EIGENVALUES OF THE HODGE LAPLACIAN ON THE HEISENBERG GROUP

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Abstract. We study the spectrum of the Hodge Laplacian Δ_1 acting on 1-forms on the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}_n$, by finding the eigenvalues of the image of Δ_1 in the Bargmann representations. As a consequence, we determine explicitely the eigenvalues for Δ_1 on some compact quotients of $\mathbf{H}_n$.

This note is part of a larger project [MPR], in which we study the question of the boundedness of spectral multipliers of Δ_1 on $\mathbf{H}_n$.

1. Introduction

The Hodge Laplacian is the collection of operators $\Delta_k = dd^* + d^*d$ acting on k-forms, for $k = 0, \ldots, 2n+1$. [1]

It is well known that Δ_k is a self-adjoint, positive, second order elliptic differential operator. The operator Δ_0 is the Laplace-Beltrami (scalar) operator. When $k \geq 1$ Δ_k is a differential operator on sections of a vector bundle (that is differential forms), and it differs from $\Delta_0 \otimes I$ in its first and zero order terms.

On a given differential manifolds M there are many relationships between the heat flow, the topology of M and the spectral properties of the Hodge Laplacians Δ_k, see e.g. [Lo, Lü, GrS, GoW]. In particular, the so-called Novikov-Shubin invariants $\alpha_k(M)$ of a closed oriented manifold M can be computed in terms of the power decay of the trace of the heat kernel for the Hodge Laplacian Δ_k on k-forms, [Lü].

In this note we study certain spectral properties of the Hodge Laplacian acting on forms on the Heisenberg group $\mathbf{H}_n$ and on the reduced Heisenberg group. We also obtain the eigenvalues of the Hodge Laplacian $\tilde{\Delta}_1$ on the a compact quotient of $\mathbf{H}_n$ thus finding explicitly the smallest non-zero eigenvalue. We thank Paolo De Bartolomeis and Adriano Tomassini for bringing this question to our attention, see also [DT].

This note is part of a larger project. In [MPR] we study spectral multipliers of Δ_1 on $\mathbf{H}_n$, determing a sharp Mihlin-Hörmander condition on a multiplier m in order that the operator $m(\Delta_1)$ be bounded on $L^p(\mathbf{H}_n)$, $1 < p < \infty$.

2000 *Mathematical Subject Classification.* 42B15, 35P15, 58A40.

Key words and phrases. Hodge Laplacian, Heisenberg group.

This work was done within the project IHP Network "HARP". We thank the European Commission for the support provided.

[1] To be precise we should write $\Delta_k = d_{k-1}d_k^* + d_{k+1}^*d_k$, but our short-hand should cause no confusion.

Let $\mathbf{H}_n$ be the $(2n+1)$-dimensional Heisenberg group with coordinates $(x,y,t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and product rule given by

$$(x,y,t)(x',y',t') = \big(x+x', y+y', t+t'+\tfrac{1}{2}(x\cdot y' - y\cdot x')\big)\,.$$

Then, a basis of left-invariant vector fields is given by

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t\ , \qquad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t\ , \qquad T = \partial_t\,,$$

where $1 \le j \le n$. The dual basis of 1-forms is given by the forms dx_j, dy_j, $j = 1,\dots,n$, and the contact form

$$\theta = dt - \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)\,.$$

It will be more convenient for us to work with vector fields and forms written in complex notation. For $j = 1,\dots,n$, we set

$$B_j = \frac{1}{\sqrt{2}}(X_j - iY_j)\ , \qquad \bar{B}_j = \frac{1}{\sqrt{2}}(X_j + iY_j)\ ,$$

and

$$\beta_j = \frac{1}{\sqrt{2}}(dx_j + idy_j)\ , \qquad \bar{\beta}_j = \frac{1}{\sqrt{2}}(dx_j - idy_j)\ .$$

The relevant commutation relation among the vector fields is

$$[B_j, \bar{B}_j] = iT\ .$$

The differential df of a smooth function is then given by

$$df = \sum_{j=1}^{n}(X_j f\, dx_j + Y_j f\, dy_j) + Tf\,\theta = \sum_{j=1}^{n}(B_j f\,\beta_j + \bar{B}_j f\,\bar{\beta}_j) + Tf\,\theta\,,$$

and similarly for exterior derivatives of differential forms. Observe that, in particular,

$$d\theta = -\sum_{j=1}^{n} dx_j \wedge dy_j = -i\sum_{j=1}^{n}\beta_j \wedge \bar{\beta}_j\,.$$

2. The Hodge Laplacian

The Hodge Laplacian is the collection of operators $\Delta_k = dd^* + d^*d$ acting on k-forms.

We are going to give a description of the Hodge Laplacian Δ_k that keeps into account the presence of the contact structure on $\mathbf{H}_n$.

A k-form ω can be uniquely written as

$$\omega = \omega_1 + \theta\wedge\omega_2\,, \tag{1}$$

where $\omega_1 \llcorner \theta = \omega_2 \llcorner \theta = 0$. We shall denote by $\llcorner$ the standard contraction operator on forms.

