ANALYSIS OF THE KOHN LAPLACIAN ON QUADRATIC CR MANIFOLDS

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Abstract. We study the Kohn Laplacian $□_b^{(q)}$ acting on $(0, q)$-forms on quadratic CR manifolds. We characterize the operators $□_b^{(q)}$ that are locally solvable and hypoelliptic, respectively, in terms of the signatures of the scalar components of the Levi form.

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Introduction

Let $V$ be an $n$-dimensional complex vector space, $W$ an $m$-dimensional real vector space, $W^C$ the complexification of $W$, and

$$\Phi : V \times V \longrightarrow W^C$$

a Hermitean map (i.e. $\Phi(z, z') = \overline{\Phi(z', z)}$ for every $z, z' \in V$, where complex conjugation in $W^C$ is referred to the real form $W$).

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We consider the associated quadratic manifold

\[ S = \{ (z, t + iu) \in V \times W^C : u = \Phi(z, z) \} \]

in \( n + m \) complex dimensions. Then \( S \) is a CR manifold of CR-dimension \( n \) and real codimension \( m \).

We consider the \( \bar{\partial}_b \)-complex on \( S \), its adjoint \( \bar{\partial}_b^* \) (with respect to the Lebesgue measure \( dz \, dt \) on \( S \) and to a fixed Hermitean inner product on \( V \)), and the Kohn Laplacians

\[ \square^{(q)}_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b \]

acting on \((0, q)\)-forms on \( S \).

We address the problem of determining under which assumptions on \( \Phi \) and \( q \) the operator \( \square^{(q)}_b \) satisfies either of the following properties:

(a) it is solvable, in the sense that, given any smooth \((0, q)\)-form \( \phi \) on \( S \) with compact support, there exists a \((0, q)\)-current \( \omega \) on \( S \) such that \( \square^{(q)}_b \omega = \phi \);

(b) it is hypoelliptic, i.e. any \((0, q)\)-current \( \omega \) on \( S \) such that \( \square^{(q)}_b \omega \) is smooth on an open set \( U \) is also smooth on \( U \).

CR manifolds appear in connection with different problems in complex analysis, such as extension theorems for CR functions or boundary behavior of holomorphic functions. Questions about solvability or hypoellipticity of (systems of) differential operators with multiple characteristics naturally arise in this context. We refer the reader to the monographs \([B]\), \([ChSh]\) and to \([AK]\) for accounts on these matters.

Analysis of the \( \bar{\partial}_b \)-complex on quadratic CR manifolds appears in \([RoV]\), see also \([T2]\) for a recent overview on this topic.

The form \( \Phi \) can be identified with the (vector-valued) Levi form on \( S \), and most of the properties of \( S \) have been recognized to depend on the signatures of the scalar-valued forms

\[ \Phi^\lambda(z, z') = \lambda(\Phi(z, z')) , \]

depending on \( \lambda \in W^* \).

In \([RoV]\) it was proved that, under the assumption that \( \Phi^\lambda \) is generically non-degenerate, the CR-equation \( \bar{\partial}_b u = f \) is solvable for any smooth \( \bar{\partial}_b \)-closed \((0, q)\)-form \( f \) if and only if there exists no \( \lambda \in W \) such that \( n^+(\lambda) = n - q \) and \( n^-(\lambda) = q \). The “only if” part of this statement was extended to general CR manifolds in \([AFN]\).

Another relevant part of the literature concerns subelliptic estimates for the Kohn Laplacian. In \([K]\) the so-called condition \( Y(q) \) was given as a sufficient condition for the subellipticity of the Kohn Laplacian on CR manifolds of codimension 1 (see also \([FK, RtS]\)). The condition stated in Theorem 2 below is equivalent to a natural extension of condition \( Y(q) \) to the present setting (see condition (v) in Theorem 7.1
and the remark that follows).\footnote{After writing this paper, we were informed of recent results of S-C. Chen and M-C. Shaw extending the $Y(q)$ condition and the relative sufficiency theorem to generic CR manifolds of higher codimension. These results seem to overlap with part of our Theorem 7.1.} Solvability of $\Box_b^{(q)}$ in absence of hypoellipticity does not seem to have been considered so far.

