n} \ . \tag{2.7}$$

By linearity, formulas (2.6) and (2.7) remain valid if u is complex-valued.

3. Poisson integrals

The *Dirichlet problem* on D consists in assigning a continuous function f on $\mathbb{T}$ and seeking for a function u continuous on $\bar{D}$ and harmonic in D, which coincides with f on $\mathbb{T}$. In other words, we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \mathbb{T} \end{cases} \tag{3.1}$$

in $u \in C(\bar{D})$. The maximum modulus principle implies that a solution, if it exists, is unique. In fact, one just has to observe that the difference of two solutions would be continuous on $\bar{D}$, harmonic in D, and identically zero on $\mathbb{T}$.

We shall prove the existence of the solution and give its expression.

If $f(e^{it}) = e^{int}$ with $n \in \mathbb{Z}$, the solution of (3.1) is easily found as

$$\begin{cases} u(z) = z^n & \text{if } n \geq 0 \ , \\ u(z) = \bar{z}^{|n|} & \text{if } n < 0 \ . \end{cases}$$

Suppose next that $f \in C^2(\mathbb{T})$, and let

$$f(e^{it}) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int}$$

be its Fourier series. It is natural to construct

$$u(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n + \sum_{n=1}^{\infty} \hat{f}(-n) \bar{z}^n \ . \tag{3.2}$$

It follows from Lemma 2.3 that this series converges uniformly on $\bar{D}$. Since each summand is harmonic in D, so is the sum (verify the mean value property). Then this is the solution of (3.1).

We use some Fourier analysis to derive an integral formula giving u. By (3.2), the Fourier series of $u_r(e^{it}) = u(re^{it})$ is, for $r < 1$,

$$u_r(e^{it}) = \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{int} \ .$$

We set

$$P_r(e^{it}) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{int} \ , \tag{3.3}$$

where the series converges uniformly on $\mathbb{T}$. Then[6]

$$u_r(e^{it}) = f * P_r(e^{it}) = \int_{\mathbb{T}} f(e^{i(t-u)}) P_r(e^{iu}) \, du \ .$$

The functions P_r in (3.3) form the *Poisson kernel* on $\mathbb{T}$.

[6] The following identity can be verified directly, expanding one of the two factors in Fourier series and integrating term by term. It is however a consequence of the general identity $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ and of the uniqueness theorem for Fourier series expansions.

Lemma 3.1. *The Poisson kernel equals*

$$P_r(e^{it}) = \frac{1-r^2}{1+r^2-2r\cos t}\ , \qquad r<1\ .$$

The function $P(re^{it}) = P_r(e^{it})$ *is harmonic in* D.

Proof. Consider the geometric series

$$\sum_{n=0}^{\infty} r^n e^{int} = \sum_{n=0}^{\infty} (re^{it})^n = \frac{1}{1-re^{it}}\ .$$

Then

$$\begin{aligned} P_r(e^{it}) &= 2\Re\mathrm{e}\Big(\sum_{n=0}^{\infty} r^n e^{int}\Big) - 1 \\ &= 2\Re\mathrm{e}\frac{1-re^{-it}}{|1-re^{it}|^2} - 1 \\ &= 2\frac{1-rcost}{1+r^2-2r\cos t} - 1 \\ &= \frac{1-r^2}{1+r^2-2r\cos t}\ . \end{aligned}$$

Moreover, the identity $P(z) = 2\Re\mathrm{e}(1-z)^{-1} - 1$ shows that P is harmonic in D. □

We have so proved that, for $f \in C^2(\mathbb{T})$ and $r<1$,

$$\begin{aligned} u(re^{it}) &= f * P_r(e^{it}) \\ &= \int_{\mathbb{T}} f(e^{i(t-u)}) \frac{1-r^2}{1+r^2-2r\cos u}\, du \\ &= \int_{\mathbb{T}} f(e^{iu}) \frac{1-r^2}{1+r^2-2r\cos(t-u)}\, du\ . \end{aligned}$$

Another way of writing the same identity is

$$u(z) = \int_{\mathbb{T}} f(e^{iu}) P(e^{-iu}z)\, du\ . \tag{3.4}$$

Observe now that (3.4) makes sense for f continuous and defines a harmonic function u in D (again, verify the mean value property using the fact that P is harmonic). It takes some work to verify that u is continuous on $\bar{D}$. We do this now, introducing some preliminary more general notions.

Definition. *Let* $\{\varphi_r\}_{r<1}$ *be a family of integrable functions on* $\mathbb{T}$. *We say that they form an* approximate identity for $r \to 1$ *if*[7]

(1) $\int_{\mathbb{T}} |\varphi_r(t)|\, dt \le C$ *for some constant* C *and every* r;
(2) $\int_{\mathbb{T}} \varphi_r(t)\, dt = 1$ *for every* r;
(3) *for every* $\delta > 0$,

$$\lim_{r\to 1} \int_{\delta \le |t| \le \pi} |\varphi_r(t)|\, dt = 0\ .$$

[7] To simplify the notation, here we think of the φ_r as periodic functions on the line. We then write $\varphi_r(t)$ instead of $\varphi_r(e^{it})$. The same for f in the next proposition.

The notion of "approximate identity" is quite general. We have chosen here to give the definition with a parameter r tending to 1. The same definition can be given, with the obvious modifications, with parameters varying on a different set, for instance for a sequence of functions. Later on, we will find approximate identities depending on a positive real parameter ε tending to 0.

The name "approximate identity" is justified by the following property.

Proposition 3.2. *Let $\{\varphi_r\}$ be an approximate identity on $\mathbb{T}$. If $f \in C(\mathbb{T})$,*

$$\lim_{r\to 1} f * \varphi_r = f \tag{3.5}$$

uniformly on $\mathbb{T}$. If $f \in L^p(\mathbb{T})$, with $1 \le p < \infty$, the limit (3.5) holds in the L^p-norm.

Proof. Suppose f is continuous. By uniform continuity, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t-u) - f(t)| < \varepsilon$ for $|u| < \delta$. By condition (3), there is $r_0 < 1$ such that for $r_0 < r < 1$,

$$\int_{\delta\le|t|\le\pi} |\varphi_r(t)|\,dt < \varepsilon\ .$$

Observe that, by condition (2), we can write

$$f(t) = \int_{\mathbb{T}} f(t)\varphi_r(u)\,du\ .$$

If $r_0 < r < 1$, we have

$$\begin{aligned}
\big|f * \varphi_r(t) - f(t)\big| &= \left|\int_{\mathbb{T}} \big(f(t-u) - f(t)\big)\varphi_r(u)\,du\right| \\
&\le \frac{1}{2\pi}\int_{|u|<\delta} \big|f(t-u) - f(t)\big|\big|\varphi_r(u)\big|\,du \\
&\qquad + \frac{1}{2\pi}\int_{\delta\le|u|\le\pi} \big|f(t-u) - f(t)\big|\big|\varphi_r(u)\big|\,du \\
&< \frac{1}{2\pi}\varepsilon\|\varphi_r\|_1 + \frac{1}{\pi}\|f\|_\infty\varepsilon \\
&\le \frac{1}{2\pi}\big(C + 2\|f\|_\infty\big)\varepsilon\ .
\end{aligned}$$

This proves the uniform convergence of $f * \varphi_r$ to f for f continuous.

Suppose now that $f \in L^p(\mathbb{T})$, with $1 \le p < \infty$. Given $\varepsilon > 0$, there is $g \in C(\mathbb{T})$ such that $\|f - g\|_p < \varepsilon$. Then

$$\|f * \varphi_r - f\|_p \le \|(f-g) * \varphi_r\|_p + \|g * \varphi_r - g\|_p + \|g - f\|_p\ .$$

We apply Young's inequality[8] to the first summand, to obtain

$$\|(f-g) * \varphi_r\|_p \le \|f - g\|_p\|\varphi_r\|_1 < C\varepsilon\ .$$

[8] More generally, Young's inequality says that if $f \in L^p(\mathbb{T})$, $g \in L^q(\mathbb{T})$, and $\frac{1}{p} + \frac{1}{q} \ge 1$, then $f * g \in L^r(\mathbb{T})$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. This property is not specific of convolution on $\mathbb{T}$, but it holds for convolution on any locally compact group (for instance on $\mathbb{R}$). Another general property is that, when p and q are conjugate exponents (hence $r = \infty$), $f * g$ is continuous.

We then observe that

$$\|g * \varphi_r - g\|_p \le \|g * \varphi_r - g\|_\infty \ ,$$

so that, by the previous part of the proof, there is r_0 such that, for $r_0 < r < 1$, $\|g * \varphi_r - g\|_p < \varepsilon$. Then,for $r_0 < r < 1$,

$$\|f * \varphi_r - f\|_p < (C + 2)\varepsilon \ .$$

This proves that $f * \varphi_r$ tends to f in $L^p(\mathbb{T})$. □

Observe that the assumption $f \in L^\infty(\mathbb{T})$ does not imply that $f * \varphi_r \to f$ in the L^∞-norm. Since each $f * \varphi_r$ is continuous, this would imply that f is continuous too.

With minor modifications to the proof, one can verify that the following more general statement holds.

Corollary 3.3. *Assume that the family* $\{\varphi_r\}_{r<1}$ *satisfies* (1), (3) *and*

(2')
$$\lim_{r\to 1} \int_{\mathbb{T}} \varphi_r(t)\, dt = c \ .$$

Then

$$\lim_{r\to 1} f * \varphi_r = cf$$

uniformly on $\mathbb{T}$ *if* $f \in C(\mathbb{T})$, *and in the* L^p*-norm if* $f \in L^p(\mathbb{T})$, *with* $1 \le p < \infty$.

Proposition 3.4. *The Poisson kernel* $\{P_r\}$ *is an approximate identity for* $r \to 1$.

Proof. Since the P_r are positive functions, (1) will be a consequence of (2). Using the power series expansion (3.3), we have, since the convergence is uniform,

$$\int_{\mathbb{T}} P_r(t)\, dt = \sum_{n\in\mathbb{Z}} r^{|n|} \int_{\mathbb{T}} e^{int}\, dt = 1 \ ,$$

because all the terms with $n \ne 0$ give integral zero.

We then prove (3). Using (1.5), we have, for $t \in [-\pi, \pi]$,

$$P_r(t) = \frac{1 - r^2}{|1 - re^{it}|^2} \le \pi^2 \frac{1 - r^2}{(|t| + 1 - r)^2} < 2\pi^2 \frac{1 - r}{(|t| + 1 - r)^2} \ . \tag{3.6}$$

So, given $\delta > 0$,

$$\begin{aligned} \int_{\delta\le|t|\le\pi} P_r(t)\, dt &< 2\pi^2 \int_{|t|\ge\delta} \frac{1 - r}{(|t| + 1 - r)^2}\, dt \\ &= 2\pi^2 \int_{|u|\ge\frac{\delta}{1-r}} \frac{1}{(|u| + 1)^2}\, du \ , \end{aligned}$$

where we have made the change of variable $t = (1 - r)u$. Since $(|u| + 1)^{-2}$ is integrable on the line,

$$\lim_{r\to 1} \int_{|u|\ge\frac{\delta}{1-r}} \frac{1}{(|u| + 1)^2}\, du = 0 \ ,$$

and this proves (3). □

Theorem 3.5. *For $f \in C(\mathbb{T})$ the unique solution to the Dirichlet problem* (3.1) *is* $u(re^{it}) = f * P_r(e^{it})$.

Proof. It follows from the uniform convergence of u_r to f and the maximum principle. □

The same approach that we have used to obtain the Poisson integral u of a continuous function f can be used to give a formula for its conjugate harmonic function $\tilde{u}$. Using the Fourier series expansion of f, (3.2) and (2.7), we find that

$$\tilde{u}(z) = -i \sum_{n=1}^{\infty} \hat{f}(n) z^n + i \sum_{n=1}^{\infty} \hat{f}(-n) \bar{z}^n \ .$$

Then

$$\tilde{u}_r(e^{it}) = -1 \sum_{n \neq 0} \hat{f}(n) \operatorname{sgn} n \, r^{|n|} e^{int} = f * \tilde{P}_r(e^{it}) \ , \tag{3.7}$$

where

$$\begin{aligned} \tilde{P}_r(e^{it}) &= -i \sum_{n \neq 0} \operatorname{sgn} n \, r^{|n|} e^{int} \\ &= 2\Im\mathrm{m} \sum_{n=0}^{\infty} r^n e^{int} \\ &= 2\Im\mathrm{m} \frac{1}{1 - re^{it}} \\ &= \frac{2r \sin t}{1 + r^2 - 2r \cos t} \ . \end{aligned} \tag{3.8}$$

The functions $\tilde{P}_r$ form the *conjugate Poisson kernel.*

4. Functions in h^p-spaces and their limits to the boundary

If u is harmonic in D, we define the means $M_p(u, r)$ according to (1.2) and (1.3). We also define $h^p(D)$ as the space of *harmonic functions* u such that

$$\|u\|_{h^p} = \sup_{0 \le r < 1} M_p(u, r) < \infty \ .$$

The statements of Proposition 1.1, Lemma 1.4, Corollary 1.5 and Theorem 1.6 remain valid with H^p replaced by h^p. Only Lemma 1.4 requires a modification in the proof, which goes as follows.

Lemma 4.1. *Let $u \in h^p(D)$. Then, for every $z \in D$,*

$$|u(z)| \le C_p \frac{\|u\|_{h_p}}{(1 - |z|)^{1/p}} \ .$$

Proof. Take ρ so that $|z| = r < \rho < 1$, and set $v(z) = u(\rho z)$. Then v is harmonic in D and continuous on $\bar{D}$. By Theorem 3.5,

$$u_r = v_{r/\rho} = v_1 * P_{r/\rho} = u_r * P_{r/\rho} \; . \tag{4.1}$$

By Young's inequality,

$$|u(z)| \leq \|u_r\|_\infty \leq \|u_\rho\|_p \|P_{r/\rho}\|_{p'} \leq \|u\|_{h^p} \|P_{r/\rho}\|_{p'} \; ,$$

where p' is the dual exponent of p. We then estimate the L^q-norm of P_r for any q. We know that $\|P_r\|_1 = 1$. Moreover

$$\|P_r\|_\infty = \max_{|t|\leq\pi} \frac{1-r^2}{1+r^2-2r\cos t} = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r} \; .$$

If $1 < q < \infty$,

$$\begin{aligned} \|P_r\|_q &= \left(\int_{\mathbb{T}} P_r(e^{it})^q \, dt \right)^{\frac{1}{q}} \\ &\leq \|P_r\|_\infty^{\frac{q-1}{q}} \, \|P_r\|_1^{\frac{1}{q}} \\ &\leq \left(\frac{1+r}{1-r} \right)^{\frac{q-1}{q}} \; . \end{aligned}$$

Majorizing $1 + r$ with 2, we then have

$$\|P_r\|_q \leq 2^{1/q'} (1-r)^{-1/q'} \; .$$

Therefore

$$|u(z)| \leq C_p \frac{\|u\|_{h^p}}{\left(1 - \frac{r}{\rho}\right)^{1/p}} \; ,$$

for every $\rho \in (r, 1)$. Letting ρ tend to 1, we have the conclusion. □

Lemma 4.2. *If u is harmonic in D, $M_p(u, r)$ is a non-decreasing function of r. In particular, $u \in h^p(D)$ if and only if*

$$\lim_{r \to 1} M_p(u, r) < \infty \; ,$$

and $\|u\|_{h^p}$ equals this limit.

Proof. If $r' < r < 1$,

$$u_{r'} = v_{r'/r} = v_1 * P_{r'/r} = u_r * P_{r'/r} \; ,$$

by (4.1). By Young's inequality[9], for any $p \in [1, \infty]$,

$$\|u_{r'}\|_p \leq \|u_r\|_p \|P_{r'/r}\|_1 = \|u_r\|_p \; . \qquad \square$$

[9]For $p = \infty$, one can simply invoke the maximum principle.

Lemma 4.2 applies in particular to holomorphic functions and H^p-spaces.

A simple way to construct h^p-functions, for any $p \in [1, \infty]$, consists in taking $f \in L^p(\mathbb{T})$ and constructing the Poisson integral

$$u(re^{it}) = f * P_r(e^{it}) \ .$$

It follows from Young's inequality that $\|u\|_{h^p} \le \|f\|_p$.

For $p = 1$, the above construction can be extended to more general objects defined on the boundary. Take a regular Borel measure μ on $\mathbb{T}$ – we write $\mu \in M(\mathbb{T})$ – and define

$$u(re^{it}) = \mu * P_r(e^{it}) = \int_{\mathbb{T}} P_r(e^{i(t-t')})\, d\mu(t') \ .$$

Since the convolution of a finite measure and an L^1-function is in L^1, $u_r \in L^1(\mathbb{T})$ for $r < 1$, and

$$\|u_r\|_1 \le \|\mu\|_M \|P_r\|_1 \le \|\mu\|_M \ . \tag{4.2}$$

The usual verification of the mean value property shows that u is harmonic in D, so that $u \in h^1(D)$ and $\|u\|_{h^1} \le \|\mu\|_M$.

The question we will discuss now is if every h^p-function can be obtained in this way, i.e. if every h^p-function is the Poisson integral of an L^p-function on $\mathbb{T}$ if $1 < p \le \infty$, and of a regular Borel measure[10] if $p = 1$.

Theorem 4.3. *Consider the operator $\mathcal{P}$ mapping f (understood as either a function or a Borel measure on $\mathbb{T}$) into the harmonic function $\mathcal{P}f$ on D given by*

$$(\mathcal{P}f)_r = f * P_r \ . \tag{4.3}$$

Then $\mathcal{P}$ maps $L^p(\mathbb{T})$ isometrically onto $h^p(D)$ for $1 < p \le \infty$, and it maps $M(\mathbb{T})$ isometrically onto $h^1(D)$.

The limit $\lim_{r\to 1}(\mathcal{P}f)_r$ exists in L^p if and only if one of the following holds

(1) *$f \in L^p(\mathbb{T})$ and $1 \le p < \infty$;*
(2) *$p = \infty$ and $f \in C(\mathbb{T})$.*

In each of these cases, $(\mathcal{P}f)_r \to f$ in the L^p-norm.

For general elements f of $M(\mathbb{T})$ or $L^\infty(\mathbb{T})$, $(\mathcal{P}f)_r$ tends to f in the corresponding weak-topology*[11].

Proof. Suppose first that $1 < p < \infty$. Take $f \in L^p(\mathbb{T})$ and set $u = \mathcal{P}f$. By Propositions 3.2 and 3.4, $u_r \to f$ in the L^p-norm. In particular,

$$\lim_{r\to 1} M_p(u, r) = \|f\|_p \ , \tag{4.4}$$

so that $\|\mathcal{P}f\|_{h^p} = \|f\|_p$. We prove next that $\mathcal{P} : L^p(\mathbb{T}) \to h^p(D)$ is onto. Take $u \in h^p(D)$ and let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of radii tending to 1. Since $\|u_{r_j}\| \le \|u\|_{h^p}$,

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[10] We regard $L^1(\mathbb{T})$ as a subspace of $M(\mathbb{T})$, identifying the function f with the measure μ such that $d\mu(t) = f(t)\, dt$.

[11] We recall that $M(\mathbb{T})$ is the dual space of $C(\mathbb{T})$ and $L^\infty(\mathbb{T})$ is the dual space of $L^1(\mathbb{T})$. The weak*-topology on the dual space X' of a Banach space X is the weakest topology induced by the elements of X as linear functionals on X'.

it follows from the Banach-Alaoglu theorem[12] that some subsequence $\{u_{r_{j_k}}\}$ has a limit $f \in L^p(\mathbb{T})$ in the weak* topology of $L^p(\mathbb{T})$. Set $v = \mathcal{P}f$ and take $r < 1$. Since $P_r \in L^{p'}(\mathbb{T})$, and by (4.3),

$$\begin{aligned}
v(re^{it}) &= f * P_r(e^{it}) \\
&= \int_{\mathbb{T}} f(e^{it'}) P_r(t-t')\,dt' \\
&= \lim_{k\to\infty} \int_{\mathbb{T}} u_{r_{j_k}}(e^{it'}) P_r(t-t')\,dt' \\
&= \lim_{k\to\infty} u_{r_{j_k}} * P_r(e^{it}) \\
&= \lim_{k\to\infty} u(r r_{j_k} e^{it}) \\
&= u(re^{it})
\end{aligned} \tag{4.5}$$

For $p = 1$, we know by (4.2) that $\mathcal{P} : M(\mathbb{T}) \to h^1(D)$ is continuous with norm at most 1. In order to prove that this map is onto, we can repeat the argument given above, using the fact that $M(\mathbb{T})$ is the dual space of $C(\mathbb{T})$, according to the Riesz representation theorem. As before, we take $u \in h^1(D)$, a sequence $\{r_j\}$ of radii tending to 1, and we regard $\{u_{r_j}\}$ as a bounded sequence in $M(\mathbb{T})$. From it, we can extract a subsequence $\{u_{r_{j_k}}\}$ having a weak* limit $\mu \in M(\mathbb{T})$. The proof that $u = \mathcal{P}\mu$ follows the same lines as in (4.5). Moreover, If $g \in C(\mathbb{T})$,

$$\begin{aligned}
\left| \int_{\mathbb{T}} g(t)\,d\mu(t) \right| &= \lim_{k\to\infty} \left| \int_{\mathbb{T}} g(t) u_{r_{j_k}}(t)\,dt \right| \\
&\le \liminf_{k\to\infty} \int_{\mathbb{T}} |g(t)||u_{r_{j_k}}(t)|\,dt \\
&\le \|g\|_\infty \liminf_{k\to\infty} \|u_{r_{j_k}}\|_1 \\
&= \|g\|_\infty \|u\|_{h^1} .
\end{aligned}$$

Therefore

$$\|\mu\|_M = \sup_{\|g\|_\infty = 1} \left| \int_{\mathbb{T}} g(t)\,d\mu(t) \right| \le \|u\|_{h^1} .$$

Putting this together with (4.2), we have that $\|\mathcal{P}\mu\|_{h^1} = \|\mu\|_M$.

Consider now $p = \infty$, and let $f \in L^\infty(\mathbb{T})$. Then $u = \mathcal{P}f$ satisfies $\|u_r\|_\infty \le \|f\|_\infty \|P_r\|_1$, so that $u \in h^\infty(D)$ and $\|u\|_{h^\infty} \le \|f\|_\infty$. Using the fact that $L^\infty(\mathbb{T})$ is the dual space of $L^1(\mathbb{T})$, the same proof given above for $p = 1$, shows that $\mathcal{P} : L^\infty(\mathbb{T}) \to h^\infty(D)$ is onto and isometric.

We pass now to the second part of the statement.

By Propositions 3.2 and 3.4, $\lim_{r\to 1} (\mathcal{P}f)_r$ exists in the L^p-norm and is equal to f in cases (1) and (2). Consider $p = 1$ and take $\mu \in M(\mathbb{T})$. In order to prove that $(\mathcal{P}\mu)_r \to \mu$ as $r \to 1$ in the weak* topology, we take $g \in C(\mathbb{T})$. Using Fubini's

[12] The Banach-Alaoglu theorem says that, if X is a separable Banach space and X' is its dual space, the closed balls in X' are sequentially compact in the weak* topology.

theorem and the parity of P_r (see Lemma 3.1),

$$\begin{aligned}\int_{\mathbb{T}} g(t)(\mathcal{P}\mu)_r(t)\,dt &= \int_{\mathbb{T}} g(t)(\mu * P_r)(t)\,dt \\ &= \int_{\mathbb{T}}\int_{\mathbb{T}} g(t)P_r(t-t')\,d\mu(t')\,dt \\ &= \int_{\mathbb{T}}\left(\int_{\mathbb{T}} g(t)P_r(t-t')\,dt\right)d\mu(t') \\ &= \int_{\mathbb{T}} (g * P_r)(t')\,d\mu(t')\ .\end{aligned}$$

Since g is continuous, $g * P_r \to g$ uniformly. Hence

$$\lim_{r\to 1}\int_{\mathbb{T}} g(t)(\mathcal{P}\mu)_r(t)\,dt = \int_{\mathbb{T}} g(t')\,d\mu(t')\ ,$$

showing that μ is the weak* limit of $(\mathcal{P}\mu)_r$. If the $(\mathcal{P}\mu)_r$ have a strong limit as $r \to 1$, this limit is in $L^1(\mathbb{T})$, and at the same time it must coincide with μ, because the norm topology is stronger than the weak* topology. This implies that $\mu \in L^1(\mathbb{T})$.

The same argument, replacing $g \in C(\mathbb{T})$ with $g \in L^1(\mathbb{T})$, shows that, if $f \in L^\infty(\mathbb{T})$, then $(\mathcal{P}f)_r \to f$ in the weak* topology, and that the convergence is in norm if and only if f is continuous. □

Observe that the inclusions $h^p(D) \subset h^q(D)$ if $q < p$ also give other consequences for convergence to the boundary. For instance, if u is bounded on D, i.e. $u \in h^\infty(D)$, and f is its boundary function, then $u_r \to f$ in any L^q-norm for $q < \infty$.

Corollary 4.4. *For $r < 1$, let $u_{(r)}(z) = u(rz)$. If $u \in h^p(D)$, with $1 < p < \infty$, $u_{(r)} \to u$ in the h^p-norm as $r \to 1$. The same is true for $p = 1$, provided $u^\sharp \in L^1(\mathbb{T})$ and for $p = \infty$, provided $u^\sharp$ is continuous. For $1 < p < \infty$, harmonic polynomials are dense in $h^p(D)$.*

Proof. The first part is a direct consequence of Theorem 4.3. Given $\varepsilon > 0$, take $r < 1$ so that $\|u^\sharp - u_r\|_p < \varepsilon$. Since each half of the Taylor series of u,

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n\ ,$$

converges to u uniformly on compact sets, there is N such that the sum $s_N(z)$ of the two partial sums of order N satisfies $|u(z) - s_N(z)| < \varepsilon$ for $|z| = r$. Then $\|u^\sharp - (u_N)_r\|_p < 2\varepsilon$. □

5. Boundary limits of conjugate harmonic functions

A natural question concerns conjugate harmonic functions: given $u \in h^p(D)$, is its conjugate harmonic function $\tilde{u}$ also in $h^p(D)$? At this stage we only give the answer for $p = 2$ (which is positive) and for $p = 1, \infty$ (which is negative).