We say that ω is *horizontal* if $\omega \llcorner \theta = 0$, that is if $\omega_2 = 0$. Notice that ω_1 and ω_2 in the decomposition above are both horizontal.

Moreover, we define the *horizontal differential* of a smooth function f as

$$d_H f = \sum_{j=1}^{n} \big(B_j f \beta_j + \overline{B}_j f \overline{\beta}_j \big) \,,$$

and we extend this definition to a horizontal form $f(\beta^I \wedge \overline{\beta}^J)$ as

$$d_H \big(f(\beta^I \wedge \overline{\beta}^J) \big) = \sum_{j=1}^{n} \Big((B_j f)\beta_j \wedge \beta^I \wedge \overline{\beta}^J + (\overline{B}_j f)\overline{\beta}_j \wedge \beta^I \wedge \overline{\beta}^J \Big) \,.$$

Then, given a k-form ω written as in (1)

$$\begin{aligned} d\omega &= d\omega_1 + (d\theta) \wedge \omega_2 - \theta \wedge d\omega_2 \\ &= d_H\omega_1 + \theta \wedge T\omega_1 + (d\theta) \wedge \omega_2 - \theta \wedge (d_H \omega_2) \\ &= \big(d_H \omega_1 + (d\theta) \wedge \omega_2 \big) + \theta \wedge \big(T\omega_1 - d_H \omega_2 \big) \,. \end{aligned}$$

We shall denote by e the exterior multiplication, so that $e(d\theta)\omega = (d\theta) \wedge \omega$, and by e^* its adjoint, that is the interior multiplication operator, so that

$$e(d\theta)^* \omega = i \sum_{j=1}^{n} e(\overline{\beta}_j)^* e(\beta_j)^* \omega = i \sum_{j=1}^{n} \overline{\beta}_j \llcorner (\beta_j \llcorner \omega) \,.$$

Thus, we can write

$$d \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \,,$$

and

$$d^* \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} d_H^* & -T \\ e(d\theta)^* & -d_H^* \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \,.$$

It is worth noticing that the identity $d^2 = 0$ implies that

$$d_H^2 \omega_1 = -e(d\theta)(T\omega_1) = -T\big(e(d\theta)\omega_1 \big) \,,$$

for any horizontal form ω_1. In particular, $d_H^2 \neq 0$ and the horizontal differentiation does not give rise to a complex.[2]

Therefore,

$$\begin{aligned} \Delta_k &= \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix} \begin{pmatrix} d_H^* & -T \\ e(d\theta)^* & -d_H^* \end{pmatrix} + \begin{pmatrix} d_H^* & -T \\ e(d\theta)^* & -d_H^* \end{pmatrix} \begin{pmatrix} d_H & e(d\theta) \\ T & -d_H \end{pmatrix} \\ &= \begin{pmatrix} \Delta_H - T^2 + e(d\theta)e(d\theta)^* & [d_H^*, e(d\theta)] \\ [e(d\theta)^*, d_H] & \Delta_H - T^2 + e(d\theta)^* e(d\theta) \end{pmatrix} \,, \end{aligned} \tag{2}$$

where we have set $\Delta_H = d_H d_H^* + d_H^* d_H$.

[2]This fact was already noticed by Rumin [R1] where an alternative complex, since then called the *Rumin complex*, was constructed. We will not take that point of view in the present work.

Thus, formula (2) above gives us the representation of Δ_k w.r. to the decomposition (1) of forms. We wish to analyse the various operators appearing in (2). In order to do so, we refer to some technical facts whose proofs appear in [MPR].

We recall that $\mathbf{H}_n$ is also endowed with a CR-structure. A horizontal form ω is said to be a (p,q)-form if

$$\omega = \sum_{|I|=p,\,|J|=q} f_{I,J}\beta^I \wedge \overline{\beta}^J ,$$

where the sum is performed over the strictly increasing multi-indices I and J,

$$I = (i_1, \dots, i_p), \quad 1 \le i_1 < \cdots < i_p \le n;$$
$$J = (j_1, \dots, j_q), \quad 1 \le j_1 < \cdots < j_q \le n.$$

The operators

$$\partial_b f = \sum_{j=1}^n B_j f\, \beta_j \ , \qquad \bar\partial_b f = \sum_{j=1}^n \bar B_j f\, \bar\beta_j \ ,$$

initially defined on functions, are naturally extended to forms. Then,

$$d_H = \partial_b + \bar\partial_b \,,$$

and setting

$$\Box = \partial_b\partial_b^* + \partial_b^*\partial_b \ , \qquad \overline{\Box} = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b \ ,$$

we obtain that

$$\Delta_H = \Box + \overline{\Box}\,.$$

Then, $\overline{\Box}$ is the Kohn Laplacian (see [FS]) acting on (p,q)-forms. It is well known that on (p,q)-forms both $\Box$ and $\overline{\Box}$ are diagonal w.r. the canonical basis $\{f(\beta^I \wedge \overline{\beta}^J) : f \in L^2(\mathbf{H}_n),\ |I| = p,\ |J| = q\}$ and