We prove that the signatures of the scalar forms $\Phi^\lambda$, as $\lambda$ varies in $W^*$, completely determine both solvability and hypoellipticity of $\Box_b^{(q)}$. One of the novelties of our results lies in the fact that we can include the case where $\Phi^\lambda$ is degenerate for every $\lambda$. Our main results are the following.

**Theorem 1.** Let $n^+(\lambda)$, resp. $n^-(\lambda)$, the number of positive, resp. negative, eigenvalues of $\Phi^\lambda$. Then $\Box_b^{(q)}$ is solvable if and only if there is no $\lambda \in W^* \setminus \{0\}$ such that $n^+(\lambda) = q$ and $n^-(\lambda) = n - q$.

**Theorem 2.** Let $n^+(\lambda)$, resp. $n^-(\lambda)$ be as in Theorem 1. Then $\Box_b^{(q)}$ is hypoelliptic if and only if there is no $\lambda \in W^* \setminus \{0\}$ such that $n^+(\lambda) \leq q$ and $n^-(\lambda) \leq n - q$.

We also prove that:

(i) property (a) is equivalent to the existence of a tempered fundamental solution for $\Box_b^{(q)}$, and also to the property that the $L^2$-null-space of $\Box_b^{(q)}$ is trivial;

(ii) when $\Box_b^{(q)}$ is not solvable, the orthogonal projection onto its $L^2$-null-space is given by convolution on $G_\Phi$ with an operator-valued distribution $C_q$ for which we give an explicit formula;

(iii) property (b) is equivalent to the fact that $\Box_b^{(q)}$ satisfies non-isotropic subelliptic estimates of order 2.

The precise statements require further notation and they can be found as Theorems 4.3, 5.2, 6.1, 6.5, and 7.1.

It is worth mentioning that there are non-trivial cases in which all the $\Phi^\lambda$ are degenerate (see the remark in Section 3a). Theorem 1 has the following consequence.

**Corollary 3.** Suppose that the Hermitean forms $\Phi^\lambda$ are degenerate for all $\lambda$. Then the operator $\Box_b^{(q)}$ is solvable for any $q$.

Theorem 1 contains some of the results in [NRS], namely Theorems 7.2.1 and 7.3.1, in the particular case where $\Phi$ is “diagonal”, i.e.

$$\Phi(z, z') = \sum_{j=1}^n z_j \overline{z_j} w_j ,$$

in an appropriate coordinate system on $V$, with $w_j \in W'$.

In the diagonal case the operator $\Box_b^{(q)}$ diagonalizes in the basis of the elementary $(0, q)$-forms $dz^I$, in the sense that

$$\Box_b^{(q)} \left( \sum_{|I|=q} f_I dz^I \right) = \sum_{|I|=q} \Box_b^{(I)} f_I dz^I ,$$
where each $\square^{(I)}_b$ acts on scalar-valued functions. This fact reduces the analysis of $\square^{(q)}_b$ to the study of each individual $\square^{(I)}_b$.

This reduction is not possible in the general case. We use the fact that a similar decoupling is possible after taking Fourier transform in the $W$-variables. However, this can be done, for each individual $\lambda \in W^*$, in a coordinate system on $V$ that depends on $\lambda$ (in fact a system that diagonalizes $\Phi^\lambda$).

Our proofs involve the identification of $S$ with a step-2 nilpotent group $G_{\Phi}$, the Fourier inversion formula on $G_{\Phi}$ and the analysis of the image of $\square^{(q)}_b$, realized as a system of harmonic oscillators, under the irreducible unitary representations of $G_{\Phi}$.

In certain cases $S$ coincides with the Šilov boundary of a Siegel domain of type $II$. This happens when the form $\Phi$ is positive w.r. to a proper cone in $W$. In fact this is equivalent to saying that there exists $\lambda \in W^*$ such that $n^+(\lambda) = n$ and $n^-(\lambda) = 0$. Under this assumption, the basic representation theory of $G_{\Phi}$ was established in [OV]. In Section 3 we give a self-contained presentation of the Fourier analysis on $G_{\Phi}$ in the general case. We note in passing that, w.r. to [OV], we prefer to privilege the Schrödinger model of the representations versus the Bargmann model.