The (positive) answer for the other values of p will be given in Chapter III. In the last part of this section we discuss under which assumptions the congiugate harmonic function of a function u continuous up to the boundary can be extended continuously to the boundary.

The case $p = 2$ is special in the sense that $h^2(D)$, being isometric with $L^2(\mathbb{T})$, is a Hilbert space. To be precise, given $u, v \in h^2(D)$, there are $f, g \in L^2(\mathbb{T})$ such that $u = \mathcal{P}f$, $v = \mathcal{P}g$. If we set

$$\langle u, v\rangle_{h^2} = \langle f, g\rangle_{L^2} \ ,$$

this inner product induces the h^2-norm[13]. It is also clear that $\mathcal{P}$ transforms orthonormal bases of $L^2(\mathbb{T})$ into orthonormal bases of $h^2(D)$. In particular, from the basis $\{e^{int}\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ we derive the orthonormal basis $\{z^n\}_{n\in\mathbb{N}} \cup \{\bar{z}^n\}_{n\geq 1}$ of $h^2(D)$.

Therefore, if

$$u = \sum_{n\geq 0} a_n z^n + \sum_{n\geq 1} a_{-n}\bar{z}^n \ ,$$

then

$$\|u\|^2_{h^2} = \sum_{n\in\mathbb{Z}} |a_n|^2 \ .$$

It follows from (2.7) that

$$\|\tilde{u}\|^2_{h^2} = \sum_{n\neq 0} |a_n|^2 \leq \|u\|^2_{h^2} \ . \tag{5.1}$$

Then $\tilde{u}$ is also in $h^2(D)$. We have so proved the following statement.

Proposition 5.1. *The* conjugate function operator $T : u \mapsto \tilde{u}$ *maps* $h^2(D)$ *continuously into itself.*

Consider now $p = 1$, and take

$$P(re^{it}) = \frac{1 - r^2}{1 + r^2 - 2r\cos t} \in h^1(D) \ .$$

Its conjugate harmonic function is

$$\tilde{P}(re^{it}) = \frac{2r\sin t}{1 + r^2 - 2r\cos t} \ .$$

[13] It follows from Lemma 4.2 and the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 + \|x + iy\|^2 - \|x - y\|^2 - \|x - iy\|^2\right)$$

holding on any complex Hilbert space, that

$$\langle u, v\rangle_{h^2} = \lim_{r\to 1}\langle u_r, v_r\rangle_{L^2} \ .$$

In fact P and $\tilde{P}$ are real, $\tilde{P}(0) = 0$, and

$$F(z) = P(z) + i\tilde{P}(z) = \frac{2}{1-z} - 1$$

is holomorphic. If $\tilde{P}$ were in $h^1(D)$, it would follow that $F \in H^1(D)$, but this is in contrast with Proposition 1.3.

This example is interesting because it emphasizes one negative property of the conjugate Poisson kernel: the functions $\tilde{P}_r$ do not form an approximate identity. More precisely, the fact that $\tilde{P} \notin h^1(D)$ means that

$$\lim_{r \to 1} \|\tilde{P}_r\|_1 = +\infty \ . \tag{5.2}$$

Passing to the case $p = \infty$, the following example shows that the harmonic conjugate of a bounded function need not be bounded.

Consider an open vertical strip $S = \{z : a < \Re e z < b\}$ in the complex plane. By the Riemann mapping theorem, there is a conformal map φ from D onto S. Adding, if necessary, an imaginary constant to φ, we can assume that $\varphi(0) \in \mathbb{R}$. If $u = \Re e \varphi$, it follows that $a < u(z) < b$, so that $u \in h^\infty(D)$. But its harmonic conjugate $\tilde{u}$ is $\Im m \varphi$, which is not bounded[14].

One may ask if the conclusion $\tilde{u} \in h^\infty(D)$ holds if u is assumed to be continuous on $\bar{D}$. The answer is negative again, but instead of giving a counterexample, we use a functional analytic argument.

Lemma 5.2. *Let X and Y be two Banach spaces of harmonic functions in D. Assume that convergence in each of these spaces implies uniform convergence on compact subsets of D, and that the conjugate function operator T maps functions in X into functions in Y. Then $T : X \to Y$ is continuous.*

Proof. By the closed graph theorem, it is sufficient to prove that if a sequence $\{(u_n, Tu_n)\}$ converges to $(u, v) \in X \times Y$, then $v = Tu$. Since T maps real-valued functions into real-valued functions, we can assume that the u_n are all real-valued. The same is therefore true for u and v.

On every compact set, $u_n + iTu_n$ converges uniformly to $u + iv$. Hence $u + iv$ is holomorphic. Since $v_n(0) = 0$ for every n, also $v(0) = 0$. We conclude that $v = \tilde{u}$. $\square$

Proposition 5.3. *There exist functions u, harmonic in D and continuous on $\bar{D}$, whose harmonic conjugate $\tilde{u}$ is not bounded in D.*

Proof. Denote by $h^\infty_c(D)$ the closed subspace of $h^\infty(D)$ consisting of those harmonic functions in D which admit a continuous extension to $\bar{D}$. Assume that T maps $h^\infty_c(D)$ into $h^\infty(D)$. By Lemma 5.2, this map would be continuous, hence there would exists a constant $C > 0$ such that

$$\|\tilde{u}\|_{h^\infty} \leq C\|u\|_{h^\infty}$$

[14]Another example is $u(z) = \Im m \log(1-z)$, which is bounded, but whose harmonic conjugate, $-\Re e \log(1-z) = -\log|1-z|$, is not.

for every $u \in h^\infty_c(D)$. Given $r < 1$, consider the linear functional on $C(\mathbb{T})$

$$\psi_r(f) = f * \tilde{P}_r(1) = \int_{\mathbb{T}} f(t)\tilde{P}_r(-t)\,dt\ .$$

If u is the Poisson integral of f, we know from Theorem 4.3 that $\|u\|_{h^\infty} = \|f\|_\infty$. Moreover, $\psi_r(f) = \tilde{u}(r)$. We would then have

$$|\psi_r(f)| \le \|\tilde{u}\|_{h^\infty} \le C\|f\|_\infty\ .$$

This would imply that

$$\begin{aligned}\|\tilde{P}_r\|_1 &= \sup_{f\in C(\mathbb{T})\,,\,\|f\|_\infty\le 1}\left|\int_{\mathbb{T}} f(t)\tilde{P}_r(-t)\,dt\right| \\ &= \sup_{f\in C(\mathbb{T})\,,\,\|f\|_\infty\le 1}|\psi_r(f)| \\ &\le C\ ,\end{aligned}$$

in contrast with (5.2). □

On the other hand, if we impose a little more regularity on the behaviour of u on the boundary, we obtain a positive result.

Proposition 5.4. *Suppose that u is harmonic in D, continuous on $\bar{D}$, and α-Lipschitz*[15] *on $\mathbb{T}$ for some $\alpha > 0$. Then $\tilde{u}$ extends continuously to $\bar{D}$.*

Proof. We shall prove that $\tilde{u}_r$ has a uniform limit $\tilde{u}_1$ for $r \to 1$. It is a simple exercise to prove that this condition is in fact equivalent to the existence of a continuous extension of $\tilde{u}$ on $\bar{D}$.

Let f be the restriction of u to $\mathbb{T}$, so that $\tilde{u}_r = f * \tilde{P}_r$. Consider

$$f * \tilde{P}_r(t) = \int_{\mathbb{T}} f(t-t')\tilde{P}_r(t')\,dt' = \int_{\mathbb{T}} f(t-t')\frac{2r\sin t'}{1+r^2-2r\cos t'}\,dt'\ .$$

Since $\tilde{P}_r$ is odd in t', $\int_{\mathbb{T}}\tilde{P}_r(t')\,dt' = 0$. We then have

$$\lim_{r\to 1} f * \tilde{P}_r(t) = \lim_{r\to 1}\int_{\mathbb{T}}\big(f(t-t') - f(t)\big)\tilde{P}_r(t')\,dt'\ . \tag{5.3}$$

We prove that this limit equals the expression formally obtained by taking the pointwise limit of the integrand, i.e.

$$\begin{aligned}\tilde{f}(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(t-t') - f(t)\big)\frac{\sin t'}{1-\cos t'}\,dt' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(t-t') - f(t)\big)\cot\frac{t'}{2}\,dt'\ .\end{aligned} \tag{5.4}$$

[15] A function f is α-Lipschitz (or α-Hölder) on a metric space X if there is a constant C such that $|f(x) - f(y)| \le Cd(x,y)^\alpha$ for every $x, y \in X$.

Observe that this integral is absolutely convergent, due to the fact that f is α-Lipschitz. In fact,

$$\big|f(t-t')-f(t)\big|\Big|\cot\frac{t'}{2}\Big|\le\frac{C}{|t'|^{1-\alpha}}\ , \tag{5.5}$$

which is integrable on $[-\pi,\pi]$.

Using (1.5), we find that

$$|\tilde{P}_r(t')|=\frac{2r|\sin t'|}{|1-re^{it'}|^2}\le\pi^2\frac{2|t'|}{(|t'|+1-r)^2}\le\frac{2\pi^2}{|t'|}\ . \tag{5.6}$$

Hence

$$\big|f(t-t')-f(t)\big|\big|\tilde{P}_r(t')\big|\le\frac{C}{|t'|^{1-\alpha}}\ , \tag{5.7}$$

and dominated convergence can be applied to show that $\lim_{r\to1}f*\tilde{P}_r(t)=\tilde{f}(t)$ pointwise. Moreover,

$$\begin{aligned}\|\tilde{f}-f*\tilde{P}_r\|_\infty&\le\max_{t\in[-\pi,\pi]}\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(t-t')-f(t)\big|\Big|\cot\frac{t'}{2}-\tilde{P}_r(t')\Big|\,dt'\\&\le\frac{C}{2\pi}\int_{-\pi}^{\pi}|t'|^\alpha\Big|\cot\frac{t'}{2}-\tilde{P}_r(t')\Big|\,dt'\ .\end{aligned}$$

We can apply dominated convergence again, because the integrand tends to zero pointwise and

$$\Big|\cot\frac{t'}{2}-\tilde{P}_r(t')\Big|\le\Big|\cot\frac{t'}{2}\Big|+\big|\tilde{P}_r(t')\big|\le\frac{C}{|t'|}\ .$$

Therefore,

$$\lim_{r\to1}\|\tilde{f}-f*\tilde{P}_r\|_\infty=0\ . \qquad\square$$

This result admits the following local version.

Proposition 5.5. *Suppose that $f\in L^1(\mathbb{T})$ is α-Lipschitz on an open interval $I\subset\mathbb{T}$ for some $\alpha>0$. Then $\tilde{f}$ in (5.4) is well defined on I and $\lim_{r\to1}f*\tilde{P}_r=\tilde{f}$ uniformly on compact subsets of I. In particular, the harmonic conjugate $\tilde{u}$ of $u=\mathcal{P}f$ admits a continuous extension to $D\cup I$.*

Proof. For fixed $t\in I$, the integral in (5.4) is absolutely convergent, because the estimate (5.5) holds for $t-t'\in I$, and $\cot\frac{t'}{2}$ is bounded for $t-t'\notin I$.

Let J be a compact subinterval of I. Without loss of generality, we can assume that $J=[-\delta,\delta]$ with $\delta<\frac{\pi}{2}$ and that the double interval $J'=[-2\delta,2\delta]$ is also contained in I. For $t\in J$, we then have

$$\begin{aligned}\big|\tilde{f}(t)-\tilde{P}_r*f(t)\big|&=\frac{1}{2\pi}\Big|\int_{-\pi}^{\pi}\big(f(t-t')-f(t)\big)\Big(\cot\frac{t'}{2}-\tilde{P}_r(t')\Big)\,dt'\Big|\\&=\frac{1}{2\pi}\Big|\int_{-\pi}^{\pi}\big(f(s)-f(t)\big)\Big(\cot\frac{t-s}{2}-\tilde{P}_r(t-s)\Big)\,ds\Big|\\&\le\frac{1}{2\pi}\Big(\int_{J'}\big|f(s)-f(t)\big|\Big|\cot\frac{t-s}{2}-\tilde{P}_r(t-s)\Big|\,ds\\&\qquad+\int_{[-\pi,\pi]\setminus J'}\big|f(s)-f(t)\big|\Big|\cot\frac{t-s}{2}-\tilde{P}_r(t-s)\Big|\,ds\Big)\\&=\frac{1}{2\pi}\big(I_1(t,r)+I_2(t,r)\big)\ .\end{aligned}$$

We now want to prove that

$$\lim_{r\to 1}\sup_{t\in J}\left|\tilde f(t)-\tilde P_r * f(t)\right| = 0\ .$$

In the integral $I_1(t,r)$, both t and s are in I, so that

$$\begin{aligned}|I_1(t,r)| &\le \int_{J'} |t-s|^\alpha \Big|\cot\frac{t-s}{2}-\tilde P_r(t-s)\Big|\,ds\\ &= \int_{t-t'\in J'} |t'|^\alpha \Big|\cot\frac{t'}{2}-\tilde P_r(t')\Big|\,dt'\\ &\le \int_{-2\pi}^{2\pi} |t'|^\alpha \Big|\cot\frac{t'}{2}-\tilde P_r(t')\Big|\,dt'\ ,\end{aligned}$$

which is independent of t. By dominated convergence, as in the proof of Proposition 5.4, this last quantity tends to zero as $r\to 1$, so that

$$\lim_{r\to 1}\sup_{t\in I'} I_1(t,r) = 0\ .$$

Passing to $I_2(t,r)$, let $M=\max_{t\in J}|f(t)|$. Then

$$\begin{aligned}I_2(t,r) &= \int_{[-\pi,\pi]\setminus J'} \big|f(s)-f(t)\big|\,\Big|\cot\frac{t-s}{2}-\tilde P_r(t-s)\Big|\,ds\\ &\le \int_{[-\pi,\pi]\setminus J'} \big(|f(s)|+M\big)\,\Big|\cot\frac{t-s}{2}-\tilde P_r(t-s)\Big|\,ds\ .\end{aligned}$$

Observe that, if $t\in J$ and $s\notin J'$, $|t-s|>\delta$. Therefore

$$\begin{aligned}\Big|\cot\frac{t-s}{2}-\tilde P_r(t-s)\Big| &= \Big|\frac{\sin(t-s)}{1-\cos(t-s)}-\frac{2r\sin(t-s)}{1+r^2-2r\cos(t-s)}\Big|\\ &= \frac{|\sin(t-s)|}{1-\cos(t-s)}\Big|1-\frac{2r\big(1-\cos(t-s)\big)}{1+r^2-2r\cos(t-s)}\Big| \qquad (5.8)\\ &= \Big|\cot\frac{t-s}{2}\Big|\,\frac{(1-r)^2}{1+r^2-2r\cos(t-s)}\end{aligned}$$

The quantity $\big|\cot\frac{t-s}{2}\big|/\big(1+r^2-2r\cos(t-s)\big)$ remains bounded for $|t-s|>\delta$. Given $\varepsilon>0$, we can then find $r_0<1$ such that, for $r>r_0$,

$$\Big|\cot\frac{t-s}{2}-\tilde P_r(t-s)\Big| < \varepsilon$$

for every t,s with $|t-s|>\delta$.

Therefore, if $r>r_0$,

$$I_2(t,r)\le \varepsilon 2\pi\big(\|f\|_1+M\big)\ .$$

This shows that

$$\lim_{r\to 1}\sup_{t\in I'} I_2(t,r) = 0\ ,$$

concluding the proof. □

6. The Cauchy projection

The Hardy space $H^p(D)$ is a closed subspace of $h^p(D)$. It is clear from Section 3 that the Poisson integral of a function (or measure) f on $\mathbb{T}$ is holomorphic in D if and only if $\hat{f}(n) = 0$ for $n < 0$. If we define

$$L^p_+(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\} ,$$

and similarly

$$M_+(\mathbb{T}) = \{\mu \in M(\mathbb{T}) : \hat{\mu}(n) = 0 \text{ for } n < 0\} ,$$

the statement of Theorem 4.3 can be repeated word by word, replacing h^p with H^p, L^p with L^p_+, and M with M_+.

In the case $p = 2$ we are working with Hilbert spaces, and we want to describe the orthogonal projection from $h^2(D)$ to $H^2(D)$, that we shall denote by C (for Cauchy).

Lemma 6.1. *For $u \in h^2(D)$, let $\tilde{u}$ be its harmonic conjugate. Then*

$$Cu = \frac{1}{2}(u + i\tilde{u}) + \frac{1}{2}u(0) . \tag{6.1}$$

Denoting by $u^\sharp \in L^2(\mathbb{T})$ the boundary function of u, i.e. $u^\sharp = \lim_{r\to 1} u_r$, then

$$Cu(z) = \int_{\mathbb{T}} \frac{u^\sharp(e^{it})}{1 - ze^{-it}}\, dt . \tag{6.2}$$

Proof. Since $\{z^n\}_{n\geq 0} \cup \{\bar{z}^n\}_{n\geq 1}$ is an orthonormal basis of $h^2(D)$, and $\{z^n\}_{n\geq 0}$ spans $H^2(D)$, if we write

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n ,$$

it follows that

$$Cu(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Then (6.1) follows from (2.6). Consider now the Fourier series of $u^\sharp$,

$$u^\sharp(e^{it}) = \sum_{n\in\mathbb{Z}} a_n e^{int} ,$$

with convergence in the $L^2(\mathbb{T})$-norm. Since

$$(Cu)_r(e^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int} ,$$

one obtains $(Cu)_r$ from $u^\sharp$ by multiplying each Fourier coefficient $\widehat{u^\sharp}(n) = a_n$ by r^n if $n \geq 0$ and by 0 if $n < 0$. Consider therefore the function

$$C_r(e^{it}) = \sum_{n=0}^{\infty} r^n e^{int} = \frac{1}{1 - re^{it}} , \tag{6.3}$$

called the *Cauchy kernel*. Then $\widehat{(Cu)_r}(n) = \widehat{u^\sharp}(n)\widehat{C_r}(n)$ for every $n \in \mathbb{Z}$, so that

$$\begin{aligned} Cu(re^{i\theta}) &= (Cu)_r(e^{i\theta}) \\ &= u^\sharp * C_r(e^{i\theta}) \\ &= \int_{\mathbb{T}} \frac{u^\sharp(e^{it})}{1 - re^{i(\theta-t)}}\, dt \ , \end{aligned}$$

giving (6.2). □

Observe that (6.2) can be rewritten as a contour integral,

$$(Cu)(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{u^\sharp(w)}{w - z}\, dw \ ,$$

an expression resembling the ordinary Cauchy integral formula. This is compatible with the fact that if u is already in $H^2(D)$ (i.e. it is holomorphic), we then have the identity, for $|z| < r < 1$,

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \oint_{\gamma_r} \frac{u(w)}{w - z}\, dw \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u_r(e^{it})}{re^{it} - z} re^{it}\, dt \\ &= \int_{\mathbb{T}} \frac{u_r(e^{it})}{1 - \frac{z}{r}e^{-it}}\, dt \end{aligned}$$

where γ_r is the circle of radius r oriented counterclockwise. Letting $r \to 1$, one obtains (6.2), since we are assuming that $Cu = u$.

The orthogonal projection of $h^2(D)$ onto $H^2(D)$ corresponds, passing to boundary values, to the orthogonal projection $C^\sharp$ of $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T})$, via the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{T}) & \xrightarrow{C^\sharp} & L^2_+(\mathbb{T}) \\ \downarrow \mathcal{P} & & \downarrow \mathcal{P} \\ h^2(D) & \xrightarrow{C} & H^2(D) \end{array}$$

So

$$C^\sharp f = \lim_{r \to 1} (C\mathcal{P}f)_r \ ,$$

where the limit is meant in the L^2-norm.

We want to give an expression of $C^\sharp$ that does not involve the harmonic extension to the interior. Because of Lemma 6.1, this problem can be reduced to that of finding a more direct formula for the operator

$$H : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T}) \ ,$$

mapping f into

$$Hf = \lim_{r \to 1} \tilde{P}_r * f \ ,$$

again in the L^2-norm. This operator is bounded, by Proposition 5.1, and

$$\begin{aligned} Hf(e^{it}) &= \lim_{r \to 1} f * \tilde{P}_r(e^{it}) \\ &= \lim_{r \to 1} \int_{\mathbb{T}} f(e^{i(t-t')}) \frac{2r \sin t'}{1 + r^2 - 2r \cos t'}\, dt' \ . \end{aligned} \tag{6.4}$$

Proposition 6.2. *If* $f \in L^2(\mathbb{T})$,

$$Hf(e^{it}) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon<|t'|<\pi} f(e^{i(t-t')})\cot\frac{t'}{2}\,dt' \ , \tag{6.5}$$

where the limit is in the L^2*-norm.*

Proof. For $0<\varepsilon<1$, define

$$H_\varepsilon f(e^{it}) = \frac{1}{2\pi}\int_{\varepsilon<|t'|<\pi} f(e^{i(t-t')})\cot\frac{t'}{2}\,dt' \ .$$

If we prove that

$$\lim_{\varepsilon\to 0}\|f*\tilde{P}_{1-\varepsilon} - H_\varepsilon f\|_2 = 0 \ , \tag{6.6}$$

this will imply the assertion, by (6.4).

We can write

$$\begin{aligned} f*\tilde{P}_{1-\varepsilon}(t) - H_\varepsilon f(t) &= \frac{1}{2\pi}\bigg(\int_{|t'|<\varepsilon} f(e^{i(t-t')})\tilde{P}_{1-\varepsilon}(t')\,dt' \\ &\quad + \int_{\varepsilon<|t'|<\pi} f(e^{i(t-t')})\Big(\tilde{P}_{1-\varepsilon}(t') - \cot\frac{t'}{2}\Big)\,dt'\bigg) \\ &= f*\varphi_\varepsilon(t) \ , \end{aligned}$$

where

$$\varphi_\varepsilon(t') = \begin{cases} \tilde{P}_{1-\varepsilon}(t') & \text{if } |t'|<\varepsilon \\ \tilde{P}_{1-\varepsilon}(t') - \cot\frac{t'}{2} & \text{if } \varepsilon\le|t'|\le\pi \ . \end{cases}$$

By (5.8),

$$\begin{aligned} \Big|\tilde{P}_{1-\varepsilon}(t') - \cot\frac{t'}{2}\Big| &= \Big|\cot\frac{t'}{2}\Big|\frac{\varepsilon^2}{1+(1-\varepsilon)^2-2(1-\varepsilon)\cos t'} \\ &= \Big|\cot\frac{t'}{2}\Big|\frac{\varepsilon^2}{\varepsilon^2+4(1-\varepsilon)\sin^2\frac{t'}{2}} \\ &\le \Big|\cot\frac{t'}{2}\Big|\frac{\varepsilon^2}{\varepsilon^2+\frac{4}{\pi^2}(1-\varepsilon)t'^2} \ . \end{aligned}$$

We are interested in this quantity for $\varepsilon\le|t'|\le\pi$, where we can use the inequality

$$\Big|\cot\frac{t'}{2}\Big| \le \frac{1}{|\sin\frac{t'}{2}|} \le \frac{\pi}{|t'|} \le \frac{\pi}{\varepsilon} \ .$$

We can also assume that ε is small enough so that $\frac{4}{\pi^2}(1-\varepsilon)>1$. For ε and t' subject to these restrictions,

$$\Big|\cot\frac{t'}{2} - \tilde{P}_{1-\varepsilon}(t')\Big| \le \frac{\pi\varepsilon}{\varepsilon^2+t'^2} \ .$$

On the other hand, if $|t'| < \varepsilon$,

$$\big|\tilde{P}_{1-\varepsilon}(t')\big| = \frac{2(1-\varepsilon)|\sin t'|}{1+(1-\varepsilon)^2 - 2(1-\varepsilon)\cos t'} \leq \frac{2\varepsilon}{\varepsilon^2 + t'^2} \ .$$

Putting these two inequalities together, we have

$$|\varphi_\varepsilon(t')| \leq \frac{\pi\varepsilon}{\varepsilon^2 + t'^2} \ ,$$

for $t' \in [-\pi, \pi]$. We can then apply Corollary 3.3 with $c = 0$. We leave the verification of conditions (1) and (3) to the reader, and observe that φ_ε has integral 0 because it is odd on $[-\pi, \pi]$. This gives (6.6). □

We cannot eliminate the limit in (6.5) and write the integral over all of $\mathbb{T}$, because the integrand would not be absolutely convergent in general. For the same reason, we cannot replace the integrals outside of the simmetric intervals $[-\varepsilon, \varepsilon]$ with integrals outside of arbitrary intervals containing 0, such as, for instance, $[-\varepsilon, 2\varepsilon]$, because the result would not be the same.

This matter can be nicely formalized in the language of distribution theory. Observe that

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<|t|<\pi} f(t)\cot\frac{t}{2}\,dt \tag{6.7}$$

does not exist, or is infinite, for a generic continuous function f on $\mathbb{T}$ (take for instance $f(t) = \frac{\operatorname{sgn} t}{\log|t|}$ near $t = 0$).