$$\Box_{p,q} = \tfrac{1}{2}L + i(\tfrac{n}{2} - p)T\,,$$

$$\overline{\Box}_{p,q} = \tfrac{1}{2}L + i(\tfrac{n}{2} - q)T\,,$$

see [FS]. Here and in what follows, we denote by L the sub-Laplacian

$$L = -\sum_{j=1}^n (B_j\bar B_j + \bar B_j B_j)\,.$$

We are in a position to obtain a description of Δ_k in the case $k = 1$. Given a 1-form ω we further refine the decomposition (1) by writing

$$\omega = \omega_1^+ + \omega_1^- + h\theta\,,$$

where ω_1^+, ω_1^- are $(1,0)$, $(0,1)$-forms, respectively.

Proposition 2.1. *The action of the Hodge Laplacian* Δ_1 *on a 1-form* $\omega = \omega_1^+ + \omega_1^- + h\theta$ *can be described as*

$$\Delta_1 \begin{pmatrix} \omega_1^+ \\ \omega_1^- \\ h \end{pmatrix} = \begin{pmatrix} (-L - T^2 + iT) & 0 & i\partial_b \\ 0 & (-L - T^2 - iT) & -i\bar{\partial}_b \\ -i\partial_b^* & i\bar{\partial}_b^* & -L - T^2 + n \end{pmatrix} \begin{pmatrix} \omega_1^+ \\ \omega_1^- \\ h \end{pmatrix} .$$

Proof. This result can be also deduced from Sec. 2 in [MPR]. We give here a direct argument, for sake of completeness.

From the discussion above, it follows that when acting on (p,q)-forms

$$\Delta_H = \tfrac{1}{2} L + i(q-p)T .$$

Starting from formula (2), we also need to compute $e(d\theta)^* e(d\theta)$ on functions and $e(d\theta)e(d\theta)^*$ on horizontal 1-forms. Notice that the latter is zero since

$$e(d\theta)^* = \Big(i \sum_{j=1}^n e(\beta_j) e(\overline{\beta}_j) \Big)^* = -i \sum_{j=1}^n (\overline{\beta}_j \llcorner)(\beta_j \llcorner)$$

is zero on 1-forms. The former is

$$e(d\theta)^* e(d\theta) h = \sum_{j,k=1}^n (\overline{\beta}_j \llcorner)(\beta_j \llcorner)(\beta_k \wedge \overline{\beta}_k) h = nh .$$

Finally, we need to compute $\big[d_H^*, e(d\theta)\big]$ when acting on functions, and $\big[e(d\theta)^*, d_H\big]$ when acting on horizontal 1-forms. We have,

$$\begin{aligned} \big[d_H^*, e(d\theta)\big] h = d_H^*(d\theta) h &= -i \sum_{j,k} (\overline{B}_j h) \beta_j \llcorner (\beta_b \wedge \overline{\beta}_k) - i \sum_{j,k} (B_j h) \overline{\beta}_j \llcorner (\beta_b \wedge \overline{\beta}_k) \\ &= i \sum_j (B_j h) \beta_j - i \sum_j (\overline{B}_j h) \overline{\beta}_j \\ &= i \partial_b h - i \overline{\partial}_b h . \end{aligned}$$

On the other hand, if $\omega_1^+ = \sum_j f_j \beta_j$,

$$\begin{aligned} \big[e(d\theta)^*, d_H\big] \omega_1^+ = e(d\theta)^*, d_H \omega_1^+ &= -i \sum_j (\overline{\beta}_j \llcorner)(\beta_j \llcorner) \Big(\sum_k (\overline{B}_k f_j) \overline{\beta}_k \wedge \beta_j \Big) \\ &= i \sum_j \overline{B}_j f_j \\ &= -i \partial_b^* \omega_1^+ . \end{aligned}$$

This gives the result. $\square$

3. The Bargmann representation

For $\lambda \in \mathbf{R} \setminus \{0\}$, we denote by $\mathcal{F}_{|\lambda|}$ the space of holomorphic functions F on $\mathbf{C}^n$ such that

$$\|F\|^2_{|\lambda|} = \frac{|\lambda|^n}{\pi^n} \int_{\mathbf{C}^n} |F(\zeta)|^2 e^{-|\lambda||\zeta|^2} dV(\zeta) < \infty\,.$$

We define the Bargmann representation π_λ by setting

(i) for $\lambda > 0$, $d\pi_\lambda(B_j) = \lambda\zeta_j$, $d\pi_\lambda(\overline{B}_j) = \partial_{\zeta_j}$, and $d\pi_\lambda(T) = i\lambda$;

(ii) for $\lambda < 0$, $d\pi_\lambda(B_j) = \partial_{\zeta_j}$, $d\pi_\lambda(\overline{B}_j) = -\lambda\zeta_j$, and $d\pi_\lambda(T) = i\lambda$.