This work has been motivated in part by the above mentioned results in [NRS]. Some of the techniques for constructing fundamental solutions and related operators are derived from [MR]; the construction of a non-smooth solution of the equation $\square^{(q)}_b \omega = 0$ in the proof of Theorem 7.1 has an analogue in [RtS].

We finally remark that, from Theorem 1, one can deduce the results in [RoV] on solvability of the CR-equation $\bar{\partial}_b u = f$, and extend them to the case where $\Phi^\lambda$ is always degenerate. We address these matters elsewhere [PR].

1. The nilpotent group associated to a quadratic manifold

Let $S$ be the quadratic manifold defined by the equation

$$\text{Im } w = \Phi(z, z),$$

with $z \in V$ and $w \in W^C$. For elements $w \in W^C$ the expressions $\text{Re } w, \text{Im } w, \bar{w}$ have the obvious meaning. For $(z', w') \in S$ the complex-affine transformation of $V \times W^C$

$$\tau_{(z', w')}(z, w) = (z + z', w + w' + 2i\Phi(z, z'))$$

maps $S$ onto itself, and

$$\tau_{(z', w')} \tau_{(z'', w'')} = \tau_{(z' + z'', w' + w'' + 2i\Phi(z', z''))}$$

$$\tau_{(z', w')}^{-1} = \tau_{(-z', -w' + 2i\Phi(z', z'))}.$$ 

Under the identification of $\tau_{(z', w')}$ with $(z', w') \in S$, this composition law defines a Lie group structure on $S$. As costumary, we introduce coordinates $(z, t) \in V \times W$ to denote the element $(z, t + i\Phi(z, z)) \in S$. Once pulled back to $V \times W$, the group multiplication takes the form

$$(z, t)(z', t') = (z + z', t + t' + 2\text{Im } \Phi(z, z')).$$
We call $G_\Phi$ this group and $\mathfrak{g}_\Phi$ its Lie algebra, that we now describe in detail.

For $v \in V$, denote by $\partial_v f$ the directional derivative of a function $f$ on $V \times W$ in the direction $v$, and let $X_v$ be the left-invariant vector field on $G_\Phi$ that coincides with $\partial_v$ at the origin. It is easy to check that

$$X_v f(z,t) = \partial_v f(z,t) + 2\text{Im} \Phi(z,v) \cdot \nabla f(z,t).$$

As we are going to introduce complex vector fields on $G_\Phi$, it is convenient to adopt the notation $Jv$ (instead of $iv$) for the complex structure on $V$. We then define $Z_v, \bar{Z}_v \in \mathfrak{g}_\Phi$ as

$$Z_v = \frac{1}{2}(X_v - iX_{Jv}) = \frac{1}{2}(\partial_v - i\partial_{Jv}) + i\Phi(z,v) \cdot \nabla_t,$$

$$\bar{Z}_v = \frac{1}{2}(X_v + iX_{Jv}) = \frac{1}{2}(\partial_v + i\partial_{Jv}) - i\Phi(z,v) \cdot \nabla_t.$$

The commutation rules are

$$[X_v, X_{v'}] = 4\text{Im} \Phi(v, v') \cdot \nabla_t,$$

$$[Z_v, Z_{v'}] = [\bar{Z}_v, \bar{Z}_{v'}] = 0,$$

$$[Z_v, \bar{Z}_{v'}] = -2i\Phi(v, v') \cdot \nabla_t.$$

Hence, $\mathfrak{g}_\Phi$ is 2-step nilpotent and, under its identification with $V \times W$,

$$[\mathfrak{g}_\Phi, \mathfrak{g}_\Phi] \subseteq \{0\} \times W \subseteq \mathfrak{j}_\Phi,$$

where $\mathfrak{j}_\Phi$ denotes the center of $\mathfrak{g}_\Phi$.