If, however, f is a C^1-function[16], we have, using the oddness of $\cot\frac{t}{2}$,

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<|t|<\pi} f(t)\cot\frac{t}{2}\,dt = \lim_{\varepsilon\to 0} \int_{\varepsilon<|t|<\pi} \big(f(t) - f(0)\big)\cot\frac{t}{2}\,dt = \int_{-\pi}^{\pi} \big(f(t) - f(0)\big)\cot\frac{t}{2}\,dt \ ,$$

where this last integral is absolutely convergent. Moreover the linear functional

$$f \longmapsto \Phi(f) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|t|<\pi} f(t)\cot\frac{t}{2}\,dt \tag{6.8}$$

is continuous w.r. to the norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$$

[16] f α-Lipschitz would be enough, but distribution theory is focused on spaces of C^k- or C^∞-functions.

on $C^1(\mathbb{T})$. In fact

$$\begin{aligned}\big|\Phi(f)\big| &= \left|\int_{-\pi}^{\pi} \big(f(t)-f(0)\big)\cot\frac{t}{2}\,dt\right| \\ &\le \int_{-\pi}^{\pi} \big|f(t)-f(0)\big|\left|\cot\frac{t}{2}\right| dt \\ &\le \|f'\|_\infty \int_{-\pi}^{\pi} |t|\left|\cot\frac{t}{2}\right| dt \\ &\le C\|f\|_{C^1}\ .\end{aligned}$$

This property is expressed by saying that Φ is a *distribution of order 1 on* $\mathbb{T}$. According to a standard terminology, a distribution such as Φ, involving limits of integrals that would not be abolutely convergent otherwise, is called a *principal value distribution.* One writes

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon<|t|<\pi} f(t)\cot\frac{t}{2}\,dt = \text{p.v.}\int_{\mathbb{T}} f(t)\cot\frac{t}{2}\,dt\ .$$

The convolution $\Phi * f$ is defined for $f\in C^1(\mathbb{T})$ as

$$\Phi * f(t) = \Phi\big(f(t-\cdot)\big) = \text{p.v.}\int_{\mathbb{T}} f(t-t')\cot\frac{t'}{2}\,dt'\ ,$$

and one can easily prove that $\Phi * f\in C(\mathbb{T})$.

One also writes

$$\text{p.v.}\cot\frac{t}{2} * f\ , \qquad \text{or} \qquad \Big(\text{p.v.}\cot\frac{t}{2}\Big) * f$$

for $\Phi * f$.

As a consequence of Proposition 6.2, we have the following extension result.

Corollary 6.3. *For $f\in C^1(\mathbb{T})$, $Hf = \Phi * f$. Therefore the convolution operator $f\mapsto \Phi * f$ extends in a unique way to a continuous operator from $L^2(\mathbb{T})$ to itself.*

Moreover

$$C^\sharp f = \frac{1}{2}\big(f + iHf + \hat{f}(0)\big)\ .$$

7. Blaschke products and the F. and M. Riesz theorem

Working with holomorphic functions, one comment must be made on products of H^p-functions[17].

[17] We do not discuss this issue for h^p-functions because, in general, the product of two harmonic functions is not harmonic.

Lemma 7.1. *Suppose $f \in H^p(D)$ and $g \in H^q(D)$, with $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} \le 1$. Then $fg \in H^s(D)$, $\|fg\|_{H^s} \le \|f\|_{H^p}\|g\|_{H^q}$. If $p, q > 1$, denoting by $f^\sharp$ and $g^\sharp$ the boundary functions of f, g respectively, then $(fg)^\sharp$ is also a function and $(fg)^\sharp = f^\sharp g^\sharp$.*

Proof. Suppose first that p and q are both finite. By Hölder's inequality,

$$M_s(fg, r) \le M_p(f, r) M_q(g, r)$$

for every $r < 1$. Therefore $fg \in H^s(D)$ and $\|fg\|_{H^s} \le \|f\|_{H^p}\|g\|_{H^q}$. Moreover,

$$\begin{aligned} \|(fg)_r - f^\sharp g^\sharp\|_s &= \|f_r g_r - f^\sharp g^\sharp\|_s \\ &\le \|(f_r - f^\sharp)g_r + (g_r - g^\sharp)f^\sharp\|_s \\ &\le \|(f_r - f^\sharp)g_r\|_s + \|(g_r - g^\sharp)f^\sharp\|_s \\ &\le \|f_r - f^\sharp\|_p \|g_r\|_q + \|g_r - g^\sharp\|_q \|f^\sharp\|_p \\ &\le \|f_r - f^\sharp\|_p \|g\|_{H^q} + \|g_r - g^\sharp\|_q \|f\|_{H^q} \ . \end{aligned}$$

This shows that $(fg)_r$ tends to $f^\sharp g^\sharp$ in $L^s(D)$, so that $(fg)^\sharp = f^\sharp g^\sharp$.

If one or both exponents are ∞, the identity $(fg)^\sharp = f^\sharp g^\sharp$ follows from the above and the inclusion $H^\infty(D) \subset H^t(D)$ for $t < \infty$. Once this is established, the inequality of norms is obvious. □

As we see, this result implies (when $q = p'$) that an H^1-function which is the product of an H^p- and an $H^{p'}$-function is the Poisson integral of an L^1-funtion on the torus. On the other hand, the picture given by Lemma 7.1 is not complete, because it does not say what happens on the boundary when one of the two exponents, say p, is equal to 1. In that case we are forced to assume that $q = \infty$, but stil we do not know if the product $f^\sharp g^\sharp$ of a Borel measure times a bounded measurable function makes sense.

All these problems are washed out by the following theorem.

Theorem 7.2 (F. and M. Riesz). *Let μ be a Borel measure in $M_+(\mathbb{T})$. Then μ is absolutely continuous with respect to Lebesgue measure.*

In other words, $M_+(\mathbb{T}) = L^1_+(\mathbb{T})$.

Corollary 7.3. *If $f \in H^1(D)$, the limit $\lim_{r\to 1} f_r$ exists in the L^1-norm. Holomorphic polynomials are dense in $H^1(D)$.*

Proof. The first part is an immediate consequence of Theorems 7.2 and 4.3. For the second part, proceed as in the proof of Corollary 4.4. □

There are several proofs of the brothers Riesz theorem. The one we give involves an important notion in H^p-theory, the *Blaschke products.*

For $\alpha \in D$, consider the function

$$\psi_\alpha = \frac{z - \alpha}{\bar\alpha z - 1} \ . \tag{7.1}$$

One easily verifies that ψ_α is holomorphic on $\mathbb{C}\setminus\{1/\bar\alpha\}$ (hence on a neighborhood of D), injective, and that on $\mathbb{T}$

$$|\psi_\alpha(e^{it})| = \frac{|e^{it} - \alpha|}{|\bar\alpha - e^{-it}|} = 1 \ ,$$

hence[18] $|\psi_\alpha(z)| < 1$ for $z \in D$.

Suppose now that $f \in H^p(D)$ is not identically zero, and that $f(\alpha) = 0$ for some $\alpha \in D$. Then the function

$$g(z) = \frac{f(z)}{\psi_\alpha(z)}$$

has a removable singularity at $z = \alpha$. Hence g is holomorphic in D.

Lemma 7.4. *$g \in H^p(D)$ and $\|g\|_{H^p} = \|f\|_{H^p}$.*

Proof. We clearly have

$$M_p(f, r) = M_p(g\psi_\alpha, r) \leq M_p(g, r) \ .$$

Given $\varepsilon > 0$, it is possible to find $r_0 > |\alpha|$ such that if $|z| > r_0$, then $|\psi_\alpha(z)| > 1 - \varepsilon$. Then, if $r > r_0$,

$$M_p(f, r) = M_p(g\psi_\alpha, r) \geq (1 - \varepsilon)M_p(g, r) \ .$$

Passing to the limit as $r \to 1$ and using the arbitrarity of ε, we have the conclusion. □

This suggests a way to "remove zeroes" from a function $f \in H^p(D)$ which is not identically 0, without modifying its norm. If the zeroes of f in D are a finite number, $\alpha_0, \alpha_1, \ldots, \alpha_n$ (counted with their multiplicities), it is sufficient to repeat the above construction $n + 1$ times to obtain a factorization of $f(z)$ as $g(z)B(z)$, with

$$B(z) = \prod_{j=0}^{n} \psi_{\alpha_j}(z) = \prod_{j=0}^{n} \frac{z - \alpha_j}{\bar{\alpha}_j z - 1} \ . \tag{7.2}$$

Then g has no zeroes in D and $\|g\|_{H^p} = \|f\|_{H^p}$.

If the zeroes of f in D are infinitely many, they form a countable set without accumulation points in D itself. Ordering these zeroes by increasing modulus, and counting them with their multiplicities, we obtain a sequence $\{\alpha_j\}$, with $|\alpha_j| \leq |\alpha_{j+1}|$, and $\lim_{j\to\infty} |\alpha_j| = 1$.

This is true for *any* non-identically zero holomorphic function in D. Something more can be said about distribution of zeroes of H^p-functions.

Lemma 7.5. *Let $f \in H^p(D)$ be not identically zero, and let $f(z) = az^k h(z)$, with $k \geq 0$ and $h(0) = 1$. Then*

$$\sum_{j\,:\,\alpha_j \neq 0} \big(1 - |\alpha_j|\big) \leq \log \|h\|_{H^p} = \log \|f\|_{H^p} - \log |a| \ .$$

Proof. The last equality being obvious, we can assume that $k = 0$ and $a = 1$, i.e. that $f = h$.

We initially suppose that f is holomorphic on a larger disc $D_R = \{z : |z| < R\}$ with $R > 1$, and also that f does not vanish for $|z| = 1$. We then have only finitely

[18] More precisely, ψ_α is a conformal map of D onto itself (i.e. a *Möbius transformation*). It can be shown that any Möbius transformation has the form $e^{i\theta}\psi_\alpha$ for some $\alpha \in D$ and some $e^{i\theta} \in \mathbb{T}$.

many zeroes $\alpha_0, \dots, \alpha_n$ in D, and we can decompose f as $f = gB$, with B given by (7.2). Observe that B is defined in a larger disc $D' = D_{1+\delta}$ and non-zero in $D' \setminus D$, so that g is holomorphic in D'.

Since g has no zero in D', $\log g$ admits a holomorphic determination[19] on D'. Then $\log|g| = \Re\mathrm{e} \log g$ is harmonic in D'. By the mean value theorem,

$$\begin{aligned}\log|g(0)| &= \int_{\mathbb{T}} \log|g(e^{it})|\,dt \\ &= \int_{\mathbb{T}} \log|f(e^{it})|\,dt - \int_{\mathbb{T}} \log|B(e^{it})|\,dt \\ &= \int_{\mathbb{T}} \log|f(e^{it})|\,dt\ .\end{aligned}$$

But

$$\begin{aligned}\log|g(0)| &= \log|f(0)| - \sum_{j=0}^{n} \log\left|\psi_{\alpha_j}(0)\right| \\ &= -\sum_{j=0}^{n} \log|\alpha_j|\ ,\end{aligned}$$

so that

$$-\sum_{j=0}^{n} \log|\alpha_j| = \int_{\mathbb{T}} \log|f(e^{it})|\,dt\ . \tag{7.3}$$

Passing to a general $f \in H^p(D)$, with $f(0) = 1$, we apply (7.3) to $f_r(z) = f(rz)$, with $r < 1$ and $r \neq |\alpha_j|$ for every j. The zeroes of f_r in D are the $\alpha'_j = \alpha_j/r$, with $|\alpha_j| < r$. Therefore,

$$-\sum_{j:|\alpha_j|<r} \log\frac{|\alpha_j|}{r} = \int_{\mathbb{T}} \log|f(re^{it})|\,dt\ .$$

Observe now that, for $0 < x < 1$, $\log x < x - 1$. Therefore

$$1 - \frac{|\alpha_j|}{r} < -\log\frac{|\alpha_j|}{r}\ ,$$

and

$$\sum_{j:|\alpha_j|<r} \left(1 - \frac{|\alpha_j|}{r}\right) < \int_{\mathbb{T}} \log|f(re^{it})|\,dt\ .$$

The left-hand side is an increasing function of r, so that, by the Beppo-Levi theorem,

$$\lim_{r\to 1} \sum_{j:|\alpha_j|<r} \left(1 - \frac{|\alpha_j|}{r}\right) = \sum_{j=0}^{\infty} \left(1 - |\alpha_j|\right)\ .$$

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[19] This is true since D' is simply connected. In fact $\frac{g'}{g}$ is holomorphic in D, and it admits a primitive h. Then $(e^h/g)' = 0$, so that $e^h = cg$. Adding an appropriate constant to h, we obtain $c = 1$.

This limit is then controlled by the supremum over r on the right-hand side, i.e.

$$\sum_{j=0}^{\infty}\big(1-|\alpha_j|\big) \le \sup_{r<1}\int_{\mathbb{T}} \log|f(re^{it})|\,dt\ .$$

It remains to prove that this supremum is finite if $f \in H^p(D)$. It is sufficient to take $p = 1$, by the inclusion properties of Hardy spaces. By Jensen's inequality[20], if $r \neq r_j$ for every j,

$$\begin{aligned}\exp\Big(\int_{\mathbb{T}} \log|f(re^{it})|\,dt\Big) &= \Big(\exp\Big(\int_{\mathbb{T}} \log|f(re^{it})|^p\,dt\Big)\Big)^{\frac{1}{p}}\\ &\le \Big(\int_{\mathbb{T}} |f(re^{it})|^p\,dt\Big)^{\frac{1}{p}}\\ &\le \|f\|_{H^p}\ ,\end{aligned}$$

so that

$$\sum_{j=0}^{\infty}\big(1-|\alpha_j|\big) \le \log\|f\|_{H^p}\ .$$

This concludes the proof. □

In principle, we would like to decompose an H^p-function $f(z)$ with infinitely many zeroes $\{\alpha_j\}$ as the product $g(z)B(z)$, with the finite product (7.2) replaced by the analogous infinite product. The requirements would be

(1) that the partial products converge uniformly on compact subsets of D,
(2) that the limit function vanish only at the zeroes of f.

These requirements already show that some modification to (7.2) must be made. Suppose that an infinite product of complex numbers a_j converges to a non-zero value, i.e.

$$\prod_{j=0}^{\infty} a_j = \lim_{n\to\infty}\prod_{j=0}^{n} a_j = \lim_{n\to\infty} A_n = A \neq 0\ .$$

Then necessarily $a_j \neq 0$ for every j and

$$\lim_{j\to\infty} a_j = \lim_{j\to\infty}\frac{A_j}{A_{j-1}} = 1\ .$$

Consider therefore a point $z \in D$ different from the α_j, i.e. such that $f(z) \neq 0$. In general, we cannot expect that $\lim_{j\to\infty}\psi_{\alpha_j}(z) = 1$. However, this difficulty is bypassed if we by multiply each ψ_{α_j} by a contant factor of modulus 1.

So, for $\alpha \in D$, define

$$\tilde{\psi}_\alpha(z) = \frac{\bar\alpha}{|\alpha|}\psi_\alpha(z) = \frac{\bar\alpha}{|\alpha|}\frac{z-\alpha}{\bar\alpha z-1} \quad \text{if } \alpha \neq 0, \text{ and } \quad \tilde{\psi}_0(z) = z\ . \tag{7.4}$$

[20] Let m be a probability measure on a set X, and let $f: X \to \mathbb{R}$ be measurable. If $\varphi: \mathbb{R} \to \mathbb{R}$ is convex, then

$$\varphi\Big(\int_X f(x)\,dm(x)\Big) \le \int_X \varphi\big(f(x)\big)\,dm(x)\ .$$

Here it is applied with $\varphi(t) = e^t$.

For $\alpha_j \neq 0$,

$$\tilde{\psi}_{\alpha_j}(z) - 1 = \frac{\bar{\alpha}_j z - |\alpha_j|^2 - |\alpha_j|\bar{\alpha}_j z + |\alpha_j|}{|\alpha_j|(\bar{\alpha}_j z - 1)} = \frac{1 - |\alpha_j|}{|\alpha_j|} \, \frac{\bar{\alpha}_j z + |\alpha_j|}{\bar{\alpha}_j z - 1} \ .$$

Since

$$|\bar{\alpha}_j z - 1| \geq 1 - |\alpha_j||z| > 1 - |z| \ ,$$

we have

$$\left|\tilde{\psi}_{\alpha_j}(z) - 1\right| < 2\frac{1 - |\alpha_j|}{1 - |z|} \ . \tag{7.5}$$

It follows that

$$\lim_{j \to \infty} \tilde{\psi}_{\alpha_j}(z) = 1 \ . \tag{7.6}$$

We then construct the *Blaschke product*

$$B(z) = \prod_{j=0}^{\infty} \tilde{\psi}_{\alpha_j}(z) \ . \tag{7.7}$$

Proposition 7.6. *Let $\{\alpha_j\}$ be a sequence of complex numbers $\alpha_j \in D$, such that $\sum_{j=0}^{\infty} \big(1 - |\alpha_j|\big) < \infty$. Then the Blaschke product* (7.7) *converges unconditionally*[21] *and uniformly on compact sets to a function $B(z) \in H^\infty(D)$ vanishing only at the points α_j. Moreover, $|B(e^{it})| = 1$ almost everywhere on $\mathbb{T}$.*

Proof. Let $B_n(z) = \prod_{j=0}^{n} \tilde{\psi}_{\alpha_j}(z)$ be the partial products of (7.4). We discuss unconditional convergence first.

If z is one of the α_j, then $B_n(z) = 0$ for n large, and this holds independently of ordering.

If z is not one of the α_j, it follows from (7.6) that for $j \geq j_0 = j_0(z)$, the points $\tilde{\psi}_{\alpha_j}(z)$ are contained in the disc $\tilde{D}$ centered at 1 and radius 1/2. Let $\log z$ be the principal determination of the logarithm in $\tilde{D}$, i.e. such that $\log 1 = 0$. Then, if $n > j_0$,

$$B_n(z) = B_{j_0-1}(z) \prod_{j=j_0}^{n} e^{\log \tilde{\psi}_{\alpha_j}(z)} = B_{j_0-1}(z) \exp\left(\sum_{j=j_0}^{n} \log \tilde{\psi}_{\alpha_j}(z) \right) \ .$$

We can then say that the Blaschke product is unconditionally convergent at z to a non-zero limit if and only if the series

$$\sum_{j=j_0}^{\infty} \log \tilde{\psi}_{\alpha_j}(z)$$

[21] This means independently of reorderings of the terms.

is unconditionally (i.e. absolutely) convergent. Since $\frac{\log w}{w-1}$ is holomorphic in $\tilde{D}$ and continuous on $\overline{\tilde{D}}$, $|\log w| \leq C|w-1|$ for $w \in \tilde{D}$. By (7.5),

$$\begin{aligned} \sum_{j=j_0}^{\infty} \left|\log \tilde{\psi}_{\alpha_j}(z)\right| &\leq C \sum_{j=j_0}^{\infty} \left|1-\tilde{\psi}_{\alpha_j}(z)\right| \\ &\leq \frac{C}{1-|z|} \sum_{j=j_0}^{\infty} \left(1-|\alpha_j|\right) . \end{aligned} \tag{7.8}$$

We prove next uniform convergence on compact sets. Given a compact subset K of D, there is $r<1$ such that $|z| \leq r$ for $z \in K$. We take now $j_0 = j_0(K)$ large enough so that $|\alpha_j| > r$ for $j \geq j_0$, and $\tilde{\psi}_{\alpha_j}(z) \in \tilde{D}$ for $j \geq j_0$ and $z \in K$. This can be done by (7.5).

We verify the Cauchy condition for the B_n in the uniform norm on K. If $z \in K$ and $j_0 < n < m$,

$$\begin{aligned} \left|B_n(z) - B_m(z)\right| &= \left|B_n(z)\right| \left|1 - \prod_{j=n+1}^{m} \tilde{\psi}_{\alpha_j}(z)\right| \\ &\leq \left|1 - \exp\Big(\sum_{j=n+1}^{m} \log \tilde{\psi}_{\alpha_j}(z)\Big)\right| . \end{aligned}$$

By (7.8),

$$\begin{aligned} \left|\sum_{j=n+1}^{m} \log \tilde{\psi}_{\alpha_j}(z)\right| &\leq \sum_{j=n+1}^{m} \left|\log \tilde{\psi}_{\alpha_j}(z)\right| \\ &\leq \frac{C}{r}\,\frac{1+r}{1-r} \sum_{j=n+1}^{m} \left(1-|\alpha_j|\right) . \end{aligned}$$

Given ε, $0<\varepsilon<1$, we can then take n_0 large enough so that, if $n_0 \leq n < m$,

$$\left|\sum_{j=n+1}^{m} \log \tilde{\psi}_{\alpha_j}(z)\right| < \varepsilon ,$$

for all $z \in K$. Let $a > 0$ be such that $|1-e^w| < a|w|$ for $|w|<1$. Then

$$\begin{aligned} \left|B_n(z) - B_m(z)\right| &\leq \left|1 - \exp\Big(\sum_{j=n+1}^{m} \log \tilde{\psi}_{\alpha_j}(z)\Big)\right| \\ &< a\left|\sum_{j=n+1}^{m} \log \tilde{\psi}_{\alpha_j}(z)\right| \\ &< a\varepsilon , \end{aligned}$$

for all $z \in K$.

Finally, we prove that $|B(e^{it})| = 1$ a.e. on $\mathbb{T}$. Since $|B(z)| < 1$ for $z \in D$, $\|B\|_{H^\infty} \leq 1$, so that $|B(e^{it})| \leq 1$ a.e. on $\mathbb{T}$.

Let

$$R_n(z) = \frac{B(z)}{B_n(z)} = \prod_{j=n+1}^{\infty} \tilde{\psi}_{\alpha_j}(z) \ .$$

Then R_n is also a Blaschke product holomorphic in D, and

$$|B(z)| \le |R_n(z)| \le 1 \ . \tag{7.9}$$

On the boundary we have the identity

$$B(e^{it}) = B_n(e^{it}) R_n(e^{it}) \ ,$$

by Lemma 7.1. Moreover, $|B_n(e^{it})| = 1$ for every t, so that $|B(e^{it})| = |R_n(e^{it})|$. Observe that

$$|R_n(0)| = M_1(R_n, 0) \le \int_{\mathbb{T}} |R_n(e^{it})| \, dt = \int_{\mathbb{T}} |B(e^{it})| \, dt \ .$$

But

$$\lim_{n\to\infty} R_n(0) = \lim_{n\to\infty} \prod_{j=n+1}^{\infty} |\alpha_j| = \lim_{n\to\infty} e^{\sum_{j=n+1}^{\infty} \log|\alpha_j|} = 1 \ .$$

Therefore,

$$\int_{\mathbb{T}} |B(e^{it})| \, dt \ge 1 \ ,$$

and, combining this with (7.9), we conclude that $|B(e^{it})| = 1$ a.e. $\square$

Theorem 7.7. *Let $f \in H^p(D)$, not identically zero, and let B be its Blaschke product. Then $g = \frac{f}{B} \in H^p(D)$ and $\|g\|_{H^p} = \|f\|_{H^p}$.*

Proof. With B_n denoting the partial product of B. By Lemma 7.4, $\|f/B_n\|_{H^p} = \|f\|_{H^p}$. Fix $r < 1$. Since the functions $|f/B_n|$ converge monotonically to $|f/B|$, it follows that, if $1 > r \ne |\alpha_j|$ for every j, then

$$M_p(f/B, r) = \lim_{n\to\infty} M_p(f/B_n, r) \le \|f\|_{H^p} \ .$$

Therefore, $g \in H^p(D)$ and $\|g\|_{H^p} \le \|f\|_{H^p}$. On the other hand, since $|f(z)| \le |g(z)|$ for every $z \in D$, we trivially have $\|f\|_{H^p} \le \|g\|_{H^p}$. $\square$

An immediate consequence is the following factorization theorem, a kind of inverse of Lemma 7.1.

Corollary 7.8. *Let $f \in H^s(D)$, not identically zero, with $1 \le s \le \infty$, and let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. Then there exist $g \in H^p(D)$ and $h \in H^q(D)$ such that $f = gh$, and $\|g\|_{H^p}^p = \|h\|_{H^q}^q = \|f\|_{H^s}^s$.*

Proof. Let B be the Blaschke product of f and $\varphi = \frac{f}{B}$. Then $\varphi \in H^s(D)$ and has no zeroes in D. Let

$$g(z) = \varphi(z)^{\frac{s}{p}} = e^{\frac{s}{p} \log \varphi(z)} \ .$$

Then $|g(z)| = |\varphi(z)|^{\frac{s}{p}}$, so that

$$M_p(g, r)^p = \int_{\mathbb{T}} |g(re^{it})|^p \, dt = \int_{\mathbb{T}} |\varphi(re^{it})|^s \, dt = M_s(\varphi, r)^s \ .$$

Therefore $g \in H^p(D)$ and $\|g\|_{H^p}^p = \|\varphi\|_{H^s}^s = \|f\|_{H^s}^s$.

If we set $h(z) = \varphi(z)^{\frac{s}{q}} B(z)$, then $g(z)h(z) = \varphi(z)^{\frac{s}{p}+\frac{s}{q}} B(z) = f(z)$, and moreover

$$M_q(h/B, r)^q = M_s(\varphi, r)^s \ ,$$

as before. Therefore,

$$\|h\|_{H^q}^q = \|h/B\|_{H^q}^q = \|\varphi\|_{H^s}^s = \|f\|_{H^s}^s \ . \qquad \square$$

$\square$

We can prove now the F. and M. Riesz theorem.

Proof of Theorem 7.2. Let $f(z)$ be the Poisson integral of μ. Since $\hat{\mu}(n) = 0$ for $n < 0$, then

$$f(z) = \sum_{n=0}^{\infty} \hat{\mu}(n) z^n$$

is holomorphic. Therefore $f \in H^1(D)$. We can assume that $\mu \neq 0$, so that f is not identically zero. We write $f = gh$, with $g, h \in H^2(D)$, according to Corollary 7.8. By Lemma 7.1, f has a boundary function

$$f(e^{it}) = g(e^{it})h(e^{it}) \in L^1(\mathbb{T}) \ .$$

It follows that $d\mu(t) = f(e^{it})\, dt$, i.e. μ is absolutely continuous w.r. to the Lebesgue measure. $\square$

8. Dual spaces

The Cauchy projection

$$C : u(z) = \sum_{n=0}^{\infty} b_n z^n + \sum_{n=1}^{\infty} b_{-n} \bar{z}^n \longmapsto Cu(z) = \sum_{n=0}^{\infty} b_n z^n$$

is well defined as a linear map from the space of all harmonic functions on D, with values in the space of homorphic functions on D. It follows from (6.1) that C maps $h^p(D)$ into $H^p(D)$ if and only if the conjugate function operator maps $h^p(D)$ into itself.