These formulas are invariant under the change

$$B_j \mapsto \overline{B}_j, \quad \overline{B}_j \mapsto B_j, \quad T \mapsto -T \quad \text{and} \quad \lambda \mapsto -\lambda\,,$$

while Δ is invariant under this transformation of the vector fields. Hence, we can assume $\lambda > 0$. We then have

$$d\pi_\lambda(L) = -2\lambda \sum_{j=1}^n \zeta_j \partial_{\zeta_j} - n\lambda\,.$$

Next, we wish to describe the image of Δ_1 under $d\pi_\lambda$.

We denote by $\Lambda^k = \Lambda^k(\mathfrak{h}_n^*)$ the k-th exterior product of the dual of the Lie algebra $\mathfrak{h}_n$ of H_n (also identifiable with the space of left-invariant k-forms on H_n). We call

$$\mathcal{D}\Lambda^k(\mathbf{H}_n) = \mathcal{D}(\mathbf{H}_n) \otimes \Lambda^k$$

the space of smooth k-forms on $\mathbf{H}_n$ with compact support. This notation will be consistently adapted to function spaces other than $\mathcal{D}(\mathbf{H}_n)$ or to subspaces of Λ^k.

Observe that $d : L^2\Lambda^k \to L^2\Lambda^{k+1}$ and that consequentely $d\pi_\lambda(d)$ must be regarded as a (unbounded) mapping

$$d\pi_\lambda(d) : \mathcal{F}_\lambda(\mathbf{C}^n) \otimes \Lambda^k \to \mathcal{F}_\lambda(\mathbf{C}^n) \otimes \Lambda^{k+1}$$

defined as follows. We identify elements in $\mathcal{F}_\lambda(\mathbf{C}^n) \otimes \Lambda^1$ with "forms"

$$\sum_j F_j \beta_j + \sum_j G_j \overline{\beta}_j + H\theta\,, \tag{3}$$

with coefficients in $\mathcal{F}_\lambda(\mathbf{C}^n)$. Then, for an element $F \in \mathcal{F}_\lambda(\mathbf{C}^n)\big(\equiv \mathcal{F}_\lambda(\mathbf{C}^n) \otimes \Lambda_0\big)$, $d\pi_\lambda(d)(F)$ is defined as

$$d\pi_\lambda(d)(F) = \sum_j \big(d\pi_\lambda(B_j)F\big)\beta_j + \sum_j \big(d\pi_\lambda(\overline{B}_j)F\big)\overline{\beta}_j + \big(d\pi_\lambda(T)F\big)\theta\,.$$

This allows to define a complex $d\pi_\lambda$ by extending this definition to higher-order "forms" in an obvious way.

Then,

$$d\pi_\lambda(\Delta_k) = d\pi_\lambda(d) d\pi_\lambda(d)^* + d\pi_\lambda(d)^* d\pi_\lambda(d)\,,$$

and in particular (recall that we are restricting ourselves to the case $\lambda > 0$) $d\pi_\lambda(\Delta_1) \in \mathrm{End}\,(\mathcal{F}_\lambda) \otimes \mathrm{End}\,(\Lambda_1)$. That is, $d\pi_\lambda(\Delta_1)$ must be understood as acting on $(2n+1)$-tuples $(F_1, \ldots, F_n, G_1, \ldots, G_n, H)$. (For some details on the Fourier transform of vector-valued functions on $\mathbf{H}_n$ see e.g. [PR], Sc. 3.)

From the discussion above and Proposition 2.1 the following result now follows.

Proposition 3.1. *The action of the image $d\pi_\lambda(\Delta_1)$ of the Hodge Laplacian Δ_1, when acting on 1-forms with coefficients in $\mathcal{F}_\lambda$ is given by*

$$d\pi_\lambda(\Delta_1) =$$
$$\begin{pmatrix} \big(2\lambda\zeta\partial_\zeta + (n-1)\lambda + \lambda^2\big) I & 0 & i\lambda\zeta \\ 0 & \big(2\lambda\zeta\partial_\zeta + (n+1)\lambda + \lambda^2\big) I & -i\partial_\zeta \\ -i\,{}^t\partial_\zeta & i\lambda\,{}^t\zeta & 2\lambda\zeta\partial_\zeta + n\lambda + \lambda^2 + n^2 \end{pmatrix},$$

where $\zeta\partial_\zeta$ denotes the scalar differential operator $\sum_j \zeta_j \partial_{\zeta_j}$.

We now wish to decompose the space $\mathcal{F}_\lambda \otimes \Lambda^1$, which from now on we identify with $\mathcal{F}_\lambda(\mathbf{C}^n)^{2n+1}$, into finite dimensional subspaces that are invariant under both the action of n-torus $\mathbf{T}^n$ and of Δ_1.