2. **The Kohn Laplacian on $G_\Phi$**

A $(0,q)$-form on $S$ is a section of the vector bundle $\Lambda^{0,q}(T^*S)$, whose fiber at each point can be identified with the exterior product $\Lambda_q = \Lambda^{0,q}(V^*)$. As every vector bundle on $S$ is trivial, we regard $(0,q)$-forms as vector-valued functions on $G_\Phi$ with values in $\Lambda_q$.

Let $\{v_1, \ldots, v_n\}$ be any orthonormal basis of $V$ with respect to the given inner product. Let $(z_1, \ldots, z_n)$ denote the coordinates on $V$ with respect to this basis. As costumary, we write

$$Z_j = \frac{1}{2}(X_{v_j} - iX_{Jv_j}), \quad \bar{Z}_j = \frac{1}{2}(X_{v_j} + iX_{Jv_j}), \quad j = 1, \ldots, n.$$

The $\bar{\partial}$ complex is defined as follows.

We denote by $d\bar{z}^I$ the $(0,q)$-form $d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$, where $I = (i_1, \ldots, i_q)$ is a strictly increasing multi-index. Given a $(0,q)$-form $\phi = \sum_{|I|=q} \phi_I d\bar{z}^I$ with smooth coefficients, we set

$$\bar{\partial}_b \phi = \sum_{|I|=q} \sum_{k=1}^n \bar{Z}_k(\phi_I) d\bar{z}_k \wedge d\bar{z}^I = \sum_{|J|=q+1} \sum_{k} \epsilon_{kl} \bar{Z}_k(\phi_I) d\bar{z}^J.$$

Here $\epsilon_{kl} = 0$ if $J \neq \{k\} \cup I$ as sets, and it equals the parity of the permutation that rearranges $(k, i_1, \ldots, i_q)$ in increasing order if $J = \{k\} \cup I$. 


The inner product on $V$ induces a Hermitian product $(\cdot, \cdot)$ on each $\Lambda_q$. In such a way that the elements $d\bar{z}^I$ form an orthonormal system.

Let $dz \, dt$ denote the left-invariant Haar measure on $G_\Phi$. On the space $L^2(G_\Phi) \otimes \Lambda_q$ of $(0, q)$-forms with coefficients in $L^2(G_\Phi)$ we consider the inner product

$$\langle \phi, \psi \rangle = \int_{G_\Phi} (\phi(z, t), \psi(z, t)) \, dzdt.$$

The formal adjoint $\bar{\partial}_b^* \bar{\partial}_b$ of $\bar{\partial}_b \bar{\partial}_b^*$ can be easily computed to yield

$$\bar{\partial}_b^* \left( \sum_{|I|=q} \phi_I d\bar{z}^I \right) = \sum_{|J|=q-1} \left( - \sum_{k, |I|=q} \epsilon_{k,j}^I Z_k \phi_I \right) d\bar{z}^J.$$

We now compute the Kohn Laplacian $\Box^{(q)}_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$.

Given two multi-indices $K$ and $L$ such that $|K| = |L| = q$ and $|\{K \cap L\}| = q - 1$, we set

$$\epsilon(K, L) = (-1)^m$$

where $m$ is the number of elements in $K \cap L$ between the unique element $k \in K \setminus L$ and the unique element $\ell \in L \setminus K$.

**Proposition 2.1.** With respect to any fixed orthonormal basis on $V$, the operator $\Box^{(q)}_b$ is represented by a matrix $(\Box_{LK})$ of scalar left-invariant differential operators on $G_\Phi$ as

$$\Box^{(q)}_b \left( \sum_K \phi_K d\bar{z}^K \right) = \sum_L \left( \sum_K \Box_{LK} \phi_K \right) d\bar{z}^L.$$