Therefore we know from Section 5 that this does not happen if $p = 1$ or ∞, and we still have to see that it happens instead if $1 < p < \infty$. The scope of this section is to remark that this issue is relevant for determining the dual space of $H^p(D)$.

Consider the inner product in $h^2(D)$:

$$(8.1) \qquad \langle u, v \rangle_{h^2} = \int_{\mathbb{T}} u^\sharp(e^{it}) \overline{v^\sharp(e^{it})}\, dt = \lim_{r \to 1} \int_{\mathbb{T}} u(re^{it}) \overline{v(re^{it})}\, dt = \sum_{n \in \mathbb{Z}} a_n \overline{b_n} \ ,$$

if $u^\sharp, v^\sharp$ are the boundary functions and a_n, b_n their Fourier coefficients (i.e. the Taylor coefficients of u and v respectively). By the general theory of Hilbert spaces, the map

$$v \longmapsto \varphi_v(u) = \langle u, v \rangle_{h^2}$$

establishes a conjugate-linear isometric isomorphism between $h^2(D)$ and its dual space $h^2(D)^*$.

In the same way, restricting to $H^2(D)$,

$$g \longmapsto \varphi_g(f) = \langle f, g\rangle_{H^2}$$

establishes a conjugate-linear isometric isomorphism between $H^2(D)$ and its dual space $H^2(D)^*$.

Let us see first how (8.1) can be extended to $u \in h^p(D)$ and $v \in h^{p'}(D)$, where p' is the conjugate exponent of p.

Lemma 8.1. *Let $u \in h^p(D)$ and $v \in h^{p'}(D)$. If $p = 1$, assume that either $u^\sharp \in L^1(\mathbb{T})$ or $v^\sharp \in C(\mathbb{T})$ (and symmetrically if $p = \infty$). Then the limit*

$$B(u, v) = \lim_{r\to 1} \int_{\mathbb{T}} u(re^{it})\overline{v(re^{it})}\, dt$$

exists and it equals $\int_{\mathbb{T}} u^\sharp(e^{it})\overline{v^\sharp(e^{it})}\, dt$.

Proof. Suppose first that $1 < p < \infty$. Then, denoting by $\| \, \|_p$ the norm in $L^p(\mathbb{T})$,

$$\begin{aligned}
&\Big| \int_{\mathbb{T}} u(re^{it})\overline{v(re^{it})}\, dt - \int_{\mathbb{T}} u^\sharp(e^{it})\overline{v^\sharp(e^{it})}\, dt \Big| \\
&\qquad \le \Big| \int_{\mathbb{T}} \big(u(re^{it}) - u^\sharp(e^{it})\big)\overline{v(re^{it})}\, dt \Big| + \Big| \int_{\mathbb{T}} u^\sharp(e^{it})\overline{\big(v(re^{it}) - v^\sharp(e^{it})\big)}\, dt \Big| \\
&\qquad \le \|u_r - u^\sharp\|_p \|v_r\|_{p'} + \|u^\sharp\|_p \|v_r - v^\sharp\|_{p'} \\
&\qquad \le \|u_r - u^\sharp\|_p \|v\|_{h^{p'}} + \|u\|_{h^p} \|v_r - v^\sharp\|_{p'} \ .
\end{aligned}$$

By Theorem 4.3, $\|u_r - u^\sharp\|_p \to 0$, and $\|v_r - v^\sharp\|_{p'} \to 0$, and this proves the statement.

If $p = 1$ (and $p = \infty$ is the same), we have

$$\begin{aligned}
&\Big| \int_{\mathbb{T}} u(re^{it})\overline{v(re^{it})}\, dt - \int_{\mathbb{T}} u^\sharp(e^{it})\overline{v^\sharp(e^{it})}\, dt \Big| \\
&\qquad\qquad \le \|u_r - u^\sharp\|_1 \|v_r\|_\infty + \Big| \int_{\mathbb{T}} \big(v_r(e^{it}) - v^\sharp(e^{it})\big)\overline{u^\sharp(e^{it})}\, dt \Big| \ .
\end{aligned}$$

In this case, the conclusion follows from the fact that $\|u_r - u^\sharp\|_1 \to 0$, and that $v_r \to v^\sharp$ in the weak* topology of $L^\infty(\mathbb{T})$, again by Theorem 4.3. $\square$

Corollary 8.2. *The sesquilinear form B in Lemma 8.1 is continuous on $h^p(D) \times h^{p'}(D)$, precisely, if $u \in h^p(D)$, $v \in h^{p'}(D)$, then*

$$\big|B(u, v)\big| \le \|u\|_{h^p} \|v\|_{h^{p'}} \ .$$

If a_n, b_n are their respective Taylor coefficients, then the series $\sum_{n\in\mathbb{Z}} a_n\overline{b_n}$ is Abel summable, and

$$\lim_{r\to 1} \sum_{n\in\mathbb{Z}} a_n\overline{b_n} r^{|n|} = B(u, v) \ .$$

Proof. The first statement follows from Lemma 8.1. For the second, it is sufficient to observe that

$$\sum_{n\in\mathbb{Z}} a_n\overline{b_n} r^{|n|} = \int_{\mathbb{T}} u(\sqrt{r}e^{it})\overline{v(\sqrt{r}e^{it})}\, dt \ . \qquad \square$$

In terms of linear functionals on $H^p(D)$, the following is an immediate consequence.

Proposition 8.3. *If $g \in H^{p'}(D)$, the linear functional*

$$\varphi_g(f) = B(f, g)$$

is continuous on $H^p(D)$, and $\|\varphi_g\|_{H^p(D)^*} \leq \|g\|_{H^{p'}}$. The map

$$\Phi : g \mapsto \varphi_g$$

is a norm-decreasing conjugate-linear immersion of $H^{p'}(D)$ into $H^p(D)^*$.

Proof. The only fact that does not follows directly from Corollary 8.2 is the injectivity of Φ. But if $\varphi_g = 0$ for some $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^{p'}(D)$, then

$$\overline{b_n} = B(z^n, g) = 0 \ ,$$

for every n. Therefore $g = 0$. □

It is then natural to ask if Φ is surjective onto $H^p(D)^*$.

For $p < \infty$ there is an interesting connection with the boundedness of the Cauchy projection.

Theorem 8.4. *Suppose $p < \infty$. Then $\Phi : H^{p'}(D) \to H^p(D)^*$ is surjective if and only if the Cauchy projection is bounded from $h^{p'}(D)$ to $H^{p'}(D)$.*

Proof. By Lemma 5.2,

$$\|g\|_{H^{p'}} \leq A\|\varphi_g\|_{H^p(D)^*} \leq A\|g\|_{H^{p'}} \ , \tag{8.3}$$

for some constant $A > 0$.

Take $u \in h^{p'}(D)$, and denote $u^\sharp$ its boundary function. The linear functional $\psi(f) = B(f, u)$ is continuous on $H^p(D)$, because

$$|\psi(f)| \leq \|f^\sharp\|_p \|u^\sharp\|_{p'} = \|u\|_{h^{p'}} \|f\|_{H^p} \ .$$

In particular, $\|\psi\|_{H^p(D)^*} \leq \|u\|_{h^{p'}}$. Being Φ surjective, there is $g \in H^{p'}(D)$ such that $\psi = \varphi_g$, and

$$\|g\|_{H^{p'}} \leq A\|\psi\|_{H^p(D)^*} \leq A\|u\|_{h^{p'}} \ ,$$

by (8.3). We show that $g = Cu$, by observing that, if

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \bar{z}^n \ ,$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \ ,$$

then $\psi(z^n)$ gives $\overline{\widehat{u^\sharp}(n)} = \overline{a_n}$ for $n \geq 0$, but at the same time it equals $\varphi_g(z^n) = \overline{b_n}$. Therefore,

$$\|Cu\|_{H^{p'}} \leq A\|u\|_{h^{p'}} \ .$$

Conversely, suppose that the Cauchy projection maps $h^{p'}(D)$ into $H^{p'}(D)$, and let ψ be a continuous linear functional on $H^p(D)$. Define

$$\psi^\sharp : L^p_+(\mathbb{T}) \longrightarrow \mathbb{C}$$

as $\psi^\sharp(f^\sharp) = \psi(f)$. By the Hahn-Banach theorem, $\psi^\sharp$ admits a continuous extension to all of $L^p(\mathbb{T})$, with the same norm. So there is $h \in L^{p'}(\mathbb{T})$ such that

$$\|h\|_{p'} = \|\psi^\sharp\|_{L^p_+(\mathbb{T})^*} = \|\psi\|_{H^p(D)^*} ,$$

and

$$\psi(f) = \int_{\mathbb{T}} f^\sharp(e^{it})\overline{h(e^{it})}\, dt ,$$

for every $f \in H^p(D)$.

If $u \in h^{p'}(D)$ is the Poisson integral of h, so that $h = u^\sharp$,

$$\psi(f) = B(f, u) .$$

Let

$$u(z) = \sum_{n=0}^\infty b_n z^n + \sum_{n=1}^\infty b_{-n}\bar{z}^n ,$$

and

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

be the series expansions of u and f. By Corollary 8.2,

$$\psi(f) = \lim_{r\to 1}\sum_{n=0}^\infty a_n \overline{b_n} r^n = B(f, Cu) .$$

Hence $\psi = \Phi(Cu)$, and $Cu \in H^{p'}(D)$. □

As we have anticipated, we shall see in Chapter III that C is bounded from $h^p(D)$ to $H^p(D)$ for $1 < p < \infty$, so that, for these values of p, Φ provides an identification between $H^p(D)^*$ and $H^{p'}(D)$.

Since C does not map bounded harmonic functions into bounded holomorphic functions (see Proposition 5.3), not all the continuous linear functionals on $H^1(D)$ can be represented as φ_g for some $g \in H^\infty(D)$.

Consider, for example, $v(z) = \arg(1-z) = \Im\mathrm{m}\log(1-z) \in h^\infty(D)$. Then φ_v is bounded on $H^1(D)$, and, for $f \in H^1(D)$, $B(f, v) = B(f, g)$ with $g(z) = \frac{1}{2i}\log(1-z) \notin H^\infty(D)$.

More general examples arise from the following *Hardy's inequality*.

Theorem 8.5. *Let* $f(z) = \sum_{n=0}^\infty a_n z^n$ *be in* $H^1(D)$. *Then*

$$\sum_{n=0}^\infty \frac{|a_n|}{n+1} \le \|f\|_{H^1} .$$

Proof. We assume first that $f(0) = a_0 = 0$, and prove in this case the stronger inequality

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \|f\|_{H^1} \ .$$

The function

$$g(t) = \int_0^t f^\sharp(e^{is})\, ds$$

is continuous on the real line, and, since $\int_{\mathbb{T}} f^\sharp(e^{it})\, dt = f(0) = 0$, it is periodic of period 2π. Hence it defines a continuous function on $\mathbb{T}$, whose derivative is $f^\sharp$, and whose Fourier coefficients are

$$\begin{aligned}
\hat{g}(0) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^t f^\sharp(e^{is})\, ds\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f^\sharp(e^{is}) \int_s^{2\pi} dt\, ds \\
&= \frac{1}{2\pi} \int_0^{2\pi} f^\sharp(e^{is})(2\pi - s)\, ds \\
&= -\frac{1}{2\pi} \int_0^{2\pi} s f^\sharp(e^{is})\, ds \ ,
\end{aligned}$$

and

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \int_0^t f^\sharp(e^{is})\, ds\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f^\sharp(e^{is}) \int_s^{2\pi} e^{-int}\, dt\, ds \\
&= \frac{1}{2\pi} \int_0^{2\pi} f^\sharp(e^{is}) \frac{1}{in}(e^{-ins} - 1)\, ds \\
&= \frac{\widehat{f^\sharp}(n)}{in} \ ,
\end{aligned}$$

if $n \neq 0$. Hence,

$$\hat{g}(n) = \begin{cases} \dfrac{a_n}{in} & \text{if } n > 0 \\ 0 & \text{if } n < 0 \ . \end{cases}$$

It follows that the function

$$F(z) = i\hat{g}(0) + \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \ ,$$

which is a holomorphic primitive of $f(z)/z$ in D, coincides with the Poisson integral of ig, hence it is continuous on $\bar{D}$.

In particular, by Theorem 4.3,

$$i\hat{g}(0) + \lim_{r \to 1} \sum_{n=1}^{\infty} \frac{a_n}{n} r^n = F(1) = 0 \ .$$

Assume first that $a_n \geq 0$ for every $n > 0$. By monotone convergence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_n}{n} &= \lim_{r\to 1} \sum_{n=1}^{\infty} \frac{a_n}{n} r^n \\ &= \frac{i}{2\pi} \int_0^{2\pi} s f^\sharp(e^{is})\, ds \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} s|f^\sharp(e^{is})|\, ds \\ &\leq \|f\|_{H^1} .\end{aligned}$$

We remove now the assumption that $a_n \geq 0$, and write $f = f_1 f_2$, with $f_1, f_2 \in H^2(D)$, and $\|f_1\|_{H^2}^2 = \|f_2\|_{H^2}^2 = \|f\|_{H^1}$, according to Corollary 7.8. Since $f(0) = 0$, one of the two factors must vanish at 0, say $f_1(0) = 0$.

If

$$f_1(z) = \sum_{n=1}^{\infty} b_n z^n , \qquad f_2(z) = \sum_{n=0}^{\infty} c_n z^n ,$$

let

$$g_1(z) = \sum_{n=1}^{\infty} |b_n| z^n , \qquad g_2(z) = \sum_{n=0}^{\infty} |c_n| z^n .$$

Then $\|g_1\|_{H^2} = \|f_1\|_{H^2}$, $\|g_2\|_{H^2} = \|f_2\|_{H^2}$. Hence $g = g_1 g_2 \in H^1(D)$ and $\|g\|_{H^1} \leq \|g_1\|_{H^2}\|g_2\|_{H^2} = \|f\|_{H^1}$.

Observe that

$$|a_n| = \Big| \sum_{j+k=n} b_j c_k \Big| \leq \sum_{j+k=n} |b_j||c_k| = \tilde{a}_n ,$$

which is the n-th Taylor coefficient of g. Therefore,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n} \leq \|g\|_{H^1} \leq \|f\|_{H^1} .$$

If $f(0) \neq 0$, we just replace f by zf, observing that $\|zf\|_{H^1} = \|f\|_{H^1}$. $\square$

Corollario 8.6. *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ with $b_n = O(n)$. Then $\varphi_g(f) = B(f, g)$ is a continuous linear functional on $H^1(D)$.*

Proof. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1(D)$, it follows from Hardy's inequality that

$$B(f, g) = \lim_{r\to 1} \sum_{n=1}^{\infty} a_n \overline{b_n} r^n = \sum_{n=1}^{\infty} a_n \overline{b_n} ,$$

by dominated convergence. Therefore,

$$\big|B(f, g)\big| \leq \|f\|_{H^1} \sup_{n\in\mathbb{N}} \frac{|b_n|}{n+1} . \qquad \square$$

To conclude, the case $p = \infty$ must be treated separately. As one can reasonably expect, the dual space of $H^\infty(D)$ is strictly bigger than $\Phi\big(H^1(D)\big)$. This can be shown in many ways.

One argument is based on the following remark. Let $HC(D)$ the space of holomorphic functions that admit a continuous extension to $\bar{D}$. Then $HC(D)$ is a closed subspace of $H^\infty(D)$, and proper, because it does not contain the infinite Blaschke products. By the Hahn-Banach theorem, there is a non-zero continuous linear functional φ on $H^\infty(D)$ which vanishes identically on $HC(D)$. Suppose that $\varphi(f) = B(f,g)$ for some $g \in H^1(D)$, and let

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \ .$$

Since $z^n \in HC(D)$ for $n \geq 0$,

$$0 = B(z^n, g) = \overline{a_n} \ ,$$

so that $g = 0$. But this contradicts the fact that φ is not identically zero.

This argument motivates a new question: can Φ be surjective from $H^1(D)$ onto the dual space of $HC(D)$? The answer is negative again. It is possible to modify the first part of the proof of Theorem 8.4 to show that, if this is the case, then the Cauchy projection maps $h^1(D)$ into $H^1(D)$, which we know to be false.

CHAPTER II
HARDY SPACES ON THE HALF-PLANE

1. Definitions and basic facts

Let now D_+ be the upper half-plane in $\mathbb{C}$, $D_+ = \{x+iy : y>0\}$.

Definition. *If f is holomorphic (resp. harmonic) in D_+, we say that $f \in H^p(D_+)$ (resp. $f \in h^p(D_+)$), for $1 \le p \le \infty$, if for every $y>0$ $f_y(x) = f(x+iy)$ is in $L^p(\mathbb{R})$, and*

$$\sup_{y>0} \|f_y\|_p < \infty \ . \tag{1.1}$$

We keep the notation $M_p(f,r)$ for $\|f_r\|_p$, and $\|f\|_{H^p}$ (resp. $\|f\|_{h^p}$) for the supremum in (1.1).

Lemma 1.1. *If $f \in h^p(D_+)$, and $z = x+iy \in D_+$, then*

$$|f(z)| \le C_p \frac{\|f\|_{h^p}}{y^{\frac{1}{p}}} \ .$$

Proof. We assume that $p<\infty$, the other case being trivial.

For every $r<y$,

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z+re^{it})\, dt \ ,$$

by the mean value property. Integrating in polar coordinates around z, we then have

$$\begin{aligned} \frac{1}{|B(z,y)|}\int_{B(z,y)} f(w)\, dw &= \frac{1}{\pi y^2}\int_0^y \int_{-\pi}^{\pi} f(z+re^{it})\, dt\, r\, dr \\ &= \frac{2}{y^2}\int_0^y f(z)\, r\, dr \\ &= f(z) \ . \end{aligned} \tag{1.2}$$

Therefore, using Hölder's inequality and the inclusion $B(z,y) \subset S_y = \{u+iv :$

$0 < v < 2y\}$,

$$\begin{aligned}
|f(z)| &\leq \frac{1}{\pi y^2} \int_{B(z,y)} |f(w)|\, dw \\
&\leq \left(\frac{1}{\pi y^2} \int_{B(z,y)} |f(w)|^p\, dw \right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{\pi y^2} \int_{S_y} |f(w)|^p\, dw \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{\pi y^2} \int_{S_y} |f(u+iv)|^p\, du\, dv \right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{\pi y^2} \int_0^{2y} M_p(f,v)^p\, dv \right)^{\frac{1}{p}} \\
&\leq \|f\|_{h^p} \left(\frac{1}{\pi y^2} \int_0^{2y} dv \right)^{\frac{1}{p}} \\
&= \left(\frac{2}{\pi y} \right)^{\frac{1}{p}} \|f\|_{h^p} . \qquad \square
\end{aligned}$$

As for the corresponding spaces on the unit disc, we then have the following immediate consequence.

Theorem 1.2. *$H^p(D_+)$ and $h^p(D_+)$ are Banach spaces, and convergence in their norm implies uniform convergence on compact sets.*

The information given by Lemma 1.1 concerns the behaviour of an h^p-function f both for $y \to 0$ and for $y \to \infty$. It does not say anything, however, on the behaviour of $f(z)$ when z tends to infinity within a horizontal strip. The best one can say is the following.

Lemma 1.3. *For $0 < a < b$, let $S_{a,b} = \{x + iy : a \leq y \leq b\}$. If $f \in h^p(D_+)$, with $p < \infty$, , then*

$$\lim_{z \to \infty,\, z \in S_{a,b}} f(z) = 0 .$$

Proof. For $z \in S_{a,b}$, let B_z the disc centered at z of radius a. By (1.2),

$$|f(z)| \leq \frac{1}{\pi a^2} \int_{B_z} |f(w)|\, dw \leq \left(\frac{1}{\pi a^2} \int_{B_z} |f(w)|^p\, dw \right)^{\frac{1}{p}} . \tag{1.3}$$

Observe that $B_z \subset S_{0,b+a}$, and that

$$\int_{S_{0,b+a}} |f(w)|^p\, dw = \int_0^{b+a} \int_{\mathbb{R}} |f(x+iy)|^p\, dx\, dy \leq (b+a)\|f\|_{h^p} < \infty .$$

Therefore, given $\varepsilon > 0$, there is $M > 0$ such that

$$\int_0^{b+a} \int_{|x|>M} |f(x+iy)|^p\, dx\, dy < \pi a^2 \varepsilon^p . \tag{1.4}$$

If $|\Re ez| > M + a$, $B_z \subset \{x + iy : |x| > M, 0 < y < b + a\}$. Putting (1.3) and (1.4) together, we obtain that $|f(z)| < \varepsilon$. □

In contrast with the corresponding spaces on the unit disc, no Hardy space on D_+ is contained in any other, as it can be seen with appropriate examples. For instance, if $\alpha > 0$,

$$f_\alpha(z) = \frac{1}{(z+i)^\alpha}$$

is in $H^p(D_+)$ if and only if $p\alpha > 1$, and

$$g_\alpha(z) = \frac{1}{z^\alpha(z+i)^2}$$

is in $H^p(D_+)$ if and only if $p\alpha < 1$.

2. Poisson integrals

Before discussing Poisson integrals in D_+, we must recall some facts about convolution on $\mathbb{R}$ on one side, and about conformal mappings on the other side.

The convolution of two continuous functions f and g with compact support is defined as

$$f * g(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt \; . \tag{2.1}$$

By a simple change of variables, one verifies that $f * g = g * f$. It is also easy to verify that

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g = \{x + x' : x \in \operatorname{supp} f, y \in \operatorname{supp} g\} \; .$$

When it will be necessary to distinguish between convolution on $\mathbb{T}$ and convolution on $\mathbb{R}$, we shall write $*_{\mathbb{T}}$ and $*_{\mathbb{R}}$ accordingly.

The integral (2.1) may not make sense for more general functions, unless certain integrability conditions are satisfied[22]. One can prove that the integral (2.1) is convergent for almost every x if $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, with $1 \le p, q \le \infty$ and

$$\frac{1}{p} + \frac{1}{q} \ge 1 \; . \tag{2.2}$$

In this case $f * g \in L^r(\mathbb{R})$, with

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \; ,$$

and the *Young inequality*

$$\|f * g\|_r \le \|f\|_p \|g\|_q \tag{2.3}$$

[22] Observe that if $f(x) = (1+|x|)^{-\alpha}$ and $g(x) = (1+|x|)^{-\beta}$, the integral (2.1) is divergent for every x if $\alpha + \beta \le 1$. Compare this example with the restrictions on the exponents p and q below.

holds. When $r = \infty$ (i.e. p and q are conjugate exponents), $f * g$ is also continuous. If, in addition, $1 < p, q < \infty$, $f * g$ vanishes at infinity.

The convolution of two finite, regular Borel measures $\mu, \nu \in M(\mathbb{R})$ is defined as the measure $\mu * \nu$ such that

$$\int_{\mathbb{R}} f(x)\, d(\mu * \nu)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s+t)\, d\mu(s)\, d\nu(t) \ , \tag{2.4}$$

for $f \in C_0(\mathbb{R})$. One has the inequality

$$\|\mu * \nu\|_1 \leq \|\mu\|_1 \|\nu\|_1 \ . \tag{2.5}$$

In a more subtle way, the convolution $\mu * f$ of a measure $\mu \in M(\mathbb{R})$ with a function $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, is also well defined, and

$$\|\mu * f\|_p \leq \|\mu\|_1 \|f\|_p \ . \tag{2.6}$$

Recall now that if A and B are two connected, simply connected, proper open subsets of $\mathbb{C}$, there exists a conformal mapping[23] φ from A onto B. For $A = D$, the unit disc, and $B = D_+$, one can explicitely write such mappings. One of them is

$$\varphi(z) = i\frac{1+z}{1-z} \ , \tag{2.7}$$

and it is called the *Cayley transform*.

Lemma 2.1. *The Cayley transform φ maps D onto D_+, it is invertible, and*

$$\varphi^{-1}(w) = \frac{w-i}{w+i} \ . \tag{2.8}$$

Moreover φ has a continuous extension to $\bar{D} \setminus \{1\}$, and

$$\varphi(e^{i\theta}) = -\cot\frac{\theta}{2} \in \mathbb{R} = \partial D_+ \ . \tag{2.9}$$

Proof. One has

$$\varphi(z) = i\frac{(1+z)(1-\bar{z})}{|1-z|^2} = i\frac{1-|z|^2 + 2i\Im\mathrm{m} z}{|1-z|^2} \ ,$$

so that

$$\Im\mathrm{m}\varphi(z) = \frac{1-|z|^2}{|1-z|^2} > 0 \ ,$$

if $z \in D$. Hence φ maps D into D_+.

It is easy to verify that φ is injective and that (2.8) gives its inverse function. If $w \in D_+$, $|w-i| < |w+i|$ by a simple geometric consideration, so that $|\varphi^{-1}(w)| < 1$. This shows that φ is onto.

The extension of φ to the boundary is easy to derive. □

We can use the Cayley transform to trasfer harmonic functions from D to D_+ and viceversa.

[23] i.e. holomorphic and invertible. The fact we are stating is called the *Riemann mapping theorem*.