Given a rotation $\big(e^{i\alpha_1}, \ldots, e^{i\alpha_n}\big) \in \mathbf{T}^n$, we know that it acts on β_j by multiplication by $e^{i\alpha_j}$, on $\overline{\beta}_j$ by multiplication by $e^{-i\alpha_j}$, and leaves θ fixed. On the other hand, $\big(e^{i\alpha_1}, \ldots, e^{i\alpha_n}\big)$ acts on $\mathcal{F}_\lambda(\mathbf{C}^n)$ when $\lambda > 0$ by

$$D_{(\alpha_1,\ldots,\alpha_n)} \circ F(\zeta_1, \ldots, \zeta_n) = F\big(e^{-i\alpha_1}\zeta_1, \ldots, e^{-i\alpha_n}\zeta_n\big)\,.$$

Therefore, we give the following definition.

Definition 3.2. Let P be the set of $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$, with $m_j \geq -1$ for all j and at most one $m_j = -1$. We say that the form (3) is of type m, with $m \in P$, if, for some $a_j, b_j, c \in \mathbf{C}$,

(i) each $F_j = a_j \zeta^{m+e_j}$;

(ii) each $G_j = b_j \zeta^{m-e_j}$;

(iii) $H = c\zeta^m$.

(Here $e_j = (0, \ldots, 1, \ldots, 0)$ denotes the j-th element of the canonical basis.)

Then we set

$$\mathcal{W}_m = \big\{ \big(a_1\zeta^{m+e_1}, \ldots, a_n\zeta^{m+e_n}, b_1\zeta^{m-e_1}, \ldots, b_n\zeta^{m-e_n}, c\zeta^m\big) : \, a, b \in \mathbf{C}^n, \, c \in \mathbf{C} \big\}\,,$$

with the convention that the coefficients a_j, b_j, c are 0 if the corresponding exponent of ζ^{m+e_j}, ζ^{m-e_j}, ζ^m resp., has a negative entry.

If we normalize the monomials in $\mathcal{F}_\lambda(\mathbf{C}^n)$, we obtain the orthonormal basis

$$\psi_m = \sqrt{\frac{\lambda^{|m|}}{m!}}\,\zeta^m\,.$$

The following result is clear from the previous discussion.

Proposition 3.3. *We have an orthogonal direct sum decomposition*

$$\mathcal{F}_\lambda(\mathbf{C}^n)^{2n+1} = \bigoplus_{m\in P} \mathcal{W}_m$$

and the operator $d\pi_\lambda(\Delta_1)$ acts invariantly on each $\mathcal{W}_m$. An orthonormal basis of $\mathcal{W}_m$ is given by

$$\Big\{ (\psi_{m+e_1}, 0, \dots, 0), \dots, (0, \dots, 0, \psi_{m-e_n}, 0), (0, \dots, 0, \psi_m) \Big\}.$$

With respect to this basis, $d\pi_\lambda(\Delta_1)$ acts on $\mathcal{W}_m$ as

$$A_m = \begin{pmatrix} \big(\lambda(2\ell(m)+n+1)+\lambda^2\big)I & 0 & i\sqrt{\lambda(m+1)} \\ 0 & \big(\lambda(2\ell(m)+n-1)+\lambda^2\big)I & -i\sqrt{\lambda m} \\ -i\,{}^t\big(\sqrt{\lambda(m+1)}\big) & i\,{}^t\big(\sqrt{\lambda m}\big) & \lambda(2\ell(m)+n)+\lambda^2+n \end{pmatrix}$$

where $\ell(m) = \sum_j m_j$, $\sqrt{\lambda(m+1)}$ denotes the column vector $\big(\sqrt{\lambda(m_j+1)}\big)_{j=1,\dots,n}$, and similarly for $\sqrt{\lambda m}$.

We remark that in general, $\mathcal{W}_m$ has dimension $2n+1$. If one of the exponents of ζ^{m+e_j}, ζ^{m-e_j}, ζ^m is negative, then the dimension of $\mathcal{W}_m$ is smaller. In this case, the rows and columns in A_m corresponding to these entries, have to be cancelled.

4. Eigenvalues of $d\pi_\lambda(\Delta_1)$ on $\mathbf{H}_n$

Using the decomposition in Prop. 3.3 we can now describe the eigenvalues of $d\pi_\lambda(\Delta_1)$ on $\mathcal{F}_\lambda(\mathbf{C}^n)^{2n+1}$.

Proposition 4.1. *Assume $\lambda > 0$. For $m \in P$, denote by m_+ the number of components of m that are ≥ 1. Then the complete list of eigenvalues of $d\pi_\lambda(\Delta_1)$ on $\mathcal{W}_m$ is the following.*

Case (I). If $m_j = -1$ for one j, then

(1) $\lambda^2 + \lambda\big(2\ell(m)+n\big) + \lambda$, *with multiplicity n;*
(2) $\lambda^2 + \lambda\big(2\ell(m)+n\big) - \lambda$, *with multiplicity m_+.*

Case (II). If $m_j = 0$ for all j, then

(1) $\lambda^2 + n\lambda$, *with multiplicity* 1*;*
(2) $\lambda^2 + (n+1)\lambda$, *with multiplicity $n-1$;*
(3) $\lambda^2 + (n+1)\lambda + n$, *with multiplicity* 1.