Then,

$$\Box_{LK} = -\delta_{LK} \mathcal{L} + M_{LK}$$

where $\delta_{LK}$ is the Kronecker delta, $\mathcal{L} = \frac{1}{2} \sum_{k=1}^n \left( \overline{Z}_k Z_k + Z_k \overline{Z}_k \right)$ and

$$M_{LK} = \begin{cases} \frac{1}{2} \left( \sum_{k \in K} [Z_k, \overline{Z}_k] - \sum_{k \notin K} [Z_k, \overline{Z}_k] \right) & \text{if } K = L, \\ \epsilon(K, L)[Z_k, \overline{Z}_\ell] & \text{if } |\{K \cap L\}| = q - 1, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** By (3) and (4) we have

$$\bar{\partial}_b(\bar{\partial}_b^* \phi) = \bar{\partial}_b \left( - \sum_{|J|=q-1} \left( \sum_{k, |I|=q} \epsilon_{k,j}^I Z_k \phi_I \right) d\bar{z}^I \right)$$

$$= - \sum_{|L|=q} \left( \sum_{k, \ell, |J|=q-1, |K|=q} \epsilon_{k,j}^I \epsilon_{\ell,j}^L Z_\ell \overline{Z}_\ell Z_k \phi_K \right) d\bar{z}^L.$$
On the other hand, 
\[ \bar{\partial}_b^*(\bar{\partial}_b \phi) = \bar{\partial}_b^* \left( \sum_{|H|=q+1} \left( \sum_{j,|K|=q} \epsilon_{jK}^H \bar{Z}_j \phi_K \right) d\bar{z}^H \right) \]
\[ = - \sum_{|L|=q} \left( \sum_{i,j,|H|=q+1,|K|=q} \epsilon_{jK}^H \epsilon_{iL}^H Z_i \bar{Z}_j \phi_K \right) d\bar{z}^L. \]

Hence,
\[ \square_b^{(q)}(\phi) = - \sum_{|L|=q} \sum_{|K|=q} \left( \sum_{\ell,k,|J|=q-1} \epsilon_{kJ}^L \epsilon_{\ellJ}^L \bar{Z}_k Z_k + \sum_{i,j,|H|=q+1} \epsilon_{iK}^H \epsilon_{iL}^H Z_i \bar{Z}_j \right) \phi_K d\bar{z}^L. \]

Then,
\[ \square_{LK} = - \sum_{\ell,k,|J|=q-1} \epsilon_{kJ}^L \epsilon_{\ellJ}^L \bar{Z}_k Z_k - \sum_{i,j,|H|=q+1} \epsilon_{iK}^H \epsilon_{iL}^H Z_i \bar{Z}_j. \]

When \( K = L \) the indices \( k \) and \( \ell \) are forced to be equal, as well as \( i \) and \( j \). Hence,
\[ \square_{LL} = - \left( \sum_{k \in L} \bar{Z}_k Z_k + \sum_{j \not\in L} Z_j \bar{Z}_j \right) \]
\[ = - \frac{1}{2} \sum_{k=1}^n (\bar{Z}_k Z_k + Z_k \bar{Z}_k) - \frac{1}{2} \left( \sum_{k \in L} [\bar{Z}_k, Z_k] + \sum_{k \not\in L} [Z_k, \bar{Z}_k] \right). \]

This proves the statement for the terms along the diagonal.

On the other hand, when \( K \neq L \), the coefficient \( \epsilon_{K,J}^L \epsilon_{\ellJ}^L \) is different from 0 if only if \( K = J \cup \{k\} \) and \( L = J \cup \{\ell\} \). Notice that, given \( K \) and \( L \) such that \( |\{K \cap L\}| = q - 1 \), they uniquely determine \( J, k \) and \( \ell \). Analogously, \( \epsilon_{jK}^H \epsilon_{iL}^H \neq 0 \) if and only if \( H = K \cup \{j\} = L \cup \{i\} \). Then, necessarily, \(|\{K \cap L\}| = q - 1 \) as before, and if \( k \) and \( \ell \) are as above, \( j = \ell \) and \( i = k \).