Lemma 2.2. *Let A, B be open subsets of $\mathbb{C}$. If $\varphi : A \to B$ is holomorphic, and u is harmonic on B, then $u \circ \varphi$ is harmonic on A.*

Proof. We can assume that u is real-valued. Fix $z \in A$, and let $U \subset B$ be a circular neighborhood of $w = \varphi(z)$. By Corollary 2.6 od Chapter I, $u_{|_B}$ is the real part of a holomorphic function f. Then $f \circ \varphi$ is holomorphic on the neighborhood $V = \varphi^{-1}(U)$ of z, and $u \circ \varphi = \Re\mathrm{e}(f \circ \varphi)$ is harmonic on V. $\square$

Lemma 2.3. *Let u be continuous on $\overline{D_+}$, harmonic in D and bounded. Then*

$$u(x+iy) = \frac{1}{\pi} \int_{\mathbb{R}} u(t) \frac{y}{(x-t)^2 + y^2}\, dt \ . \tag{2.10}$$

Proof. If φ is the Cayley transform, the function $v(z) = u \circ \varphi(z)$ is in $h^\infty(D)$. In addition, v is continuous on $\bar{D} \setminus \{1\}$. By dominated convergence,

$$\int_{\mathbb{T}} v(e^{i\theta})\, d\theta = \lim_{r \to 1} \int_{\mathbb{T}} v(re^{i\theta})\, d\theta = v(0) \ .$$

Since $v(0) = u(i)$ and $v(e^{i\theta}) = u(-\cot\frac{\theta}{2})$ by (2.9), the change of variable $-\cot\frac{\theta}{2} = t$ gives

$$u(i) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\theta})\, d\theta = \frac{1}{\pi} \int_{-\infty}^{+\infty} u(t) \frac{1}{t^2+1}\, dt \ ,$$

which is (2.10) for $x + iy = i$.

For a general point $x_0 + iy_0 \in D_+$, consider the function $\tilde{u}(z) = u(x_0 + y_0 z)$, which satisfies the same hypotheses as u. Then, with simple changes of variables,

$$\begin{aligned} u(x_0 + iy_0) &= \tilde{u}(i) \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} u(x_0 + y_0 t) \frac{1}{t^2+1}\, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} u(x_0 + t) \frac{y_0}{t^2 + y_0^2}\, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{+\infty} u(t) \frac{y_0}{(x_0 - t)^2 + y_0^2}\, dt \ . \end{aligned}$$

$\square$

Definition. *For $y > 0$, the function*

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

is called the Poisson kernel *on* $\mathbb{R}$.

Formula (2.10) can be written as

$$u_y = u_0 * P_y \ . \tag{2.11}$$

Corollary 2.4. *Let $u \in h^p(D_+)$, with $1 \le p \le \infty$. Given $0 < y_1 < y_2$, then*

$$u_{y_2} = u_{y_1} * P_{y_2 - y_1} \ .$$

In particular, the Poisson kernel has the semigroup property

$$P_{y+y'} = P_y * P_{y'} \ . \tag{2.12}$$

Proof. Let $v(z) = u(z + iy_1)$. By Lemma 1.1, v satisfies the assumptions of Lemma 2.3. Therefore

$$u_{y_2} = v_{y_2 - y_1} = v_0 * P_{y_2 - y_1} = u_{y_1} * P_{y_2 - y_1} \ .$$

Applying this identity to

$$u(x + iy) = P_y(x) \in h^1(D_+)$$

(observe that u is harmonic because it equals $-\frac{1}{\pi}\Im\mathrm{m}\frac{1}{x+iy}$), we obtain (2.12). ☐

A further important property of the Poisson kernel is that it forms an approximate identity on the real line. By definition, an approximate identity on $\mathbb{R}$ (for $\varepsilon \to 0$) is a family of functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ satisfying

(1) $\int_{\mathbb{R}} |\varphi_\varepsilon(t)|\, dt \le C$ for some constant C and every ε;
(2) $\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \varphi_\varepsilon(t)\, dt = 1$ for every r;
(3) for every $\delta > 0$,

$$\lim_{\varepsilon\to 0} \int_{|t|>\delta} |\varphi_\varepsilon(t)|\, dt = 0 \ .$$

Proposition 3.2 of Chapter I extends to approximate identities on $\mathbb{R}$.

Lemma 2.5. *Take $\varphi \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. Then the functions $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\big(\frac{x}{\varepsilon}\big)$ form an approximate identity for $\varepsilon \to 0$. In particular, the Poisson kernels P_y form an approximate identity for $y \to 0$.*

Proof. A simple change of variable shows that $\|\varphi_\varepsilon\|_1 = \|\varphi\|_1$ and $\int_{\mathbb{R}} \varphi_\varepsilon(t)\, dt = \int_{\mathbb{R}} \varphi(t)\, dt = 1$. This proves (1) and (2). By the same change of variable,

$$\int_{|t|>\delta} |\varphi_\varepsilon(t)|\, dt = \int_{|t|>\frac{\delta}{\varepsilon}} |\varphi(t)|\, dt \ ,$$

which tends to 0 with ε.

For the Poisson kernel, observe that

$$P_y(x) = \frac{1}{y} P_1\big(\frac{x}{y}\big) \ , \tag{2.13}$$

and

$$\|P_1\| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x^2+1}\, dx = 1 \ . \qquad \square$$

We are now in a position to extend to the spaces $h^p(D)$ the results proved about norm or weak* convergence to the boundary. We summarize them in the following statement.

Theorem 2.6. *If $u \in h^p(D_+)$, then $M_p(u,y) = \|u_y\|_p$ is decreasing in y, so that*

$$\|u\|_{h^p} = \lim_{y\to 0} M_p(u,y) \ .$$

The operator $\mathcal{P}$ mapping f (understood as either a function or a Borel measure on $\mathbb{R}$) into the harmonic function $\mathcal{P}f$ on D_+ given by

$$(\mathcal{P}f)(x+iy) = f * P_y(x) \ ,$$

maps $L^p(\mathbb{R})$ isometrically onto $h^p(D_+)$ for $1 < p \le \infty$, and it maps $M(\mathbb{R})$ isometrically onto $h^1(D_+)$.

The limit $\lim_{y\to 0}(\mathcal{P}f)_y$ exists in L^p if and only if one of the following holds

(1) *$f \in L^p(\mathbb{R})$ and $1 \le p < \infty$;*
(2) *$p = \infty$ and $f \in C(\mathbb{R})$.*

In each of these cases, $(\mathcal{P}f)_y \to f$ in the L^p-norm.

For general elements f of $M(\mathbb{R})$ or $L^\infty(\mathbb{R})$, $(\mathcal{P}f)_y$ tends to f in the corresponding weak-topology.*

Proof. Once we have proven that $\mathcal{P}f$ is harmonic, it is a matter of adapting the proof of Theorem 4.3 in Chapter I.

We verify the mean value property for $u = \mathcal{P}f$ assuming that f is a function (the case $f \in M(\mathbb{R})$ is left to the reader). If $\overline{B(z_0,r)} \subset D_+$, then

$$\int_{\mathbb{T}} u(z_0 + re^{i\theta})\, d\theta = \int_{\mathbb{T}}\int_{\mathbb{R}} P(z_0 + re^{i\theta} - t) f(t)\, dt\, d\theta \ . \tag{4.6}$$

We can apply Fubini's theorem and change the order of integration for the following reason. For fixed θ, write $z_0 + re^{i\theta} = x_\theta + iy_\theta$. Then

$$P(z_0 + re^{i\theta} - t) f(t) = P(x_\theta + iy_\theta - t) f(t)$$

is integrable in t, because it is the product of an $L^{p'}$- and an L^p-function. By (2.13),

$$\|P_y\|_q = \left(\int_{\mathbb{R}} \frac{1}{y^q}\Big|P_1\Big(\frac{x}{y}\Big)\Big|^q\, dx\right)^{\frac{1}{q}} = \frac{C_q}{y^{1-\frac{1}{q}}} \ .$$

if $1 \le q < \infty$, and similarly for $q = \infty$.

There is a $\delta > 0$ such that $\Im\mathrm{m} y_\theta \ge \delta$ for all θ. Therefore

$$\int_{\mathbb{R}} \big|P(z_0 + re^{i\theta} - t)\big|\big|f(t)\big|\, dt \le C_{p,\delta}\|f\|_p$$

uniformly in θ. Therefore the double integral in (4.6) is absolutely convergent.

We apply Fubini's theorem, and use the fact that the function $P(x+iy)$ is harmonic in D_+, as observed in the proof of Corollary 2.4. Then

$$\begin{aligned}\int_{\mathbb{T}} u(z_0 + re^{i\theta})\, d\theta &= \int_{\mathbb{R}} f(t) \int_{\mathbb{T}} P(z_0 + re^{i\theta} - t)\, d\theta\, dt \\ &= \int_{\mathbb{R}} f(t) P(z_0 - t)\, dt \\ &= u(z_0) \ . \qquad \square\end{aligned}$$

The following corollary can be seen as a replacement for the (non-existing) inclusion relations among the $h^p(D_+)$.

Corollary 2.7. *If $1 < p < \infty$, and $1 \le q \le \infty$, then $h^p(D_+) \cap h^q(D_+)$ is dense in $h^p(D_+)$. This does not hold if $p = 1, \infty$ and $q \neq p$.*

Proof. By Theorem 2.6, it is sufficient to prove the corresponding statement for $L^p(\mathbb{R})$ $(1 < p \le \infty)$ and $M(\mathbb{R})$.

If $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, and $1 \le q < p$, then $f_R = f\chi_{[-R,R]}$ is in both $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, and $f_R \to f$ in the L^p-norm as $R \to \infty$. If $p < q \le \infty$, the same holds for $f_R = f\chi_{\{|f|<R\}}$.

For $p = 1 \neq q$, observe that $M(\mathbb{R}) \cap L^q(\mathbb{R}) = L^1(\mathbb{R}) \cap L^q(\mathbb{R})$, so that its closure in $M(\mathbb{R})$ is only $L^1(\mathbb{R})$.

For $p = \infty$, observe that the constant function 1 cannot be the uniform limit of functions in $L^q(\mathbb{R})$ if $q < \infty$. □

We must also mention the behaviour of $M_p(u, y)$ as $y \to +\infty$.

Proposition 2.8. *If $1 < p < \infty$ and $u \in h^p(D_+)$, then $\lim_{y\to+\infty} M_p(u, y) = 0$.*

Proof. Given $\varepsilon > 0$, there is a continuous function φ on $\mathbb{R}$ with compact support such that $\|u^\sharp - \varphi\|_p < \varepsilon$. Then, for $y > 0$,

$$\begin{aligned} M_p(u, y) &= \|u^\sharp * P_y\|_p \\ &\le \|(u^\sharp - \varphi) * P_y\|_p + \|\varphi * P_y\|_p \\ &< \varepsilon + \|\varphi\|_1 \|P_y\|_p \\ &\le \varepsilon + y^{-\frac{1}{p'}} \|\varphi\|_1 \ . \end{aligned}$$

So, if y is large enough, $M_p(u, y) < 2\varepsilon$. □

3. The Fourier transform and the Paley-Wiener theorem

We begin this section by recalling some of the basic facts about the Fourier transform on $\mathbb{R}$.

If $f \in L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$, one defines the Fourier transform of f at ξ as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx \ . \tag{3.1}$$

The inequality $|\hat{f}(\xi)| \le \|f\|_1$ is an immediate consequence of the definition. The function $\hat{f}$ so defined is continuous on $\mathbb{R}$ and vanishes at infinity (this last fact is known as the Riemann-Lebesgue theorem). Then the linear operator $\mathcal{F}$ mapping f into $\mathcal{F}f = \hat{f}$ is continuous from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$.

One also defines the Fourier transform of a finite Borel measure $\mu \in M(\mathbb{R})$ as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \ .$$

Then $|\hat{\mu}(\xi)| \le \|\mu\|_1$, $\hat{\mu}$ is continuous, but we can no longer say that it vanishes at infinity.

Further properies of $\mathcal{F}$ are the following.

(1) if $\check{f}(x) = f(-x)$, then $\widehat{\check{f}}(\xi) = \hat{f}(-\xi)$;
(2) $\widehat{\overline{f}}(\xi) = \overline{\hat{f}(-\xi)}$;
(3) $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$;
(4) if $f_\varepsilon(x) = \frac{1}{\varepsilon} f(\frac{x}{\varepsilon})$, then $\widehat{f_\varepsilon}(\xi) = \hat{f}(\varepsilon\xi)$;
(5) if f is absolutely continuous, so that also $f' \in L^1(\mathbb{R})$, then $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$;
(6) if both f and xf are integrable, than $\hat{f}$ is C^1 and $(\hat{f})'(\xi) = -i\,\widehat{(xf)}(\xi)$;
(7) if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and the Plancherel formula holds:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 \ ; \tag{3.2}$$

(8) because of (3.2), $\mathcal{F}$ extends to every $f \in L^2(\mathbb{R})$, and $\|\hat{f}\|_2 = \sqrt{2\pi}\|f\|_2$;
(9) since $L^p(\mathbb{R}) \subset L^1(\mathbb{R}) + L^2(\mathbb{R})$, if $1 < p < 2$, $\mathcal{F}$ is well defined on $L^p(\mathbb{R})$, $\hat{f} \in L^{p'}(\mathbb{R})$, and the Hausdorff-Young inequality holds:

$$\|\hat{f}\|_{p'} \le \|f\|_p \ ; \tag{3.3}$$

(10) if $\hat{f}$ is integrable[24], the inversion formula holds:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi \ . \tag{3.4}$$

Lemma 3.1. *The Fourier transform of P_y is*

$$\widehat{P_y}(\xi) = e^{-y|\xi|} \ .$$

If $u \in h^p(D_+)$, with $1 \le p \le 2$, and $u^\sharp$ is its boundary function (or measure), then

$$\widehat{u_y}(\xi) = \widehat{u^\sharp}(\xi) e^{-y|\xi|} \ . \tag{3.5}$$

Proof. By (4) above and (2.13),

$$\widehat{P_y}(\xi) = \widehat{P_1}(y\xi) \ ,$$

and by (3) and (2.12),

$$\widehat{P_{y+y'}}(\xi) = \widehat{P_1}\big((y+y')\xi\big) = \widehat{P_1}(y\xi)\widehat{P_1}(y'\xi) \ .$$

Moreover, by (1) and (2), $\widehat{P_1}$ is real and even. Putting these information together with the fact that $\widehat{P_1} \in C_0(\mathbb{R})$, we find that $\widehat{P_1}(\xi) = e^{-a|\xi|}$ for some $a > 0$.

[24]This happens, in particular, if f is differentiable and $f' \in L^2(\mathbb{R})$.

In particular, $\widehat{P_1} \in L^1(\mathbb{R})$, so that we can use the inversion formula in order to determine a. We are so led to compute

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-a|\xi|}e^{ix\xi}\,d\xi &= \frac{1}{\pi}\int_0^\infty e^{-a\xi}\cos(x\xi)\,d\xi \\ &= \frac{1}{\pi}\Re\mathrm{e}\left(\int_0^\infty e^{-(a-ix)\xi}\,d\xi\right) \\ &= \frac{1}{\pi}\Re\mathrm{e}\frac{1}{a-ix} \\ &= \frac{1}{\pi}\frac{a}{x^2+a^2}\ .\end{aligned}$$

This must be equal to $P_1(x)$, hence $a = 1$. The rest of the proof is obvious. □

The Fourier transform allows to describe for which functions $f \in L^p(\mathbb{R})$, the Poisson integral $u(x+iy) = f * P_y(x)$ is holomorphic. The main result concerns $p = 2$, and is called the *Paley-Wiener theorem.*

Theorem 3.2. *A function $u \in h^2(D_+)$ is holomorphic if and only if the Fourier transform $\widehat{u^\sharp}$ of its boundary function $u^\sharp \in L^2(\mathbb{R})$ is zero on the negative half-line. Hence a holomorphic function f on D_+ is in $H^2(D_+)$ if and only if f is the* Fourier-Laplace transform*,*

$$f(z) = \frac{1}{2\pi}\int_0^\infty g(\xi)e^{iz\xi}\,d\xi\ , \tag{3.6}$$

of some $g \in L^2(\mathbb{R}_+)$, and in this case $\|f\|_{H^2} = \frac{1}{\sqrt{2\pi}}\|g\|_2$.

Proof. Suppose that u is holomorphic. Calling γ_a the horizontal curve $\gamma_a(t) = t+ia$, consider the line integral

$$I_a = \int_{\gamma_a} u(z)e^{-i\xi z}\,dz = \int_{-\infty}^\infty u(x+ia)e^{-i\xi(x+ia)}\,dx = e^{a\xi}\widehat{u_a}(\xi)\ .$$

We claim that $I_a = I_b$ if $a, b > 0$. Suppose $a < b$, fix $M > 0$ and let R_M be the rectangle with vertices $-M+ia$, $M+ia$, $M+ib$, $-M+ib$. Denote by $\gamma_{a,M}$, $\gamma_{b,M}$ the arcs of γ_a and γ_b describing the horizontal edges of R_M, and by γ_M^+, γ_M^- the arcs describing the vertical edges, oriented upwards. By the Cauchy integral formula applied to the boundary of R_M, we have

$$\begin{aligned}0 &= \int_{\gamma_{a,M}} u(z)e^{-i\xi z}\,dz + \int_{\gamma_M^+} u(z)e^{-i\xi z}\,dz \\ &\quad - \int_{\gamma_{b,M}} u(z)e^{-i\xi z}\,dz - \int_{\gamma_M^-} u(z)e^{-i\xi z}\,dz\ .\end{aligned} \tag{3.7}$$

Now,

$$\int_{\gamma_M^+} u(z)e^{-i\xi z}\,dz = i\int_a^b u(M+it)e^{-i\xi(M+it)}\,dt\ ,$$

so that

$$\left| \int_{\gamma_M^+} u(z) e^{-i\xi z}\, dz \right| \le \int_a^b |u(M+it)| e^{\xi t}\, dt$$
$$\le e^{b|\xi|}(b-a) \max_{t\in[a,b]} |u(M+it)| \ ,$$

which tends to 0 as $M \to \infty$ by Lemma 1.3. Then, passing to the limit as $M \to \infty$ in (3.7), we obtain that $I_a = I_b$.

So, by (3.5),

$$e^{a\xi} \widehat{u_a}(\xi) = e^{-a(|\xi|-\xi)} \widehat{u^\sharp}(\xi)$$

is independent of a. This forces $\widehat{u^\sharp}(\xi)$ to be a.e. zero for $\xi < 0$.

Since $\widehat{u^\sharp} \in L^2(\mathbb{R})$, then $\widehat{u_y} = e^{-y|\xi|}\widehat{u^\sharp} \in L^1(\mathbb{R})$ for $y > 0$, and the inversion formula (3.6) gives

$$u(x+iy) = \frac{1}{2\pi} \int_0^\infty \widehat{u^\sharp}(\xi) e^{-y\xi} e^{ix\xi}\, d\xi = \frac{1}{2\pi} \int_0^\infty \widehat{u^\sharp}(\xi) e^{i(x+iy)\xi}\, d\xi \ .$$

Conversely, take $g \in L^2(\mathbb{R}_+)$ and define f by (3.6). The integral is absolutely convergent, because $|e^{iz\xi}| = e^{-\xi \Im\mathrm{m} z} \in L^2(\mathbb{R}_+)$. If γ is a closed arc in D_+, then

$$\int_\gamma f(z)\, dz = \frac{1}{2\pi} \int_\gamma \int_0^\infty g(\xi) e^{iz\xi}\, d\xi\, dz \ .$$

If the support of γ is contained in the half-plane $\Im\mathrm{m} z \ge a > 0$, then $|e^{iz\xi}| \le e^{-a\xi}$, and $g(\xi)e^{iz\xi}$ is integrable on $\gamma \times \mathbb{R}_+$. We can then change order of integration and obtain that

$$\int_\gamma f(z)\, dz = \frac{1}{2\pi} \int_0^\infty g(\xi) \int_\gamma e^{iz\xi}\, dz\, d\xi = 0 \ ,$$

because $e^{iz\xi}$ is holomorphic in z. Then f is holomorphic by Morera's theorem.

Observe that f_y is the inverse Fourier transform of $g(\xi)e^{-y\xi}\chi_{\mathbb{R}_+}$. Therefore, by Plancherel's formula (7),

$$\|f_y\|_2 = \frac{1}{\sqrt{2\pi}} \|e^{-y\cdot} g\|_2 \ ,$$

and

$$\|f\|_{H^2} = \frac{1}{\sqrt{2\pi}} \lim_{y\to 0} \|e^{-y\cdot} g\|_2 = \frac{1}{\sqrt{2\pi}} \|g\|_2 \ . \qquad \square$$

Corollary 3.3. *Let* $1 \le p \le 2$, $u \in h^p(D_+)$ *and* $u^\sharp$ *be its boundary function (or measure). Then* u *is holomorphic if and only if* $\widehat{u^\sharp}$ *is zero on the negative half-line. If this is the case, then* u *is the Fourier-Laplace transform* (3.6) *of* $g = \widehat{u^\sharp} \in L^{p'}(\mathbb{R}_+)$.

Proof. Let $v_{(\varepsilon)}(z) = u(z+i\varepsilon)$. Then $v_{(\varepsilon)} \in h^p(D_+) \cap h^\infty(D_+)$, hence in $h^2(D_+)$. If we assume that $u \in H^p(D_+)$, then $v_{(\varepsilon)} \in H^2(D_+)$. Therefore,

$$\widehat{v^\sharp_{(\varepsilon)}}(\xi) = \widehat{u_\varepsilon}(\xi) = e^{-\varepsilon|\xi|} \widehat{u^\sharp}(\xi)$$

is zero on the negative half-line. By Hölder's inequality, $\widehat{u_y} = e^{-y|\cdot|}\widehat{u^\sharp} \in L^1(\mathbb{R})$ for $y > 0$, so that the Fourier inversion formula gives (3.6) as before.

Conversely, if $u \in h^p(D_+)$ and $\operatorname{supp} \widehat{u^\sharp} \subset [0, +\infty)$, the same is true for $v_{(\varepsilon)}$, which belongs to $h^2(D_+)$. Therefore, $v_{(\varepsilon)}$ is holomorphic in D_+, i.e. u is holomorphic in $D_+ + i\varepsilon$. By the arbitrarity of ε, u is holomorphic in D_+. □

4. The Cauchy projection and the Hilbert transform

By the Paley-Wiener theorem and Theorem 2.6, the map which assigns to a harmonic function $u \in h^2(D_+)$ the Fourier transform $\widehat{u^\sharp}$ is, up to a constant, an isometry of $h^2(D_+)$ onto $L^2(\mathbb{R})$, and the image of $H^2(D_+)$ is

$$L^2_+ = \{g \in L^2(\mathbb{R}) : \operatorname{supp} g \subseteq [0, +\infty)\} \sim L^2(\mathbb{R}_+) \ .$$

Since the orthogonal projection of $L^2(\mathbb{R})$ onto L^2_+ is obviously the map $g \mapsto g\chi_{\mathbb{R}_+}$, we derive the following recipe for constructing the orthogonal projection from $h^2(D_+)$ onto $H^2(D_+)$: starting with $u \in h^2(D_+)$, take $\widehat{u^\sharp}$, multiply it by $\chi_{\mathbb{R}_+}$, and apply (3.6). This gives the *Cauchy projection* on D_+,

$$\begin{aligned} Cu(x+iy) &= \frac{1}{2\pi}\int_0^\infty \widehat{u^\sharp}(\xi)e^{i(x+iy)\xi}\,d\xi \\ &= \mathcal{F}^{-1}\big(\widehat{u^\sharp}e^{-y\xi}\chi_{\mathbb{R}_+}\big)(x) \\ &= u^\sharp * \mathcal{F}^{-1}(e^{-y\xi}\chi_{\mathbb{R}_+})(x) \ . \end{aligned} \tag{4.1}$$

Define the *Cauchy kernel* $C_y(x) = \mathcal{F}^{-1}(e^{-y\xi}\chi_{\mathbb{R}_+})(x)$. Explicitely,

$$\begin{aligned} C_y(x) &= \frac{1}{2\pi}\int_0^\infty e^{-y\xi}e^{ix\xi}\,d\xi \\ &= -\frac{1}{2\pi i}\frac{1}{x+iy} \ . \end{aligned} \tag{4.2}$$

We can then write (4.1) in the following way.

Theorem 4.1. *The orthogonal projection C from $h^2(D_+)$ onto $H^2(D_+)$ is given by*

$$Cu(z) = u^\sharp * C_y(x) = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{u^\sharp(t)}{t-z}\,dt \ , \tag{4.3}$$

with $z = x + iy$.

If u is already in $H^2(D_+)$, then $Cu = u$, and (4.3) formally coincides with the ordinary Cauchy integral formula

$$u(z) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{u^\sharp(w)}{w-z}\,dw \ ,$$

where we denote by γ_0 the curve describing the real axis with its natural orientation. Notice however that the curve is not bounded[25] and it is not contained in the interior of the domain where u is holomorphic.

Write now

$$C_y(x) = -\frac{1}{2\pi i}\frac{x-iy}{x^2+y^2} = \frac{1}{2\pi}\frac{y}{x^2+y^2} + \frac{i}{2\pi}\frac{x}{x^2+y^2} = \frac{1}{2}\big(P_y(x) + i\tilde{P}_y(x)\big) \ ,$$

noticing that the real part of $2C_y$ is the Poisson kernel P_y and setting

$$\tilde{P}_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2} \ . \tag{4.4}$$

Therefore (4.3) takes the form

$$Cu(x+iy) = \frac{1}{2}u(x+iy) + \frac{i}{2}u^\sharp * \tilde{P}_y(x) \ . \tag{4.5}$$

Observe however that $\tilde{P}_y$ is not integrable for any $y>0$, so that the convolution $u^\sharp * \tilde{P}_y$ is not defined for $u \in h^\infty(D_+)$. On the other hand, $\tilde{P}_y$ is in $L^q(\mathbb{R})$ for any $q>1$ and in $C_0(\mathbb{R})$, so that the convolution is well defined for $u \in h^p(D_+)$ with $p<\infty$.

Proposition 4.2. *Let $u \in h^p(D_+)$ with $p<\infty$. Then*

$$\tilde{u}(x+iy) = u^\sharp * \tilde{P}_y(x)$$

is harmonic, $u + i\tilde{u}$ is holomorphic, and

$$\lim_{z\to\infty\,,\,\Im\mathrm{m} z\geq a} \tilde{u}(z) = 0 \ ,$$

for every $a>0$. If u is real-valued, the $\tilde{u}$ is the only real-valued function satisfying the above properties.