Case (III). If $m_j \geq 0$ for all j and $m_+ \geq 1$, then

(1) $\lambda^2 + \lambda\big(2\ell(m)+n\big)$, *with multiplicity* 1*;*
(2) $\lambda^2 + \lambda\big(2\ell(m)+n\big) + \lambda$, *with multiplicity $n-1$;*
(2') $\lambda^2 + \lambda\big(2\ell(m)+n\big) - \lambda$, *with multiplicity $m_+ - 1$;*

(3) $\lambda^2+\lambda\big(2\ell(m)+n\big)+\frac{n}{2}\pm\sqrt{\lambda^2+\lambda\big(2\ell(m)+n\big)+\frac{n^2}{4}}$, *each with multiplicity* 1.

Proof. We observe that, from Prop. 3.3, we can write the matrix A_m as

$$A_m=\Big(\lambda^2+\lambda\big(2\ell(m)+n\big)\Big)I+B_m\ ,$$

where

$$B_m=\begin{pmatrix}\lambda I & 0 & i\sqrt{\lambda(m+1)}\\ 0 & -\lambda I & -i\sqrt{\lambda m}\\ -i\,{}^t\big(\sqrt{\lambda(m+1)}\big) & i\,{}^t\big(\sqrt{\lambda m}\big) & n\end{pmatrix}\ . \tag{4}$$

We begin from the simplest case in which one $m_j=-1$. Then, in particular the last row and column have to be canceled, and one can easily compute that the characteristic polynomial for B_m is

$$(\lambda-\mu)^n(-\lambda-\mu)^{m_+}\ .$$

From this we immediately obtain the eigenvalues for A_m in Case (I).

Also when $m_j\geq 0$ for all j, it is not difficult to compute the characteristic polynomial for B_m. For sake of simplicity, we first assume that $m_j\geq 1$ for all j.

We compute $\det(B_m-\mu I)$ by "developing" from the last row. Notice that, when deleting the last row and the j-th column, we obtain a matrix that can be brought into an upper triangular matrix by a permutation of the rows of same parity as j, so that its determinant can be computed easily. Explicitly,

$$\begin{aligned}
&\det(B_m-\mu I)\\
&\quad=\sum_{j=1}^n(-1)^{j+1}i\sqrt{\lambda(m_j+1)}\Big[(-1)^{j+1}i\sqrt{\lambda(m_j+1)}(\lambda-\mu)^{n-1}(-\lambda-\mu)^n\Big]\\
&\qquad+\sum_{j=1}^n(-1)^{j+n+1}i\sqrt{\lambda m_j}\Big[(-1)^{j+n+1}i\sqrt{\lambda m_j}(\lambda-\mu)^n(-\lambda-\mu)^{n-1}\Big]\\
&\qquad+(n-\mu)(\lambda-\mu)^n(-\lambda-\mu)^n\\
&\quad=(\lambda-\mu)^{n-1}(-\lambda-\mu)^{n-1}\Big[-\big(\ell(m)+n\big)\lambda(-\lambda-\mu)-\ell(m)\lambda(\lambda-\mu)\\
&\qquad+(n-\mu)(\lambda-\mu)(-\lambda-\mu)\Big]\\
&\quad=(-1)^n\mu(\lambda-\mu)^{n-1}(\lambda+\mu)^{n-1}\big[\mu^2-n\mu-\lambda\big(2\ell(m)+n+\lambda\big)\big]\ .
\end{aligned}$$

From here it is easy to conclude that in this case (i.e. $m_+=n$), the eigenvalues of A_m are

(1) $\lambda^2+\lambda\big(2\ell(m)+n\big)$, with multiplicity 1;

(2–2') $\lambda^2+\lambda\big(2\ell(m)+n\big)\pm\lambda$, each with multiplicity $n-1$;

(3) $\lambda^2+\lambda\big(2\ell(m)+n\big)+\frac{n}{2}\pm\sqrt{\lambda^2+\lambda\big(2\ell(m)+n\big)+\frac{n^2}{4}}$, each with multiplicity 1, as we claimed.

Consider now the case in which $m_j \geq 0$ for all j, but $m_j = 0$ for some j, i.e. $m_+ < n$. Then, the matrix B_m admits a block decomposition

$$B_m = \begin{pmatrix} M_1 & 0 & v_1 \\ 0 & M_2 & v_2 \\ {}^t v_1 & {}^t v_2 & M_3 \end{pmatrix}$$

in which M_1 is of type $n \times n$, M_2 of type $m_+ \times m_+$ and M_3 of type 1×1.

Assume first $m_+ \geq 1$, then a computation similar to the one above shows that the characteristic polynomial of B_m in this case is

$$(-1)^{m_+}\mu(\lambda-\mu)^{n-1}(\lambda+\mu)^{m_+-1}\big[\mu^2 - n\mu - \lambda\big(2\ell(m)+n+\lambda\big)\big]\ .$$

From this, it is easy to complete the proof of Case (III).