Proof. We verify the mean value property for $\tilde{u}$. If $\overline{B(z_0,r)} \subset D_+$, then

$$\int_{\mathbb{T}} \tilde{u}(z_0 + re^{i\theta})\,d\theta = \int_{\mathbb{T}}\int_{\mathbb{R}} \tilde{P}(z_0 + re^{i\theta} - t)u^\sharp(t)\,dt\,d\theta \ . \tag{4.6}$$

We can apply Fubini's theorem and change the order of integration for the following reason. For fixed θ, write $z_0 + re^{i\theta} = x_\theta + iy_\theta$. Then

$$\tilde{P}(z_0 + re^{i\theta} - t)u^\sharp(t) = \tilde{P}(x_\theta + iy_\theta - t)u^\sharp(t)$$

is integrable in t, because it is the product of an $L^{p'}$- and an L^p-function. Since

$$\tilde{P}_y(x) = \frac{1}{y}\tilde{P}_1\big(\frac{x}{y}\big) \ ,$$

[25] It becomes a nice closed curve if embedded in the Riemann sphere.

and $\tilde{P}_1 \in L^q(\mathbb{R})$ for $q > 1$, we have, as in the proof of Theorem 2.6,

$$\|\tilde{P}_y\|_{p'} \leq C_p y^{-\frac{1}{p}} \ ,$$

for $p < \infty$.

The proof continues as for Theorem 2.6. $\square$

Observe that in the proof we have implicitely obtained the inequality

$$|\tilde{u}(x+iy)| \leq C_p y^{-\frac{1}{p}} \|u\|_{h^p} \ . \tag{4.7}$$

for $u \in h^p(D_+)$ and $p < \infty$.

By (4.5),

$$\tilde{u} = iu - 2iCu \ ,$$

so that, if $u \in h^2(D_+)$, also $\tilde{u} \in h^2(D_+)$. If $\tilde{u}^\sharp$ is the boundary function, we have

$$\widehat{\tilde{u}^\sharp}(\xi) = i\widehat{u^\sharp}(\xi) - 2i\chi_{\mathbb{R}_+}(\xi)\widehat{u^\sharp}(\xi) = -i(\operatorname{sgn}\xi)\widehat{u^\sharp}(\xi) \ , \tag{4.8}$$

so that, by Plancherel's inequality,

$$\|\tilde{u}\|_{h^2} = \|u\|_{h^2} \ . \tag{4.9}$$

The operations described in (4.8) take place on the real line, and define a linear operator $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$Hf = \mathcal{F}^{-1}\big(-i \operatorname{sgn}\xi \hat{f}(\xi)\big) \ .$$

This operator is called the *Hilbert transform*. Proposition 6.2 in Chapter I has the following analogue (with a very similar proof, that we omit).

Proposition 4.3. *For $f \in L^2(\mathbb{R})$,*

$$Hf(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|>\varepsilon} f(x-t)\frac{1}{t}\, dt \ ,$$

where the limit is in the L^2-norm.

Much of the content of Sections 5 and 6 of Chapter I can be repeated here, with the unit disc replaced by the upper half-plane.

The Cauchy projection is bounded on $h^p(D_+)$ for $p \neq 2$ if and only if the same is true for the conjugate function operator mapping u to $\tilde{u}$. As for the unit disc (and with the same arguments) this turns out to be false if $p = 1$. Also for $p = \infty$, when the formulas above do not hold, it is not true in general that the conjugate functions of a bounded harmonic function are bounded[26]. This is not even true if u is bounded and continuous on $\overline{D_+}$.

We shall see later that C is bounded from $h^p(D_+)$ onto $H^p(D_+)$ for $1 < p < \infty$.

[26] Take $u(x+iy) = \operatorname{arccot}\frac{x}{y} = \arg z = \Im\mathrm{m}\log z$. It is bounded, but its harmonic conjugates, $-\log|z|$ + constant, are unbounded.

5. Transference of H^p-functions and applications

We shall now establish simple relations between H^p-functions on the unit disc and the upper half-plane. This will allow us to transfer to D_+ many results obtained in Chapter I for D.

The simplest situation occurs for $p = \infty$: if φ denotes the Cayley transform (2.7), the map

$$f \longmapsto f \circ \varphi^{-1} = T_\infty f \tag{5.1}$$

is an isometry of $H^\infty(D)$ onto $H^\infty(D_+)$, for the simple reason that f is bounded if and only if $f \circ \varphi^{-1}$ is bounded, and the two ∞-norms are equal[27].

Clearly the same does not hold for $p < \infty$ (take $f = 1$). In order to see what needs to be modified, let us consider the simple case where f is continuous on $\bar{D}$ and holomorphic in the interior.

Then $g = f \circ \varphi^{-1}$ is continuous on $\overline{D_+}$, and let us try to compute the L^p-norm of $g_{|\mathbb{R}}$. By (2.9), with the change of variable $x = -\cot \frac{\theta}{2}$ we have

$$\begin{aligned}
\|g_{|\mathbb{R}}\|_p &= \left(\int_{-\infty}^{\infty} |f \circ \varphi^{-1}(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p \, \frac{d\theta}{\sin^2 \frac{\theta}{2}} \right)^{\frac{1}{p}} ,
\end{aligned}$$

and this differs from the L^p-norm of g by the factor $\sin^{-2} \frac{\theta}{2}$ coming from the change of variable. This suggest that we must multiply g by a power of $(\varphi^{-1})'(z) = \frac{-2i}{(z+i)^2}$. Precisely, for $p < \infty$ and $f \in H^p(D)$, we set

$$T_p f(z) = \frac{1}{\pi^{\frac{1}{p}} (z+i)^{\frac{2}{p}}} f \circ \varphi^{-1}(z) \ . \tag{5.2}$$

Proposition 5.1. *If $1 < p \leq \infty$, T_p is an isometry of $H^p(D)$ onto $H^p(D_+)$. If $p = 1$, T_1 is an isometry of $H^1(D)$ onto the subspace $H^1_0(D_+)$ of those $f \in H^1(D_+)$ such that $f^\sharp \in L^1(\mathbb{R})$.*

At the end of this section we shall prove that in fact $H^1_0(D_+) = H^1(D_+)$, as a consequence of the F. and M. Riesz theorem for $\mathbb{R}$.

Proof. The case $p = \infty$ having been discussed already, we take $p < \infty$. Assume first that f is continuous on $\bar{D}$, bounded and holomorphic in the interior, and let $g = T_p f$. Then g is continuous on $\overline{D_+}$ and

$$M_p(g, y)^p \leq \frac{1}{\pi} \|f\|_\infty^p \int_{\mathbb{R}} \frac{1}{x^2 + (y+1)^2} \, dx \leq \|f\|_\infty^p \ ,$$

so that $g \in H^p(D_+)$. Let $g^\sharp$ be its boundary function (or measure) on $\mathbb{R}$, in the sense of Theorem 2.6. If $1 < p < \infty$, $g_y \to g^\sharp$ in the L^p-norm as $y \to 0$, there is a sequence $y_n \to 0$ such that $g_{y_n} \to g^\sharp$ almost everywhere. Therefore $g^\sharp = g_{|\mathbb{R}}$.

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[27] This is true also for bounded harmonic functions, but what we shall do next for H^p-functions with $p < \infty$ has no analogue for harmonic functions.

If $p = 1$ we get to the same conclusion by a different argument. Since $g_y \to g^\sharp$ in the weak* topology, for any $\varphi \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(x)\, dg^\sharp(x) = \lim_{y\to 0} \int_{\mathbb{R}} \varphi(x) g(x+iy)\, dx = \int_{\mathbb{R}} \varphi(x) g(x)\, dx \ .$$

Hence $dg^\sharp(x) = g(x)\, dx$.

We now have

$$\|g_{|\mathbb{R}}\|_p = \left(\frac{1}{\pi} \int_{-\infty}^{\infty} |f \circ \varphi^{-1}(x)|^p \, \frac{dx}{|x+i|^2} \right)^{\frac{1}{p}} \ .$$

If $x = \varphi(e^{i\theta})$, then

$$x + i = i \frac{1+e^{i\theta}}{1-e^{i\theta}} + i = \frac{2i}{1-e^{i\theta}} \ ,$$

so that

$$\frac{1}{|x+i|^2} = \sin^2 \frac{\theta}{2} \ .$$

Therefore the change of variable $x = -\cot \frac{\theta}{2}$ gives $\frac{dx}{|x+i|^2} = \frac{1}{2} d\theta$ and then

$$\|g_{|\mathbb{R}}\|_p = \|f_{|\mathbb{T}}\|_p \ .$$

We have so proved that T_p is an isometry from a dense subspace of $H^p(D)$ (see Corollaries 4.3 and 7.3 in Chapter I) onto its image, call it V, in $H^p(D_+)$. Therefore T_p extends to an isometry (also denoted by T_p) of all $H^p(D)$ onto the closure of V in $H^p(D_+)$. The proof will be finished if we prove that T_p has a dense image in $H^p(D_+)$ if $p > 1$ or in $H^1_0(D_+)$ if $p = 1$.

Take $g \in H^p(D_+)$, resp. in $H^1_0(D_+)$, $\varepsilon > 0$, and define

$$h_\varepsilon(z) = \frac{g(z+i\varepsilon)}{(1-i\varepsilon z)^2} \ .$$

Observe that for $z = x+iy \in \overline{D_+}$, $|1-i\varepsilon z| \geq 1+\varepsilon y \geq 1$, so that the denominator does not vanish, and moreover

$$|h_\varepsilon(z)| \leq |g(z+i\varepsilon)| \ .$$

This implies that $h_\varepsilon \in H^p(D_+)$. We prove that $\lim_{\varepsilon\to 0} h_\varepsilon = g$ in $H^p(D_+)$. In fact

$$\begin{aligned}
\|h_\varepsilon - g\|_{H^p} &= \left(\int_{\mathbb{R}} \left| \frac{g_\varepsilon(x)}{(1-i\varepsilon x)^2} - g^\sharp(x) \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}} \frac{|g_\varepsilon(x) - g^\sharp(x)|^p}{|1-i\varepsilon x|^{2p}} \, dx \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_{\mathbb{R}} |g^\sharp(x)|^p \left| \frac{1}{(1-i\varepsilon x)^2} - 1 \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}} |g_\varepsilon(x) - g^\sharp(x)|^p \, dx \right)^{\frac{1}{p}} \\
&\qquad + \left(\int_{\mathbb{R}} |g^\sharp(x)|^p \left| \frac{1}{(1-i\varepsilon x)^2} - 1 \right|^p dx \right)^{\frac{1}{p}} \ .
\end{aligned}$$

As $\varepsilon \to 0$, the first term tends to 0 by Theorem 2.6, and the second by dominated convergence, since $\left|\frac{1}{(1-i\varepsilon x)^2} - 1\right| \le 2$.

We show that h_ε is the image under T_p of a function in $H^p(D)$. We compute therefore

$$\begin{aligned}
T_p^{-1} h_\varepsilon(z) &= \pi^{\frac{1}{p}} \big(\varphi(z) + i\big)^{\frac{2}{p}} h_\varepsilon \circ \varphi(z) \\
&= \pi^{\frac{1}{p}} \Big(\frac{2i}{1-z}\Big)^{\frac{2}{p}} \frac{1}{\big(1 - i\varepsilon\varphi(z)\big)^2} g\big(\varphi(z) + i\varepsilon\big) \\
&= \pi^{\frac{1}{p}} \Big(\frac{2i}{1-z}\Big)^{\frac{2}{p}} \frac{(1-z)^2}{\big(1 + \varepsilon - (1-\varepsilon)z\big)^2} g\big(\varphi(z) + i\varepsilon\big) \\
&= \pi^{\frac{1}{p}} (2i)^{\frac{2}{p}} \frac{(1-z)^{\frac{2}{p'}}}{\big(1 + \varepsilon - (1-\varepsilon)z\big)^2} g\big(\varphi(z) + i\varepsilon\big) \ .
\end{aligned}$$

The factor $g\big(\varphi(z) + i\varepsilon\big)$ is bounded because $\Im\mathrm{m}\big(\varphi(z) + i\varepsilon\big) > \varepsilon$, and the denominator does not vanish on $\bar{D}$. therefore $T_p^{-1} h_\varepsilon$ is bounded, hence in $H^p(D)$. □

This fact will allow us to transfer the factorization results proved in the unit disc to D_+.

Lemma 5.2. *Let $1 \le p \le \infty$. If $f \in H^p(D_+)$ is not identically zero and $\{\alpha_j\}_{j\in\mathbb{N}}$ is the sequence of its zeroes (repeated according to their multiplicities), then*

$$\sum_{j=0}^{\infty} \frac{\Im\mathrm{m}\alpha_j}{|\alpha_j|^2 + 1} < \infty \ . \tag{5.3}$$

Proof. For $p > 1$, we take $g = T_p^{-1} f \in H^p(D)$, and apply Lemma 7.5 in Chapter I. The zeroes of g are the points

$$\beta_j = \varphi^{-1}(\alpha_j) = \frac{\alpha_j - i}{\alpha_j + i} \ .$$

Then

$$\sum_{j=0}^{\infty} (1 - |\beta_j|^2) \asymp \sum_{j=0}^{\infty} (1 - |\beta_j|) < \infty \ ,$$

and, if $\alpha_j = a_j + ib_j$,

$$\begin{aligned}
1 - |\beta_j|^2 &= \frac{|\alpha_j + i|^2 - |\alpha_j - i|^2}{|\alpha_j + i|^2} \\
&= \frac{4\Im\mathrm{m}\alpha_j}{|\alpha_j + i|^2} \\
&= \frac{4b_j}{a_j^2 + (b_j + 1)^2} \\
&\asymp \frac{b_j}{a_j^2 + b_j^2 + 1} \\
&= \frac{\Im\mathrm{m}\alpha_j}{|\alpha_j|^2 + 1} \ .
\end{aligned}$$

Take now $f \in H^1(D_+)$. Replacing, if necessary, $f(z)$ by $f\big((1+\delta)z\big)$, we can assume that $f(i) \neq 0$. Notice that, in doing so, the zeroes change from α_j to $(1+\delta)^{-1}\alpha_j$, but this change does not affect the convergence of the series (5.3).

For $\varepsilon > 0$, let $f^\varepsilon(z) = f(z + i\varepsilon)$. Then $f^\varepsilon \in H^1(D_+)$, $\|f^\varepsilon\|_{H^1} \le \|f\|_{H^1}$, and $(f^\varepsilon)^\sharp(x) = f(x + i\varepsilon) = f_\varepsilon(x) \in L^1(\mathbb{R})$. Therefore, $T_1^{-1} f^\varepsilon \in H^1(D)$. If ε is small enough, say $\varepsilon < \varepsilon_0$, $T_1^{-1} f^\varepsilon(0) = -4\pi f\big((1+\varepsilon)i\big) \neq 0$. We then set

$$g^\varepsilon = \frac{1}{T_1^{-1} f^\varepsilon(0)} T_1^{-1} f^\varepsilon \ .$$

If $E_\varepsilon = \{j : \Im\mathrm{m}\alpha_j > \varepsilon\}$, the zeroes of f^ε in D_+ are the $\alpha_j^\varepsilon = \alpha_j - i\varepsilon$ with $j \in E_\varepsilon$, and the zeroes of g^ε in D are $\beta_j^\varepsilon = \varphi^{-1}(\alpha_j^\varepsilon)$, $j \in E_\varepsilon$.

Since $g^\varepsilon(0) = 1$, we can apply Lemma 7.5 in Chapter I to obtain that

$$\begin{aligned} \sum_{j \in E_\varepsilon} \big(1 - |\beta_j^\varepsilon|\big) &\le \sup_{r<1} \int_{\mathbb{T}} \log |g^\varepsilon(re^{it})| \, dt \\ &\le \log \|g^\varepsilon\|_{H^1(D)} \\ &= \log \|f^\varepsilon\|_{H^1(D_+)} - \log \big|4\pi f\big((1+\varepsilon)i\big)\big| \end{aligned}$$

Setting $\alpha_j = a_j + ib_j$ and taking $\varepsilon < \varepsilon_0$, we have

$$1 - |\beta_j^\varepsilon| \asymp \frac{\Im\mathrm{m}\alpha_j^\varepsilon}{|\alpha_j^\varepsilon + i|^2} = \frac{b_j - \varepsilon}{a_j^2 + (b_j - \varepsilon + 1)^2} \asymp \frac{b_j - \varepsilon}{a_j^2 + b_j^2 + 1} \ .$$

Therefore,

$$\sum_{j \in E_\varepsilon} \frac{b_j - \varepsilon}{a_j^2 + b_j^2 + 1} \le C\Big(\log \|f^\varepsilon\|_{H^1(D_+)} - \log \big|4\pi f\big((1+\varepsilon)i\big)\big| \Big) \ .$$

If ε decreases, the left-hand side increases for each $j \in \mathbb{N}$ (taking into account that the sets E_ε also increase). Passing to the limit by monotone convergence,

$$\sum_{j \in \mathbb{N}} \frac{b_j}{a_j^2 + b_j^2 + 1} \le C\big(\log \|f\|_{H^1(D_+)} - \log |4\pi f(i)| \big) < \infty \ . \qquad \square$$

Blaschke products on D_+ are defined as compositions $B \circ \varphi^{-1} = T_\infty B$ of Blaschke products on D with the inverse Cayley transform. If $w = \varphi^{-1}(z)$ and $\beta = \varphi^{-1}(\alpha)$, then, according to (7.4) in Chapter I, we define

$$\sigma_\alpha(z) = \tilde{\psi}_\beta(w) = \frac{\bar\beta}{|\beta|} \frac{w - \beta}{\bar\beta w - 1} = \frac{|\alpha^2 + 1|}{\alpha^2 + 1} \frac{z - \alpha}{z - \bar\alpha}$$

if $\alpha \neq i$, and

$$\sigma_i(z) = \varphi^{-1}(z) = \frac{z - i}{z + i} \ .$$

Once we are at this stage, we can simply state the analogues of the results of Section 7 in Chapter I. They can be transfered directly from D via the maps T_p for $p > 1$, or reproved in exactly the same way for $p = 1$.

Proposition 5.3. *Let* $\{\alpha_j\}$ *be a sequence of points in* D_+ *satisfying* (5.3). *The Blaschke product*

$$B(z) = \prod_{j=0}^{\infty} \sigma_{\alpha_j}(z)$$

converges unconditionally and uniformly on compact subsets of D_+ *to a function* $B \in H^\infty(D_+)$ *vanishing in* D_+ *only at the points* α_j. *Moreover,* $|B^\sharp(x)| = 1$ *for almost every* $x \in \mathbb{R}$.

Theorem 5.4. *Let* $f \in H^p(D_+)$, $1 \le p \le \infty$, *be not identically zero, and let* B *be its Blaschke product. Then* $g = \frac{f}{B} \in H^p(D_+)$ *and* $\|g\|_{H^p} = \|f\|_{H^p}$.

Corollary 5.5. *Let* $f \in H^s(D_+)$, *not identically zero, with* $1 \le s \le \infty$, *and let* $p, q \in [1, \infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. *Then there exist* $g \in H^p(D_+)$ *and* $h \in H^q(D_+)$ *such that* $f = gh$, *and* $\|g\|_{H^p}^p = \|h\|_{H^q}^q = \|f\|_{H^s}^s$.

Finally, we can state the F. and M. Riesz theorem for the line.

Theorem 5.6 (F. and M. Riesz). *Let* $\mu \in M(\mathbb{R})$ *be a measure such that* $\hat{\mu}(\xi) = 0$ *for* $\xi \le 0$. *Then* μ *is absolutely continuous w.r. to Lebesgue measure.*

The proof follows from the factorization theorem 5.5 like in the disc.

Corollary 5.7. *If* $f \in H^1(D_+)$, *then* $f^\sharp \in L^1(\mathbb{R})$, *and therefore* $\lim_{y\to 0} f_y = f^\sharp$ *in the* L^1*-norm. The map* T_1 *of Proposition 5.1 is an isometry of* $H^1(D)$ *onto* $H^1(D_+)$.

CHAPTER III
POINTWISE CONVERGENCE TO THE BOUNDARY AND CONJUGATE HARMONIC FUNCTIONS IN h^p

1. The Hardy-Littlewood maximal function

We attack now a question that has been in the background so far, i.e. the pointwise behaviour of h^p- or H^p-functions. In particular, in the cases where $f^\sharp$ is a function, we wnt to know (e.g. in the disc, to fix the notation) if

$$\lim_{r\to 1} f(re^{it}) = f^\sharp(e^{it})$$

for almost every $e^{it} \in \mathbb{T}$. If $f \in H^\infty(D)$ and $f^\sharp$ is continuous, the answer is positive, since we know that the functions f_r tend to $f^\sharp$ uniformly. Partial answers can be given easily also in other cases. If $f \in H^p(D)$ with $1 < p < \infty$, then $f_r \to f^\sharp$ in norm, and therefore there is a sequence $r_n \to 1$ such that $f_{r_n} \to f$ almost everywhere. But this is much less than what we are looking for.

Sharp answers to our question follow from considerations about *maximal functions*. This notion makes sense and is useful in many different situations, and it is worth therefore to discuss their properties in a general context.

Let X be a topological space. A *quasi-distance* on X is a function d from $X \times X$ to $\mathbb{R}$ such that

(1) $d(x,y) \geq 0$ for all $x, y \in X$;
(2) $d(x,y) = 0$ if and only if $x = y$;
(3) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(4) there is a constant $c \geq 1$ such that

$$d(x,z) \leq c\big(d(x,y) + d(y,z)\big) \tag{1.1}$$

for all $x, y, z \in X$;
(5) the balls $B(x,r) = \{y : d(x,y) < r\}$ form a fundamental neighborhood system at each $x \in X$.

Let m be a positive regular Borel measure on X. One says that m is *doubling* (more precisely, d-doubling) if there is a constant c' such that

$$m\big(B(x,2r)\big) \leq c' m\big(B(x,r)\big) \tag{1.2}$$

for all $x \in X$ and $r > 0$.

Definition. *A triple* (X, d, m)*, where* d *is a quasi-distance on* X *and* m *is a* d*-doubling measure, is called a* space of homogeneous type[28].

Examples.

The following are spaces of homogeneous type[29]:

(1) The unit circle $\mathbb{T}$, with the distance $d(e^{it}, e^{it'}) = |e^{it} - e^{it'}|$ (or equivalently with the quotient distance of $\mathbb{R}/2\pi\mathbb{Z}$) and the normalized Lebesgue measure.
(2) $\mathbb{R}^n$, with Euclidean distance and the Lebesgue measure.
(3) $\mathbb{R}$, with the Euclidean distance and the measure $dm(x) = |x|^\alpha dx$, with $\alpha > -n$.
(4) $\mathbb{R}^n$, with the Lebesgue measure and the distance

$$d(x, y) = \max\left\{|x_1 - y_1|^{1/d_1}, \dots, |x_n - y_n|^{1/d_n}\right\} ,$$

where $d_1, \dots, d_n > 0$.
(5) $\mathbb{Z}$, with the distance $d(n, m) = |n - m|$ and the counting measure $m(E) = \mathrm{card}E$.
(6) The unit sphere $S^{n-1} \subset \mathbb{R}^n$, with the induced distance from $\mathbb{R}^n$ and the Hausdorff measure.
(7) The unit sphere $S^{2n-1} \subset \mathbb{C}^n$, with the Hausdorff measure and the distance $d(\zeta, \zeta') = |1 - \langle\zeta, \zeta'\rangle|$, where $\langle\zeta, \zeta'\rangle$ is the Hermitean inner product in $\mathbb{C}^n$.

Let X be a space of homogeneous type.

Definition. *Let* f *be locally integrable w.r. to* m*. The function*

$$Mf(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|\, dy , \tag{1.3}$$

is called the Hardy-Littlewood maximal function *of* f *and the operator* $M : f \longmapsto Mf$ *is called the* Hardy-Littlewood maximal operator.

Clearly M is not linear (observe that $Mf(x) \geq 0$ for every f), but only *sub-linear*, in the sense that

$$M(f + g) \leq Mf + Mg , \qquad M(\lambda f) = |\lambda| Mf . \tag{1.4}$$

Lemma 1.1. *The function* Mf *is lower-semicontinuous, hence measurable.*

Proof. Let $M(x_0) > \alpha$. Then there is a ball B containing x such that

$$\frac{1}{m(B)} \int_B |f(y)|\, dy > \alpha .$$

Then $Mf(x) > \alpha$ for every $x \in B$. □

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[28]Sometimes the definition of space of homogeneous type is given by requiring that X be just a set, and d a function satisfying conditions (1)-(4) only. A topology is then introduced on X by stating that a set A is open if for every $x \in A$ there is a ball $B(x, r) \subset A$. However, this does not guarantee that the balls are open, not even that they are Borel sets. Therefore, one must impose that the doubling measure m be defined on a σ-algebra containing both the open sets and the balls.

[29]Some of the following statements would require a proof, that we omit.

Remark. The classical definition of Hardy-Littlewood maximal function (for $X = \mathbb{R}^n$, with Euclidean distance and Lebesgue measure) is the following:

$$M'f(x) = \sup_{r>0} \frac{1}{m\big(B(x,r)\big)} \int_{B(x,r)} |f(y)|\, dy \ , \tag{1.5}$$

i.e. limited to the averages of $|f|$ over balls centered at x. Clearly $M'f(x) \le Mf(x)$; however $M'f$ is not necessarily lower-semicontinuous. The measurability of $M'f$ follows in this case from the fact that the map

$$F(x,r) = \frac{1}{m\big(B(x,r)\big)} \int_{B(x,r)} |f(y)|\, dy$$

is continuous in r, so that the sup in (1.5) can be limited to $r \in \mathbb{Q}$.

In a general space of homogeneous type, $F(x,r)$ need not be continuous in r, hence the definition (1.3) is preferable.

We want to discuss boundedness of M on the spaces $L^p(X)$, i.e. inequalities of the form

$$\|Mf\|_p \le C\|f\|_p \ .$$

Notice that it follows from (1.4) that $|Mf - Mg| \le M(f-g)$, hence boundedness on L^p is equivalent to continuity, as for linear operators.

Obviously M is bounded on $L^\infty(X)$. At the other extreme, $p = 1$, M is not bounded in general. For instance, in the classical siuation ($X = \mathbb{R}^n$, etc.), taking f the characteristic function of the unit ball, one has

$$Mf(x) \ge \frac{C}{1 + |x|^n} \ ,$$

so that $Mf \notin L^1$.

Nevertheless, the starting point of our proof is that, for $p = 1$, M satisfies a weaker form of boundedness. We first recall some properties of the *distribution function*, defined for $\alpha > 0$,

$$\delta_f(\alpha) = m\big(\{x : |f(x)| > \alpha\}\big) \tag{1.6}$$

of an m-measurable function f on X.

Lemma 1.2. *Let (X,m) be a measure space. If $f \in L^p(X,m)$, $1 \le p < \infty$, then the following hold:*

(1) *the* Chebishev inequality

$$\delta_f(\alpha) \le \Big(\frac{\|f\|_p}{\alpha}\Big)^p \ ;$$

(2) *the identity*

$$\|f\|_p^p = p \int_0^\infty \delta_f(\alpha)\alpha^{p-1}\, d\alpha \ .$$

Proof. Let $E_\alpha = \{x : f(x) > \alpha\}$. Then

$$m(E_\alpha) \le \int_{E_\alpha} \frac{|f(x)|^p}{\alpha^p}\, dm(x) \le \Big(\frac{\|f\|_p}{\alpha}\Big)^p .$$

This gives (1). Suppose now that $f \in L^p(X,m)$ assumes a finite number of values. Then $|f| = \sum_{j=1}^n c_j \chi_{A_j}$, with $0 < c_1 < \cdots < c_n$ and the A_j pairwise disjoint. Then

$$m(E_\alpha) = \sum_{j: c_j > \alpha} m(A_j) ,$$

and, setting $c_0 = 0$,

$$\begin{aligned} p\int_0^\infty m(E_\alpha)\alpha^{p-1}\, d\alpha &= p\sum_{j=1}^n \int_{c_{j-1}}^{c_j} \alpha^{p-1} \sum_{k=j}^n m(A_k)\, d\alpha \\ &= \sum_{j=1}^n (c_j^p - c_{j-1}^p) \sum_{k=j}^n m(A_k) \\ &= \sum_{j=1}^n c_j^p m(A_k) \\ &= \|f\|_p^p . \end{aligned}$$

For general f, one approximates $|f|$ from below by finite-valued functions. □

Definition. *Let T be a linear, or sub-linear, operator defined on $L^p(X)$, $p < \infty$, and taking values in the space of measurable functions on X. One says that T is* weak-type (p,p) *if, for every $\alpha > 0$*

$$\delta_{Tf}(\alpha) \le C\Big(\frac{\|f\|_p}{\alpha}\Big)^p . \tag{1.7}$$

The expression "weak-type" comes from the fact that (1.7) is a weaker condition than boundedness on L^p, as a consequence of Chebishev's inequality. In fact, if T is bounded on $L^p(X)$,

$$\delta_{Tf}(\alpha) \le \Big(\frac{\|Tf\|_p}{\alpha}\Big)^p \le \|T\|^p_{L^p\to L^p}\Big(\frac{\|f\|_p}{\alpha}\Big)^p .$$

One also says that an operator T is *strong type* (p,p) if it is bounded on $L^p(X)$.

Going back to the maximal operator M, we shall see that it is weak-type $(1,1)$. The proof is based on the following *Vitali covering lemma.*

Lemma 1.3. *Inside a space X of homogeneous type, let $\{B_j\}_{j\in J}$ be a finite family of balls covering a measurable set E. There exists a sub-family $\{B_j\}_{j\in J'}$ such that $B_j \cap B_k = \emptyset$ for $j,k \in J'$, $j \ne k$, and*

$$m\Big(\bigcup_{j\in J'} B_j\Big) \ge \kappa m(E) ,$$

where κ dipends only on the constants c, c' in (1.1) *and* (1.2).

Proof. Start with a ball B_{j_1} of maximum radius. Inductively, take $B_{j_{k+1}}$ with maximum radius among the balls disjoint from $B_{j_1}\cup\cdots\cup B_{j_k}$. This procedure stops after a finite number of steps, precisely when there are no more balls left which are disjoint from $B_{j_1}\cup\cdots\cup B_{j_k}$. Then set $J'=\{j_1,\dots,j_k\}$ and call $B_{j_1},\dots B_{j_k}$ the "selected" balls, and the remaining ones the "excluded" balls.

If B is a ball of radius r, denote by B^* the ball with the same center and radius $3c^2r$, where c is the constant in (1.1). Observe that if two balls B, B' have non-empty intersection and the radius of B' is not larger than the radius of B, then $B'\subseteq B^*$ by (1.1).

Let B' one of the excluded balls. It necessarily intersects one of the selected ones. Let $\bar\ell$ be the smallest integer ℓ such that $B'\cap B_{j_\ell}\neq\emptyset$. Then the radius $B_{j_{\bar\ell}}$ is greater than or equal to the radius of B', so that

$$B'\subseteq B^*_{j_{\bar\ell}}\ .$$

Therefore

$$E\subseteq\bigcup_{j\in J}B_j\subseteq\bigcup_{j\in J'}B^*_j\ ,$$

Take ν so that $2^\nu\geq 2c$. Then

$$m(B^*_j)\leq m\big(B(x,2^kr)\big)\leq {c'}^\nu m(B)\ ,$$

so that, with $\kappa=\frac{1}{{c'}^\nu}$,

$$m(E)\leq\sum_{j\in J'}m(B^*_j)=\kappa^{-1}\sum_{j\in J'}m(B_j)=\kappa^{-1}m\Big(\bigcup_{j\in J'}B_j\Big)\ . \qquad\square$$

Theorem 1.4. *The operator M is weak-type* $(1,1)$.

Proof. Given $f\in L^1(X)$ and $\alpha>0$, let $E_\alpha=\{x: Mf(x)>\alpha\}$. By Lemma 1.1, E_α is open and its measure is the supremum of the measures of its compact subsets.

Let E be a compact subset of E_α. Given $x\in E$, $Mf(x)>\alpha$, so that there is a ball B_x containing x such that

$$\frac{1}{m(B_x)}\int_{B_x}|f(y)|\,dy>\alpha\ ,$$

i.e.

$$m(B_x)\leq\frac{1}{\alpha}\int_{B_x}|f(y)|\,dy\ . \tag{1.8}$$

Since E is compact, we can extract a finite subcovering $\{B_{x_j}\}_{j\in J}$ of E from $\{B_x\}_{x\in E}$. By Lemma 1.2, we can further extract a finite family $\{B_{x_j}\}_{j\in J'}$ of mutually disjoint balls such that

$$\sum_{j\in J'}m(B_{x_j})\geq\kappa m\Big(\bigcup_{j\in J}B_{x_j}\Big)\geq\kappa m(E)\ .$$

Combining this with (1.8), we have

$$m(E) \le \kappa^{-1}\sum_j m(B_{x_j}) \le \frac{\kappa^{-1}}{\alpha}\sum_j \int_{B_{x_j}} |f(y)|\,dy \le \kappa^{-1}\frac{\|f\|_1}{\alpha}\ .$$

Taking the supremum over $E \subset E_\alpha$, we obtain that $m(E_\alpha) \le \kappa^{-1}\|f\|_1/\alpha$. □

Combining together the weak-type (1,1) property of M and its boundedness on L^∞, we can prove that it is bounded on L^p for $1 < p < \infty$. What follows is a special case of the *Marcinkiewicz interpolation theorem*[30].

Theorem 1.5. *Let T be a linear, or sub-linear, operator which is weak-type $(1,1)$ and bounded on $L^\infty(X,m)$. Then T is bounded on $L^p(X,m)$ for $1<p<\infty$.*

Proof. Given $f \in L^p(X,m)$ and $\alpha > 0$, define

$$f_\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| \le \alpha \\ 0 & \text{if } |f(x)| > \alpha\ , \end{cases} \qquad f^\alpha(x) = \begin{cases} f(x) & \text{if } |f(x)| > \alpha \\ 0 & \text{if } |f(x)| \le \alpha\ . \end{cases}$$

Then $f^\alpha \in L^\infty(X,m)$ with $\|f_\alpha\|_\infty \le \alpha$, and $f^\alpha \in L^1(X,m)$. In fact,

$$\begin{aligned}\|f^\alpha\|_1 &= \int_{\{|f(x)|>\alpha\}} |f(x)|\,dm(x)\\ &\le \int_{\{|f(x)|>\alpha\}} |f(x)|\frac{|f(x)|^{p-1}}{\alpha^{p-1}}\,dm(x)\\ &\le \frac{1}{\alpha^{p-1}}\|f\|_p^p\ .\end{aligned}$$

If $C_\infty = \|T\|_{L^\infty\to L^\infty}$, then $\|Tf_\alpha\|_\infty \le C_\infty\alpha$. Since $f = f_\alpha + f^\alpha$,

$$|Tf(x)| \le |Tf_\alpha(x)| + |Tf^\alpha(x)|\ ,$$

so that

$$|Tf(x)| > 2C_\infty\alpha \Longrightarrow |Tf^\alpha(x)| > C_\infty\alpha\ ,$$

in other words,

$$\{x : |Tf(x)| > 2C_\infty\alpha\} \subseteq \{x : |Tf^\alpha(x)| > C_\infty\alpha\}\ .$$

We use now (2) in Lemma 1.2 and the weak-type (1,1) of T to obtain

$$\begin{aligned}\|Tf\|_p^p &= p\int_0^\infty m\big(\{x : |Tf(x)| > \alpha\}\big)\alpha^{p-1}\,d\alpha\\ &= p(2C_\infty)^p\int_0^\infty m\big(\{x : |Tf(x)| > 2C_\infty\alpha\}\big)\alpha^{p-1}\,d\alpha\\ &\le p(2C_\infty)^p\int_0^\infty m\big(\{x : |Tf^\alpha(x)| > C_\infty\alpha\}\big)\alpha^{p-1}\,d\alpha\\ &\le p2^pC_\infty^{p-1}C_1\int_0^\infty \|f^\alpha\|_1\alpha^{p-2}\,d\alpha\\ &= p2^pC_\infty^{p-1}C_1\int_0^\infty \alpha^{p-2}\int_{\{x:|f(x)|>\alpha\}} |f(x)|\,dx\,d\alpha\\ &= p2^pC_\infty^{p-1}C_1\int_X |f(x)|\int_0^{|f(x)|}\alpha^{p-2}\,d\alpha\,dx\\ &= C'\|f\|_p^p\ ,\end{aligned}$$

[30] For its general formulation, see E.M. Stein, G. Weiss, *An introduction to Fourier analysis on Euclidean spaces*

having denoted by C_1 the weak-type (1,1) constant for T. $\square$

Corollary 1.6. *If $1 < p \leq \infty$, M is bounded on $L^p(X)$.*

2. Poisson integrals and maximal function

The reason for introducing the Hardy-Littlewood maximal function is that it controls other quantities that intervene in the analysis of the boundary behaviour of h^p- (or H^p-) functions. We begin with the upper half-plane D_+, where the geometric picture is more clear.

For every point $x_0 \in \mathbb{R} = \partial D_+$, we define different "approach regions" from the interior of D_+, and for each of them we introduce a maximal operator. The most natural approach region is the vertical line $x = x_0$. Correspondingly, for a function f in some $L^p(\mathbb{R})$, we call *vertical maximal function* of f the function

$$M_{\mathrm{vert}} f(x) = \sup_{y>0} \left|\mathcal{P}f(x+iy)\right| , \tag{2.1}$$

where $\mathcal{P}$ denotes the operator assigning to f its Poisson integral.

Other approach regions are the non-tangential angles. Given a point $x_0 \in \mathbb{R}$, consider the open infinite angle $\Gamma_\alpha(x_0)$ inside D_+ with vertex in x_0, symmetric w.r. to the vertical line $x = x_0$, and with semi-aperture α, $0 < \alpha < \frac{\pi}{2}$. Explicitely,

$$\Gamma_\alpha(x_0) = \{x + iy : |x - x_0| < y \tan\alpha\} .$$

For a fixed $\alpha \in (0, \pi/2)$, we define the *non-tangential maximal function* of $f \in L^p(\mathbb{R})$ as

$$M_{\mathrm{nt},\alpha} f(x) = \sup_{z\in\Gamma_\alpha(x)} |\mathcal{P}f(z)| .$$

Obviously,

$$M_{\mathrm{vert}} f(x) \leq M_{\mathrm{nt},\alpha} f(x) \leq M_{\mathrm{nt},\alpha'} f(x) ,$$

if $\alpha < \alpha'$.

Lemma 2.1. *For every $\alpha \in (0, \pi/2)$ there is a constant $C_\alpha > 0$ such that, if $f \in L^p(\mathbb{R})$,*

$$M_{\mathrm{nt},\alpha} f(x) \leq C_\alpha M f(x) .$$

In particular, each $M_{\mathrm{nt},\alpha}$ and M_{vert} are weak-type $(1,1)$ and bounded on L^p for $1 < p < \infty$.

Proof. We can assume that $\alpha > \pi/4$, so that $a = \tan\alpha > 1$. Consider the dyadic intervals $I_j^a = [-a2^j, a2^j]$ in the real line, with $j \geq 0$. We set $E_j^a = I_j^a \setminus I_{j-1}^a$ for

$j > 0$, and $E_0^a = I_0^a$. Then

$$
\begin{aligned}
P_1(x) &= \frac{1}{\pi}\frac{1}{1+x^2} \\
&\le \frac{1}{\pi}\chi_{E_0^a}(x) + \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{1}{1+a^2 2^{2(j-1)}}\chi_{E_j^a}(x) \\
&\le \frac{1}{\pi}\chi_{E_0^a}(x) + \frac{4}{\pi a^2}\sum_{j=1}^{\infty}2^{-2j}\chi_{E_j^a}(x) \\
&\le \frac{4}{\pi}\chi_{I_0^a}(x) + \frac{4}{\pi}\sum_{j=1}^{\infty}2^{-2j}\big(\chi_{I_j^a}(x) - \chi_{I_{j-1}^a}(x)\big) \qquad (2.2)\\
&= \frac{4}{\pi}\sum_{j=0}^{\infty}(2^{-2j} - 2^{-2(j+1)})\chi_{I_j^a}(x) \\
&= C\sum_{j=0}^{\infty}2^{-2j}\chi_{I_j^a}(x)\ .
\end{aligned}
$$

For a general $y > 0$, (2.2) and the identity

$$P_y(x) = \frac{1}{y}P_1\Big(\frac{x}{y}\Big)\ ,$$

give

$$P_y(x) \le C\sum_{j=0}^{\infty}\frac{2^{-2j}}{y}\chi_{I_j^a}\Big(\frac{x}{y}\Big) = C\sum_{j=0}^{\infty}\frac{2^{-2j}}{y}\chi_{I_j^{ay}}(x)\ . \qquad (2.3)$$

It follows that

$$
\begin{aligned}
\big|f * P_y(x)\big| &\le \int_{\mathbb{R}}|f(x-t)|P_y(t)\,dt \\
&\le C\sum_{j=0}^{\infty}\frac{2^{-2j}}{y}\int_{I_j^{ay}}|f(x-t)|\,dt \qquad (2.4)\\
&= Ca\sum_{j=0}^{\infty}\frac{2^{-2j}}{y}\int_{x-ay2^j}^{x+ay2^j}|f(t)|\,dt\ .
\end{aligned}
$$

If $x+iy \in \Gamma_\alpha(x_0)$, then $x_0 \in [x-ay2^j, x+ay2^j]$, so that

$$\frac{1}{2ay2^j}\int_{[x-ay2^j,x+ay2^j]}|f(t)|\,dt \le Mf(x_0)\ .$$

Therefore

$$\big|f * P_y(x)\big| \le Ca^2\sum_{j=0}^{\infty}2^{-j}Mf(x_0)\ ,$$

and

$$M_{\mathrm{nt},\alpha}f(x_0) \le Ca^2 Mf(x_0)\ . \qquad \square$$

Theorem 2.2. *Suppose that u is in $h^p(D_+)$, $1 < p \le \infty$, or in $H^p(D_+)$, $1 \le p \le \infty$. Then the functions*

$$u^*_\alpha(x) \stackrel{\text{def}}{=} \sup_{z\in\Gamma_\alpha(x)} \big|u(z)\big| \ ,$$

are in $L^p(\mathbb{R})$ and $\|u^*_\alpha\|_p \le C_{p,\alpha}\|u\|_{h^p}$. The same is true for

$$u^*(x) \stackrel{\text{def}}{=} \sup_{y>0} \big|u(x+iy)\big| \ .$$

Proof. For $p > 1$, $u^*_\alpha \in L^p(\mathbb{R})$ because $u^\sharp \in L^p(\mathbb{R})$ and by Lemma 2.1.

It remains to discuss the case $u \in H^1(D_+)$. By Corollary 5.5 in Chapter II, we can factorize u as $u = vw$, with $v, w \in H^2(D_+)$ and $\|v\|_{H^2} = \|w\|_{H^2} = \|u\|^{\frac{1}{2}}_{H^1}$. Then

$$u^*_\alpha(x) = \sup_{z\in\Gamma_\alpha(x)} \big|v(z)\big|\big|w(z)\big| \le v^*_\alpha(x) w^*_\alpha(x) \ .$$

Therefore,

$$\|u^*_\alpha\|_1 \le \|v^*_\alpha\|_2 \|w^*_\alpha\|_2 \le C^2_{2,\alpha}\|v\|_{H^2}\|w\|_{H^2} = C^2_{2,\alpha}\|u\|_{H^1} \ . \qquad \square$$

We pass now to the unit disc. The substitute for the vertical maximal function is the *radial maximal function*,

$$M_{\text{rad}} f(e^{it}) = \sup_{r<1} \big|\mathcal{P}f(re^{it})\big| \ .$$

defined for $f \in L^1(\mathbb{T})$.

As non-tangential access regions, we take the *Stolz regions* $S_\rho(e^{it})$, with $\rho \in (0,1)$, defines as the open convex envelop of the point e^{it} and the disc with center 0 and radius ρ. Near e^{it}, the Stolz region $S_\rho(e^{it})$ is the angle pointing towards the interior of the disc, symmetric w.r. to the radius, and of semi-aperture $\alpha = \arcsin \rho$.

We then define the *non-tangential maximal function* of $f \in L^1(\mathbb{T})$ as

$$M_{\text{nt},\rho} f(e^{it}) = \sup_{z\in S_\rho(e^{it})} |\mathcal{P}f(z)| \ .$$

Lemma 2.3. *For every $\rho \in (0,1)$ there is a constant $C_\rho > 0$ such that, if $f \in L^1(\mathbb{T})$,*

$$M_{\text{nt},\rho} f(e^{it}) \le C_\rho M f(e^{it}) \ .$$

In particular, each $M_{\text{nt},\rho}$ and M_{rad} are weak-type $(1,1)$ and bounded on L^p for $1 < p < \infty$.

Proof. We use (3.6) in Chapter I,

$$P_r(e^{it}) \le C \frac{1-r}{t^2 + (1-r)^2}$$

with $t \in [-\pi, \pi]$, and proceed as in the proof of Lemma 1.1, just replacing y by $1-r$, and stopping the sums in j as soon as the intervals $I_j^{a(1-r)}$ do not intersect $[-\pi, \pi]$. $\square$

Theorem 2.4. *Suppose that u is in $h^p(D)$, $1 < p \le \infty$, or in $H^p(D)$, $1 \le p \le \infty$. Then the functions*

$$u^*_\rho(e^{it}) \stackrel{\text{def}}{=} \sup_{z\in S_\rho(e^{it})} \left|u(z)\right| ,$$

are in $L^p(\mathbb{T})$ and $\|u^*_\alpha\|_p \le C_{p,\rho}\|u\|_{h^p}$. The same is true for

$$u^*(x) \stackrel{\text{def}}{=} \sup_{r<1} \left|u(re^{it})\right| .$$

3. Pointwise convergence to the boundary

In this section we discuss pointwise convergence to the boundary of harmonic and holomorphic functions on D or on D_+. In this case we will begin with the unit disc, where the situation is simplified by the inclusion of all L^p-spaces into L^1.

Our initial problem is the *radial convergence* of a function u to the boundary, i.e. the existence of

$$\lim_{r\to 1} u(re^{it}) = u^\sharp(e^{it}) ,$$

almost everywhere. The results of the previous section, however, induce us to consider the stronger problem of *non-tangential convergence*, i.e. the existence of

$$\lim_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} u(z) = u^\sharp(e^{it}) ,$$

for every $\rho < 1$.

Theorem 3.1. *If $u = \mathcal{P}f$, with $f \in L^1(\mathbb{T})$, then u converges to f non-tangentially a.e.*

Proof. We can assume that f is real-valued. Consider then the quantity

$$\delta u(e^{it}) = \limsup_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} u(z) - \liminf_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} u(z) ,$$

which is non-negative. Obviously,

$$\delta u(e^{it}) \le 2M_{\mathrm{nt},\rho} f(e^{it}) .$$

Therefore, using Lemma 2.3 and denoting by $|E|$ the normalized Lebesgue measure of $E \subset \mathbb{T}$, we have, for $\alpha > 0$,

$$\left|\{e^{it} : \delta u(e^{it}) > \alpha\}\right| \le C_\rho \frac{\|f\|_1}{\alpha} . \tag{3.1}$$

Given $\varepsilon > 0$, there is $g \in C(\mathbb{T})$ such that $\|f - g\|_1 < \varepsilon$. By Theorem 4.3 in Chapter I, $v = \mathcal{P}g$ is continuous on $\bar{D}$, so that

$$\begin{aligned}\limsup_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} (u - v)(z) &= \limsup_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} u(z) - \lim_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} v(z) \\ &= \limsup_{z\to e^{it}\,,\, z\in S_\rho(e^{it})} u(z) - g(e^{it}) .\end{aligned}$$

In the same way,

$$\liminf_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}(u-v)(z)=\liminf_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)-g(e^{it})\ ,$$

and therefore

$$\delta(u-v)(e^{it})=\delta u(e^{it})\ .$$

Applying (3.1) to $u-v$, we obtain that

$$\big|\{e^{it}:\delta u(e^{it})>\alpha\}\big|\le C_\rho\frac{\varepsilon}{\alpha}\ , \tag{3.2}$$

and this holds for every $\varepsilon>0$. Hence

$$\big|\{e^{it}:\delta u(e^{it})>\alpha\}\big|=0\ ,$$

for every $\alpha>0$.

Then the set

$$\bigcup_{n\ge1}\{e^{it}:\delta u(e^{it})>\frac{1}{n}\}=\{e^{it}:\delta u(e^{it})>0\}$$

has measure zero, i.e.

$$\limsup_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)=\liminf_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)\quad\Big(=\lim_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)\Big)$$

almost everywhere. On the other hand, $\lim_{r\to1}\|u_r-f\|_1=0$, so that there is a subsequence $r_j\to1$ such that $u(r_je^{it})\to f(e^{it})$ a.e. We conclude that

$$\lim_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)=f(e^{it})$$

almost everywhere. □

Corollary 3.2. *If $\rho<1$ and $u\in h^p(D)$, $1<p\le\infty$, or in $H^p(D)$, $1\le p\le\infty$, then for almost every $e^{it}\in\mathbb{T}$, $\lim_{z\to e^{it}\,,\,z\in S_\rho(e^{it})}u(z)=u^\sharp(e^{it})$ for every $\rho<1$.*

In the proof of Theorem 3.1 we have used in a crucial way the density of continuous functions in L^1, in order to have (3.2). The same proof can be adapted to prove the analogous result in the upper half-plane. In this case, we approximate $f\in L^1(\mathbb{R})$ by functions $g\in C_0(\mathbb{R})$. This also works for $f\in L^p(\mathbb{R})$ as long as $p<\infty$, but it breaks down for $p=\infty$. For this case we use a different argument.

Theorem 3.3. *If $u=\mathcal{P}f$, with $f\in L^p(\mathbb{R})$ and $1\le p\le\infty$, then u converges to f non-tangentially a.e.*

Proof. If $f\in L^1(\mathbb{R})$, we proceed as described above. Take now $f\in L^\infty(\mathbb{R})$. Then $u\in h^\infty(D_+)$. If φ is the Cayley transform in (2.7) of Chapter II, then $v=u\circ\varphi\in h^\infty(D)$. We know by Corollary 3.2 that v converges to $v^\sharp(e^{it})$ non-tangentially a.e.

Fix now $x\in\mathbb{R}$ and $\alpha>0$, let $\Gamma'_\alpha(x)=\Gamma_\alpha(x)\cap\{z:\Im\mathrm{m}z<1\}$ and $e^{it}=\varphi^{-1}(x)$. Since $\varphi'(e^{it})\ne0$, φ is a diffeomorphism of a neighborhood of e^{it} onto a neighborhood of x. Therefore $\varphi^{-1}\big(\Gamma'_\alpha(x)\big)$ is contained in a Stolz angle $S_\rho(e^{it})$.

This implies that for a.e. $x \in \mathbb{R}$,

$$\lim_{z\to x\,,\,z\in\Gamma_\alpha(x)} u(z) = \lim_{z\to x\,,\,z\in\Gamma'_\alpha(x)} u(z) = \lim_{z\to e^{it}\,,\,z\in S_\rho(e^{it})} v(z) = v^\sharp\big(\varphi^{-1}(x)\big)\ . \tag{3.3}$$

In particular,

$$\lim_{y\to 0} u(x+iy) = v^\sharp\big(\varphi^{-1}(x)\big)$$

almost everywhere. Since u is bounded, we can apply dominated convergence to prove that, if $g \in L^1(\mathbb{R})$,

$$\lim_{y\to 0}\int_{\mathbb{R}} u_y(x)g(x)\,dx = \int_{\mathbb{R}} v^\sharp\big(\varphi^{-1}(x)\big)g(x)\,dx\ ,$$

i.e. $u_y \to v^\sharp\circ\varphi^{-1}$ in the weak* topology. Hence $v^\sharp\circ\varphi^{-1} = u^\sharp = f$, and (3.3) then says that u converges to f non-tangentially a.e.