Finally, in Case (II), $m_j = 0$ for all j so that B_m is of type $(n+1)\times(n+1)$ and its the characteristic polynomial is

$$(n-\mu)(\lambda-\mu)^n - n\lambda(\lambda-\mu)^{n-1} = \mu(\lambda-\mu)^{n-1}\big(\mu-(n+\lambda)\big)\ .$$

From this the conclusion for Case (II) follows at once. □

5. Eigenfunctions of Δ_1 on the reduced Heisenberg group $\tilde{\mathbf{H}}_n$

Given $\lambda > 0$, $k \in \mathbf{N}^n$, and $m \in P$, we consider the space of matrix entries

$$\mathcal{W}_{m,\lambda,k} = \Big\{\sum_{j=1}^n a_j\langle\psi_{m+e_j},\pi_\lambda(\cdot)\psi_k\rangle\beta_j + \sum_{j=1}^n b_j\langle\psi_{m-e_j},\pi_\lambda(\cdot)\psi_k\rangle\overline{\beta}_j + c\langle\psi_m,\pi_\lambda(\cdot)\psi_k\rangle\theta\Big\} \tag{5}$$

with the same convention on $a, b \in \mathbf{C}^n$ and $c \in \mathbf{C}$ as before. Then $\mathcal{W}_{m,\lambda,k}$ has the same dimension as $\mathcal{W}_m$, and, in terms of the parameter (a,b,c), Δ_1 acts on it by the matrix A_m.

For $\lambda \in \mathbf{R}$, we denote by E_λ the space of 1-forms

$$\omega(x,y,t) = e^{-i\lambda t}\tilde{\omega}(x,y) \tag{6}$$

where $\tilde{\omega}$ has coefficients in $L^2(\mathbf{R}^{2n})$, that is

$$\tilde{\omega} = \sum_{j=1}^n f_j(x,y)\beta_j + \sum_{j=1}^n g_j(x,y)\overline{\beta}_j + h(x,y)\theta,\ \text{with } f_j, g_j, h \in L^2(\mathbf{R}^2)\,.$$

The next statement is rather obvious.

Lemma 5.1. *Assume that* $\lambda > 0$.

(i) $\mathcal{W}_{m,\lambda,k} \subset E_\lambda$. *More precisely, for the elements of* $\mathcal{W}_{m,\lambda,k}$, *we have* $f_j, g_j, h \in \mathcal{S}(\mathbf{R}^2)$.

(ii) The direct sum $\sum_{m\in P, k\in \mathbf{N}^n} \mathcal{W}_{m,\lambda,k}$ *is dense in* E_λ, *once this space is equipped with the natural norm. Hence the action of* Δ_1 *on* E_λ *has discrete spectrum with eigenvalues given by Cases (I-III) in Proposition 4.1, with* λ *fixed.*

A similar statement holds for $\lambda < 0$.

We now consider the reduced Heisenberg group $\tilde{\mathbf{H}}_n = \mathbf{H}_n/\Gamma_0$, where Γ_0 consists of the elements of the form $(0,0,r)$ with $r \in \mathbf{Z}$. The space of L^2-forms on $\tilde{\mathbf{H}}_n$ is the direct sum of the subspaces $E_{2\pi\nu}$ with $\nu \in \mathbf{Z}$.

The quotient $\tilde{\mathbf{H}}_n$ inherits a left-invariant Riemannian structure from $\mathbf{H}_n$. As long as we do not distinguish between forms on $\tilde{\mathbf{H}}_n$ and Γ_0-periodic forms on $\mathbf{H}_n$, we can use the same symbol Δ_1 to denote the Hodge Laplacian on $\tilde{\mathbf{H}}_n$. It follows that Δ_1 has a discrete spectrum on each $E_{2\pi\nu}$ for $\nu \neq 0$, described according to Proposition 4.1.

On the other hand, the action of Δ_1 on E_0 is given by

$$\begin{pmatrix} -4\,{}^t\partial_z\partial_{\bar z} I & 0 & i\sqrt{2}\partial_z \\ 0 & -4\,{}^t\partial_{\bar z}\partial_z I & -i\sqrt{2}\partial_{\bar z} \\ i\sqrt{2}\,{}^t\partial_{\bar z} & -i\sqrt{2}\,{}^t\partial_z & -4\,{}^t\partial_z\partial_{\bar z} + n \end{pmatrix}.$$

By taking the Fourier transform at $\zeta \in \mathbf{C}^n$ we obtain

$$\begin{pmatrix} |\zeta|^2 I & 0 & -\frac{1}{\sqrt{2}}\bar\zeta \\ 0 & |\zeta|^2 I & \frac{1}{\sqrt{2}}\zeta \\ -\frac{1}{\sqrt{2}}\,{}^t\zeta & \frac{1}{\sqrt{2}}\,{}^t\bar\zeta & |\zeta|^2 + n \end{pmatrix}.$$

The eigenvalues of this matrix are

$$|\zeta|^2,\ |\zeta|^2 + \frac{n}{2} \pm \sqrt{|\zeta|^2 + \tfrac{n^2}{4}}\,, \tag{7}$$

which are exactly the limits eigenvalues in Cases (I-III) in Proposition 4.1 for $\lambda \to 0$, $\lambda\big(2\ell(m) + n\big) \to |\zeta|^2$. It must be noted that Δ_1 has a continuous spectrum on E_0.

6. The spectrum of Δ_1 on a compact quotient of $\mathbf{H}_n$

In this final section we compute the eigenvalues for the Hodge Laplacian acting on 1-forms on a compact nil-manifold.

We identify the reduced Heisenberg group $\tilde{\mathbf{H}}_n$ with $\mathbf{R}^{2n} \times \mathbf{T}$ as manifolds, thus endowing $\tilde{\mathbf{H}}_n$ with the product rule

$$\big(x, y, e^{2\pi i s}\big)\big(x', y', e^{2\pi i s'}\big) = \big(x + x', y + y', e^{2\pi i((s+s') + \frac{1}{2}(xy' - yx'))}\big)\,.$$

We consider the quotient of $\tilde{\mathbf{H}}_n$ modulo the subgroup

$$\tilde\Gamma = \big\{ \big(p, q, (-1)^{p\cdot q}\big) \,:\, p, q \in \mathbf{Z}^n \big\}\,.$$

The Riemaniann structure on $\tilde{\mathbf{H}}_n$ further projects to $\tilde{\mathbf{H}}_n/\tilde\Gamma$. (We remark that, in the realization of $\mathbf{H}_n$ as the group of $(2n+1)\times(2n+1)$ real matrices

$$\begin{pmatrix} 1 & {}^t x & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

then $\tilde\Gamma$ corresponds, via the exponential map, to the subgroup of matrices such that $x, y \in \mathbf{Z}^n$, $u \in \mathbf{Z}$.)

Proposition 6.1. *The eigenvalues of the Hodge Laplacian acting on* 1*-forms on* $\tilde{\mathbf{H}}_n/\tilde\Gamma$ *are the following. Setting* $\lambda = 2\pi\nu$ *with* $\nu \in \mathbf{Z}$, $\nu \neq 0$, *and* $\mu \in \mathbf{N}$,

(1) $\lambda^2 + |\lambda|(2\mu+n) + \frac{n}{2} \pm \sqrt{\lambda^2 + |\lambda|(2\mu+n) + \frac{n^2}{4}}$,
(2) $\lambda^2 + |\lambda|(2\mu+n)$,
(3) $\lambda^2 + |\lambda|(2\mu+n-1)$,

and in addition (corresponding to $\nu = 0$*),*

(4) $|\zeta|^2$, $|\zeta|^2 + \frac{n}{2} \pm \sqrt{|\zeta|^2 + \frac{n^2}{4}}$

for $\zeta \in 2\pi(\mathbf{Z} + i\mathbf{Z})$.

In particular, the smallest non-zero eigenvalue is n.

Proof. Observe that the elements of each subspace $\mathcal{W}_{m,2\pi\nu,k}$ $(\nu \neq 0)$ can be summed over each coset of $\tilde\Gamma$ since they are rapidly decreasing. Denote by $\tilde{\mathcal{W}}_{m,2\pi\nu,k}$ the corresponding subset of the space of L^2-forms on $\tilde{\mathbf{H}}_n/\tilde\Gamma$. The space of L^2-forms on $\tilde{\mathbf{H}}_n/\tilde\Gamma$ is the direct sum of the spaces $\tilde E_{2\pi\nu}$ consisting of the forms satisfying (6) for $\lambda = 2\pi\nu$; moreover, for $\nu \neq 0$, $\tilde E_{2\pi\nu}$ is the direct sum of finite dimensional subspaces $\tilde{\mathcal{W}}_{m,2\pi\nu,k}$, whereas $\tilde E_0$ consists of the forms not depending on the last variable. Then the list of eigenvalues can be obtained directly from Proposition 4.1 and from (7). Observe that the eigenvalues of Case (I) appear in (3), and that the eigenvalues of Case (II) are in (1-2) and (3) with $\mu = 0$ and $\mu = 1$ respectively.

In order to find the smallest non-zero eigenvalue, notice that the eigenvalues in (1), (2), (3) are increasing in μ and $|\nu|$, so it suffices to consider the values corresponding to $\mu = 0$ and $|\nu| = 1$ (and the "$-$" sign in (1)), i.e.

(1) $(2\pi)^2 + 2\pi n + \frac{n}{2} - \sqrt{(2\pi)^2 + 2\pi n + \frac{n^2}{4}} = (2\pi)^2 + 2\pi(n-1)$;
(2) $(2\pi)^2 + 2\pi n$;
(3) $(2\pi)^2 + 2\pi(n-1)$.

Among the eigenvalues in (4), the smallest ones are obtained for $|\zeta| = 0$, and they are 0 and n. $\square$

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