At this point, the simplest argument to prove the statement for $1 < p < \infty$ is to observe that any $f \in L^p(\mathbb{R})$ decomposes as a sum $f = f_1 + f_\infty$, with $f_1 \in L^1(\mathbb{R})$ and $f_\infty \in L^\infty(\mathbb{R})$. To see this, take

$$f_\infty(x) = \begin{cases} f(x) & \text{if } |f(x)| \le 1\ ,\\ 0 & \text{if } |f(x)| > 1\ ,\end{cases}$$

and $f_1 = f - f_\infty$. $\square$

Corollary 3.4. *If $\alpha < \pi/2$ and $u \in h^p(D_+)$, $1 < p \le \infty$, or in $H^p(D_+)$, $1 \le p \le \infty$, then $\lim_{z\to x\,,\,z\in\Gamma_\alpha(x)} u(z) = u^\sharp(x)$ almost everywhere.*

The case $p = \infty$ in Corollary 3.2 and 3.4 is referred to as "Fatou's theorem".

4. Poisson integrals of singular measures

The discussion in the previous section does not say anything about non-tangential limits of general h^1-functions. We complete the picture here, proving that Corollary 3.4 can be extended to $h^1(\mathbb{R})$ (the same can be done on $\mathbb{T}$, but we omit the proof). It must be noted however that no maximal function is involved in the proof.

Every h^1-function on D_+ is the Poisson integral of a measure $\mu \in M(\mathbb{R})$. We recall the *Lebesgue decomposition* of μ as

$$\mu = \mu_a + \mu_s\ ,$$

where μ_a is absolutely continuous with respect to Lebesgue measure m (or $\mu_a \ll m$), i.e. $d\mu_a(x) = h(x)\,dx$ with $h \in L^1(\mathbb{R})$, and μ_s is singular with respect to the Lebesgue measure (or $\mu_s \perp m$). This means that there is a set E such that $m(\mathbb{R}\setminus E) = 0$ and $|\mu_s|(E) = 0$.

The function

$$\varphi(x) = |\mu_s|(-\infty, x)$$

is non-decreasing, hence differentiable at (Lebesgue-) almost every point, and its being singular implies that $\varphi'(x) = 0$ a.e. This means that at a.e. $x \in \mathbb{R}$,

$$\lim_{h\to 0}\frac{|\mu_s|(x-h, x+h)}{h} = 0\ . \tag{4.1}$$

Lemma 4.1. *Let μ be a singular measure on $\mathbb{R}$, and $u = \mathcal{P}\mu$. Then for almost every $x \in \mathbb{R}$,*

$$\lim_{z\to x\,,\,z\in\Gamma_\alpha(x)} u(z) = 0$$

for every $\alpha > 0$.

Proof. Take a point where (4.1) holds. We can assume that this point is the origin. Given $\varepsilon > 0$, take δ such that $|\mu_s|(-h,h) < \varepsilon h$ for $h < \delta$.

With $a = \tan\alpha$, which we can assume to be greater than 1, take x with $|x| < ay$. Then

$$\begin{aligned} |u(x+iy)| &\le \int_{x-\frac{\delta}{4}}^{x+\frac{\delta}{4}} P_y(x-t)\,d|\mu|(t) + \int_{|t-x|>\frac{\delta}{4}} P_y(x-t)\,d|\mu|(t) \\ &\le \int_{x-\frac{\delta}{4}}^{x+\frac{\delta}{4}} P_y(x-t)\,d|\mu|(t) + P_y\Big(\frac{\delta}{4}\Big)\|\mu\|_1 \ . \end{aligned}$$

Since $P_y(\delta/4) = \frac{2y}{\delta^2+4y^2}$, the last term tends to 0 as $y \to 0$. We must then show that

$$\lim_{y\to 0\,,\,|x|<ay} \int_{x-\frac{\delta}{4}}^{x+\frac{\delta}{4}} P_y(x-t)\,d|\mu|(t) = 0 \ .$$

Take $y < \frac{\delta}{4a}$, so that $(x-\frac{\delta}{4}, x+\frac{\delta}{4}) \subset (-\frac{\delta}{2}, \frac{\delta}{2})$. Define $I_j = (x-2^jy, x+2^jy)$ for $j \ge 0$ and $2^jy < \frac{\delta}{2}$. Then $I_j \subset (-\delta,\delta)$ for every such j, and $(x-\frac{\delta}{4}, x+\frac{\delta}{4})$ is covered by the I_j.

Since $\sup_{t\in I_j\setminus I_{j-1}} P_y(x-t) = P_y(2^{j-1}y) < \frac{4}{2^{2j}y}$ and $(a+2^j)y < \delta$,

$$\begin{aligned} \int_{x-\frac{\delta}{4}}^{x+\frac{\delta}{4}} P_y(x-t)\,d|\mu|(t) &\le \frac{|\mu|(I_0)}{y} + \sum_{j\ge 1} \frac{4}{2^{2j}y}|\mu|(I_j\setminus I_{j-1}) \\ &\le C\sum_j \frac{|\mu|\big(-(a+2^j)y,(a+2^j)y\big)}{2^{-2j}y} \\ &\le C\varepsilon \sum_j \frac{(a+2^j)y}{2^{-2j}y} \\ &\le C_a\varepsilon \ . \end{aligned}$$

Therefore

$$\limsup_{y\to 0\,,\,|x|<ay} \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} P_y(x-t)\,d|\mu|(t) \le C_a\varepsilon \ .$$

The conclusion follows by the arbitrarity of ε. □

Theorem 4.2. *Let $u = \mathcal{P}\mu \in h^1(D_+)$, and let $\mu = hm + \mu_s$ be the Lebesgue decomposition of μ, with $h \in L^1(\mathbb{R})$. Then for a.e. $x \in \mathbb{R}$,*

$$\lim_{z\to x\,,\,z\in\Gamma_\alpha(x)} u(z) = h(x)$$

for every $\alpha > 0$.

The proof follows easily from Theorem 3.3 and Lemma 4.1.

5. L^p-estimates for the conjugate harmonic function

We discuss now a crucial point in the theory of Hardy spaces, that has been left aside in the previous chapters: the fact that for $1 < p < \infty$ the conjugate harmonic function of an h^p-function is also in h^p.

We begin with the unit disc, and recall that if

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \bar{z}^n$$

is holomorphic in the unit disc, its harmonic conjugate is

$$\tilde{u}(z) = -i \sum_{n=1}^{\infty} a_n z^n + i \sum_{n=1}^{\infty} a_{-n} \bar{z}^n \ . \tag{5.1}$$

When u is real-valued, $\tilde{u}$ is characterized by the properties that it is real-valued, $u + i\tilde{u}$ is holomorphic, and $\tilde{u}(0) = 0$.

Lemma 5.1. *If u is harmonic and positive on some open set, and $p > 0$, then*

$$\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2 \ . \tag{5.2}$$

If f is holomorphic and non-zero in some open set, and $p > 0$, then

$$\Delta\big(|f|^p\big) = p^2|f|^{p-2}|f'|^2 \ . \tag{5.3}$$

Proof. We have

$$\partial_x^2(u^p) = \partial_x\big(pu^{p-1}\partial_x u\big) = p(p-1)u^{p-2}(\partial_x u)^2 + pu^{p-1}\partial_x^2 u \ .$$

summing with the corresponding formula for $\partial_y^2(u^p)$, we have (5.2).

As to (5.3), setting the standard notation $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, we have $\Delta = 4\partial_z\partial_{\bar{z}}$. We fix a point z_0 in the domain of f, and a determination of $\log f$ in a neighborhood of z_0. We then set $f^{p/2}(z) = e^{\frac{p}{2}\log f(z)}$. Then, in this neighbourhood of z_0,

$$\begin{aligned}\Delta|f|^p &= 4\partial_z\partial_{\bar{z}} f^{p/2}\overline{f^{p/2}} \\ &= 4(\partial_z f^{p/2})(\partial_{\bar{z}}\overline{f^{p/2}}) \\ &= 4|\partial_z f^{p/2}|^2 \\ &= p^2|f|^{p-2}|f'|^2 \ . \qquad \square\end{aligned}$$

Teorema 5.2. *If $1 < p < \infty$, there is a constant C_p such that $\|\tilde{u}\|_{h^p} \le C_p\|u\|_{h^p}$ for every $u \in h^p(D)$.*

Proof. The case $p = 2$ is already known, with $C_2 = 1$.

Consider first the case where $1 < p < 2$, and $u > 0$ in D. Let $f = u + i\tilde{u}$. Then f is holomorphic and non-zero in D.

We apply Green's formula to the two functions u^p and $|f|^p$ on the disc D_r centered at the origin and with radius $r < 1$. We have

$$(5.4)\qquad \begin{aligned} r\int_0^{2\pi} \partial_r\big(u(re^{it})^p\big)\, dt &= p(p-1)\int_{D_r} u(z)^{p-2}|\nabla u(z)|^2\, dz\ ,\\ r\int_0^{2\pi} \partial_r|f(re^{it}|^p\, dt &= p^2\int_{D_r} |f(z)|^{p-2}|f'(z)|^2\, dz\ . \end{aligned}$$

But

$$(5.5)\qquad |\nabla u|^2 = (\partial_x u)^2 + (\partial_y u)^2 = (\partial_x u)^2 + (\partial_x \tilde u)^2 = |\partial_x f|^2 = |f'|^2\ .$$

Since $p < 2$ and $|u| \le |f|$, we have $|f|^{p-2} \le |u|^{p-2}$. Therefore

$$(5.6)\qquad \begin{aligned} r\int_0^{2\pi} \partial_r|f(re^{it}|^p\, dt &\le p^2\int_{D_r} |u(z)|^{p-2}|\nabla u(z)|^2\, dz\\ &= \frac{p}{p-1} r\int_0^{2\pi} \partial_r|u(re^{it}|^p\, dt\ . \end{aligned}$$

In other words,

$$\partial_r\big(M_p(f,r)^p\big) \le p'\partial_r\big(M_p(u,r)^p\big)\ .$$

Since $M_p(u,0) = M_p(f,0) = u(0)$, we conclude that $M_p(f,r)^p \le p' M_p(u,r)^p$ for every $r < 1$, hence $\|f\|_{H^p} \le (p')^{1/p}\|u\|_{h^p}$.

Since $|\tilde u| \le |f|$, we also have

$$\|\tilde u\|_{h^p} \le (p')^{1/p}\|u\|_{h^p}\ .$$

If u is a generic function in $h^p(D)$, we decompose $u^\sharp \in L^p(\mathbb{T})$ as the combination $u^\sharp = \varphi_1 - \varphi_2 + i\varphi_3 - i\varphi_4$ of four non-negative functions, with the supports of φ_1 and φ_2 disjoint, and similarly for φ_3 and φ_4. Then

$$u = u_1 - u_2 + iu_3 - iu_4\ ,$$

where $u_j = \mathcal{P}\varphi_j$ is either identically zero (if so is φ_j), or strictly positive in D, as a consequence of the fact that the Poisson kernels P_r are strictly positive everywhere. By construction, $\sum_{j=1}^4 \|\varphi_j\|_p \le C\|u^\sharp\|_p$, so that

$$\sum_{j=1}^4 \|u_j\|_{h^p} \le C\|u\|_{h^p}\ .$$

Therefore

$$\|\tilde u\|_{h^p} \le \sum_{j=1}^4 \|\tilde u_j\|_{h^p} \le C_p\|u\|_{h^p}\ ,$$

which gives the conclusion for $1 < p < 2$.

Consider now the duality between $h^p(D)$ and $h^{p'}(D)$ of Section 8 in Chapter I,

$$B(u,v) = \lim_{r\to 1}\int_{\mathbb{T}} u(re^{it})\overline{v(re^{it})}\, dt = \int_{\mathbb{T}} u^\sharp(e^{it})\overline{v^\sharp(e^{it})}\, dt = \lim_{r\to 1}\sum_{n\in\mathbb{Z}} a_n\overline{b_n} r^{2|n|}\ ,$$

if a_n, b_n are the Taylor coefficients of u and v respectively. If $1 < p < \infty$, we have

$$|B(u,v)| \le \|u\|_{h^p}\|v\|_{h^{p'}}\ , \qquad \|u\|_{h^p} = \sup_{\|v\|_{h^{p'}}\le 1} |B(u,v)|\ .$$

By (5.1), the operator mapping u into $\tilde{u}$ is skew-hermitian w.r. to B, that is

$$B(\tilde{u},v) = \lim_{r\to 1}\sum_{n\in\mathbb{Z}}(-i\,\mathrm{sgn}\,n)a_n\overline{b_n}r^{2|n|} = -B(u,\tilde{v})\ .$$

Therefore, if $u \in h^p(D)$ and $2 < p < \infty$,

$$\begin{aligned}\|\tilde{u}\|_{h^p} &= \sup_{\|v\|_{h^{p'}}\le 1} |B(\tilde{u},v)| \\ &= \sup_{\|v\|_{h^{p'}}\le 1} |B(u,\tilde{v})| \\ &\le \|u\|_{h^p}\sup_{\|v\|_{h^{p'}}\le 1}\|\tilde{v}\|_{h^{p'}} \\ &\le C_{p'}\|u\|_{h^p}\ .\end{aligned}$$

This concludes the proof. □

We switch now to the upper half-plane. We take a function $u \in h^p(D_+)$ and we suppose that $p < \infty$. By Proposition 4.2 in Chapter II, the harmonic conjugate

$$\tilde{u}(x+iy) = u^\sharp * \tilde{P}_y(x) = \frac{1}{\pi}\int_{\mathbb{R}} u^\sharp(x-t)\frac{t}{t^2+y^2}\,dt$$

is well defined, and, if u is real-valued, it is characterized as the only real-valued harmonic function such that $u+i\tilde{u}$ is holomorphic and tends to 0 at infinity in each half-plane $\{z : \Im\mathrm{m} z \ge a\}$ with $a > 0$.

We know by (4.9) in Chapter II that, for $u \in h^2(D_+)$, $\tilde{u} \in h^2(D_+)$ and $\|\tilde{u}\|_{h^2} = \|u\|_{h^2}$. We prove now the analogue of Theorem 5.2.

Theorem 5.3. *If $1 < p < \infty$, there is a constant C_p such that $\|\tilde{u}\|_{h^p} \le C_p\|u\|_{h^p}$ for every $u \in h^p(D_+)$.*

Following the proof of Theorem 5.2, we want to use Green's formula to prove that, if $f = u + i\tilde{u}$, then $|\partial_y M_p(f,y)^p| \le C|\partial_y M_p(u,y)^p|$. We shall then apply Green's formula over rectangles, and use the fact that the integrals over certain edges tend to 0 if the edge is moved to infinity. This requires some preliminary proof.

Lemma 5.4. *Suppose $u \in h^p(D_+)$ with $p < \infty$. Then*

$$\lim_{x\to\infty}|\nabla u(x+iy)| = 0\ ,$$

uniformly in $y \ge a$, for every $a > 0$. Moreover,

$$M_p(\nabla u, y) \le \frac{1}{y}M_p\big(\mathcal{P}|u^\sharp|, y\big)\ ;$$

in particular, $\lim_{y\to\infty} M_p(\nabla u, y) = 0$.

Proof. We have

$$\nabla u(x+iy) = \nabla \int_{\mathbb{R}} u^\sharp(t) P_y(x-t)\, dt = \int_{\mathbb{R}} u^\sharp(t) \nabla P_y(x-t)\, dt \ , \tag{5.7}$$

Since $P_y(x) = -\frac{1}{\pi}\Im\mathrm{m}\frac{1}{x+iy}$, using the Cauchy-Riemann equations for $(x+iy)^{-1}$, we have

$$|\nabla P_y(x)| = \frac{1}{\pi |x+iy|^2} = \frac{1}{y} P_y(x) \ . \tag{5.8}$$

By (5.7),

$$|\nabla u(x+iy)| \le \frac{1}{y} \int_{\mathbb{R}} |u^\sharp(t)| P_y(x-t)\, dt = \frac{1}{y} \big(\mathcal{P}|u^\sharp|\big)(x+iy) \ .$$

The first part of the statement then follows from Lemma 1.3 in Chapter II, since $\mathcal{P}|u^\sharp| \in h^p(D_+)$, and the last part is now obvious. □

Proof of Theorem 5.3. As in the proof of Theorem 5.2, we consider first the case $1 < p < 2$. We further suppose that $u = \mathcal{P}u^\sharp$, with $u^\sharp \ge 0$, continuous with compact support, and non-identically zero. Then u is strictly positive on D_+.

In this hypotheses, we can say right now that $\tilde{u}$ is "not far from" being in $h^p(D_+)$. In fact

$$M_p(\tilde{u}, y) = \|u^\sharp * \tilde{P}_y\|_p \le \|u^\sharp\|_1 \|\tilde{P}_y\|_p \le C_p y^{-\frac{1}{p'}} \|u^\sharp\|_1 \ ; \tag{5.9}$$

(observe that the same estimate also holds for u:

$$M_p(u, y) \le \|u^\sharp\|_1 \|P_y\|_p \le C_p y^{-\frac{1}{p'}} \|u^\sharp\|_1 \ , \tag{5.9'}$$

a fact that we shall use later).

Therefore

$$v_{(\varepsilon)}(z) = \tilde{u}(z + i\varepsilon) \in h^p(D_+) \tag{5.10}$$

for every $\varepsilon > 0$.

For $a, b, R > 0$ with $a < b$, let $Q_{a,b,R}$ be the rectangle $[-R, R] \times [a, b]$. Applying Green's formula, the analogues of (5.4) become

$$\begin{aligned} &-\int_{-R}^{R} \partial_y \big(u(x+ia)^p\big)\, dx + \int_a^b \partial_x \big(u(R+iy)^p\big)\, dy \\ &\qquad + \int_{-R}^{R} \partial_y \big(u(x+ib)^p\big)\, dx - \int_a^b \partial_x \big(u(-R+iy)^p\big)\, dy \\ &\qquad = p(p-1) \int_{Q_{a,b,R}} u(z)^{p-2} |\nabla u(z)|^2\, dz \ , \end{aligned} \tag{5.11}$$

and

$$
\begin{aligned}
(5.12)\qquad & -\int_{-R}^{R} \partial_y |f(x+ia)|^p \, dx + \int_a^b \partial_x |f(R+iy)^p| \, dy \\
& \qquad + \int_{-R}^{R} \partial_y |f(x+ib)|^p \, dx - \int_a^b \partial_x |f(-R+iy)^p| \, dy \\
& \qquad = p^2 \int_{Q_{a,b,R}} |f(z)|^{p-2} |f'(z)|^2 \, dz \ .
\end{aligned}
$$

We first fix a and b and let R tend to infinity. By Lemma 5.4 and Lemma 1.3 in Chapter II,

$$\partial_x\big(u(\pm R+iy)^p\big) = pu(\pm R+iy)^{p-1}\partial_x u(\pm R+iy)$$

tends to zero as $R \to +\infty$, uniformly in $y \in [a,b]$.

To see that the same holds for $\partial_x|f(-R+iy)^p|$, observe that

$$\partial_x|f^p| = \Re\mathrm{e}\,\partial_z f^{\frac{p}{2}}\overline{f^{\frac{p}{2}}} = \frac{p}{2}\Re\mathrm{e} f^{\frac{p}{2}-1}\overline{f^{\frac{p}{2}}} f' \ ,$$

so that

$$\big|\partial_x|f^p|\big| \le \frac{p}{2}|f|^{p-1}|f'| \ . \qquad (5.13)$$

By (5.5), $f'(x+iy)$ tends to zero as $x \to \pm\infty$ uniformly in $y \in [a,b]$. The same is true for $|f(z)|$, as a consequence of (5.10).

We can then let $R \to +\infty$ in (5.11) and (5.12), to obtain

$$
\begin{aligned}
(5.14)\qquad & -\int_{\mathbb{R}} \partial_y\big(u(x+ia)^p\big) \, dx + \int_{\mathbb{R}} \partial_y\big(u(x+ib)^p\big) \, dx \\
& \qquad = p(p-1) \int_{x\in\mathbb{R}\,,\, y\in[a,b]} u(z)^{p-2} |\nabla u(z)|^2 \, dz \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
(5.15)\qquad & -\int_{\mathbb{R}} \partial_y |f(x+ia)|^p \, dx + \int_{\mathbb{R}} \partial_y |f(x+ib)|^p \, dx \\
& \qquad = p^2 \int_{x\in\mathbb{R}\,,\, y\in[a,b]} |f(z)|^{p-2} |f'(z)|^2 \, dz \ .
\end{aligned}
$$

We now make tend b to $+\infty$, and show that the integrals containg b tend to zero. In (5.14) we use Hölder's inequality to obtain

$$
\begin{aligned}
\left| \int_{\mathbb{R}} \partial_y\big(u(x+ib)^p\big) \, dx \right| &\le p \int_{\mathbb{R}} u(x+ib)^{p-1} |\partial_y u(x+ib)| \, dx \\
&\le p M_p(u,b)^{p-1} M_p(\partial_y u, b) \ ,
\end{aligned}
$$

which tends to 0 by Lemma 5.5. For the second integral in (5.15), we use the analogue of (5.13) for $\partial_y(|f|^p)$ to obtain that

$$
\begin{aligned}
\left| \int_{\mathbb{R}} \partial_y |f(x+ib)|^p \, dx \right| &\le \frac{p}{2} M_p(f,b)^{p-1} M_p(f',b) \\
&\le \frac{p}{2}\big(M_p(u,b) + M_p(\tilde{u},b)\big)^{p-1} M_p(\nabla u, b) \ .
\end{aligned}
$$

This last quantity tends to zero by (5.10) and Lemma 5.4. So (5.14) and (5.15) respectively imply that

$$
(5.16)\qquad
\begin{aligned}
-\int_{\mathbb{R}} \partial_y\big(u(x+ia)^p\big)\,dx &= p(p-1)\int_{x\in\mathbb{R}\,,\,y\geq a} u(z)^{p-2}|\nabla u(z)|^2\,dz\\
-\int_{\mathbb{R}} \partial_y|f(x+ia)|^p\,dx &= p^2\int_{x\in\mathbb{R}\,,\,y\geq a} |f(z)|^{p-2}|f'(z)|^2\,dz\ .
\end{aligned}
$$

We claim that the two left-hand sides in (5.16) are $-\partial_y\big(M_p(u,y)^p\big)_{|y=b}$ and $-\partial_y\big(M_p(f,y)^p\big)_{|y=b}$ respectively. We postpone this proof, and assume this fact to be true for the moment.

Since $M_p(u,y)$ and $M_p(f,y)$ are decreasing in y, the left-hand sides in (5.16) are in fact positive quantities. Since

$$\lim_{y\to 0} M_p(u,y)^p = \lim_{y\to 0} M_p(f,y)^p = 0$$

by (5.9) and (5.9'), the same argument used in the proof of Theorem 5.2 shows that

$$M_p(f,y)^p \leq p' M_p(u,y)^p$$

for every $y>0$. The rest of the proof will then proceed as in Theorem 5.2, taking initially u as the Poisson integral of a continuous, complex-valued function $u^\sharp$ on $\mathbb{R}$ with compact support. Once we have the estimate

$$\|\tilde u\|_{h^p} \leq C_p\|u\|_{h^p}$$

for such functions, we use a density argument to extend it to a generic $u\in h^p(D_+)$.

We are so left with the proof of our previous claim, concerning

$$\partial_y\big(M_p(u,y)^p\big)_{|y=b} = \lim_{h\to 0}\int_{\mathbb{R}} \frac{u\big(x+i(b+h)\big)^p - u(x+ib)^p}{h}\,dx\ .$$

In order to be allowed to move the limit inside the integral, we use dominated convergence. We shall use vertical maximal functions for this purpose.

Fix $\varepsilon>0$ and smaller than $b/2$, and call $u_{(\varepsilon)}(z)$, resp. $f_{(\varepsilon)}(z)$, the functions $u(z+i\varepsilon)$, resp. $f(z+i\varepsilon)$. Notice that $u_{(\varepsilon)}, \partial_y u_{(\varepsilon)} \in h^p(D_+)$ and $f_{(\varepsilon)}, f'_{(\varepsilon)} \in H^p(D_+)$, by (5.9), (5.9') and Lemma 5.4.

We take $h > -(b-\varepsilon)$, as we can, in order to stay above the level ε.

By the mean value theorem, for every x there is $t=t(x)\in(b,b+h)$ such that

$$\frac{u\big(x+i(b+h)\big)^p - u(x+ib)^p}{h} = \partial_y\big(u(x+iy)^p\big)_{|y=t} = pu(x+it)^{p-1}\partial_y u(x+it)\ .$$

Therefore,

$$\left|\frac{u\big(x+i(b+h)\big)^p - u(x+ib)^p}{h}\right| \leq p\big(M_{\text{vert}}u_{(\varepsilon)}(x)\big)^{p-1}M_{\text{vert}}(\partial_y u_{(\varepsilon)})(x)\ .$$

Applying Hölder's inequality and Theorem 2.2, we can see that this last function is integrable. The same applies to f. □

We can now complete some statements left incomplete in the previous chapters. We give a list of them.

(i) The Cauchy projection is bounded from h^p onto H^p for $1 < p < \infty$, both in D and in D_+.

(ii) The conjugate function operator on $\mathbb{T}$ (see Propsoition 6.2 in Chapter I) and the Hilbert transform on $\mathbb{R}$ (see Proposition 4.3 in Chapter II) are bounded on L^p for $1 < p < \infty$.

(iii) Under the sesquilinear map B in Proposition 8.3 in Chapter I, the dual of $H^p(D)$ is identified with $H^{p'}(D)$ for $1 < p < \infty$. The same can be verified on D_+.