It follows that \( \square_{LK} = 0 \) unless \( |\{K \cap L\}| = q - 1 \). In this case, each of the sums in (6) reduces to one single term, and
\[ \square_{LK} = - \epsilon_{kj}^L \epsilon_{\ellj}^L \bar{Z}_k Z_k - \epsilon_{iK}^H \epsilon_{iL}^H Z_i \bar{Z}_j, \]
with \( J = K \cap L \) and \( H = K \cup L \). Furthermore,
\[ \epsilon_{K,J}^L \epsilon_{\ellJ}^L = - \epsilon_{iK}^H \epsilon_{iL}^H = \epsilon(K, L), \]
where \( \epsilon(K, L) \) is defined in (5).

Thus,
\[ \square_{LK} = \epsilon(K, L)[Z_k, \bar{Z}_j], \]
which proves the proposition. \( \square \)
3a. Representations and Plancherel formula. The irreducible unitary representations of $G_\Phi$ can be described as follows.

By Schur’s lemma, if $\pi$ is an irreducible unitary representation of $G_\Phi$, there is $\lambda \in W^*$ such that $\pi(0, t) = e^{i\lambda(t)}$. Then, by (2),

$$d\pi([Z_v, \bar{Z}_v]) = 2\lambda(\Phi(v, v')) I = 2\Phi^\lambda(v, v') I.$$  

(7)

We diagonalize $\Phi^\lambda$ with respect to an orthonormal basis $\{v_1^\lambda, \ldots, v_n^\lambda\}$ of $V$, in such a way that

$$\Phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk} \mu_j(\lambda),$$

with $\mu_j = \mu_j(\lambda) \neq 0$ for $j \leq \nu(\lambda)$ and $\mu_j = 0$ for $j > \nu(\lambda)$, where $0 \leq \nu(\lambda) = \text{rank} \Phi^\lambda \leq n$. We call

$$X_j^\lambda = X_{j,v_j^\lambda}, \quad Z_j^\lambda = \frac{1}{2}(X_j^\lambda - iY_j^\lambda), \quad \bar{Z}_j^\lambda = \frac{1}{2}(X_j^\lambda + iY_j^\lambda).$$

Then

$$d\pi([X_j^\lambda, X_k^\lambda]) = d\pi([Y_j^\lambda, Y_k^\lambda]) = 0$$

$$d\pi([X_j^\lambda, Y_k^\lambda]) = -4i\mu_j \delta_{jk} I,$$

for every $j, k$. It follows from the Stone-von Neumann theorem that there is $\eta = a + ib \in \mathbb{C}^{n-\nu(\lambda)}$ such that $\pi$ is unitarily equivalent to the representation $\pi_{\lambda, \eta}$ of $G_\phi$ on $L^2(\mathbb{R}^{\nu(\lambda)})$ such that

$$d\pi_{\lambda, \eta}(X_j^\lambda) = 2\partial_j,$$

$$d\pi_{\lambda, \eta}(Y_j^\lambda) = -2i\mu_j \xi_j$$

$$d\pi_{\lambda, \eta}(X_j^\lambda) = 2i\alpha_{j-\nu(\lambda)}$$

$$d\pi_{\lambda, \eta}(Y_j^\lambda) = 2ib_{j-\nu(\lambda)}$$

(8)

Given $\lambda$, let $(z_1^\lambda, \ldots, z_n^\lambda)$ be the coordinates on $V$ induced by the the basis $\{v_j^\lambda\}$, with $z_j^\lambda = x_j^\lambda + iy_j^\lambda$. In order to simplify the notation, we set

$$x^\lambda = (x_1^\lambda, \ldots, x_n^\lambda), \quad x' = (x_1^\lambda, \ldots, x_{\nu(\lambda)}^\lambda), \quad x'' = (x_{\nu(\lambda)+1}^\lambda, \ldots, x_n^\lambda),$$

and similarly for $y^\lambda, y', y''$. We also set $z'' = x'' + iy''$. In doing so, we must remember that $x', x''$ etc. are components that depend on $\lambda$.

The integrated form of $\pi_{\lambda, \eta}$ is, because of (8),

$$e^{i\lambda(t) + 2\text{Re}(z''t)} e^{-2i\sum_{\nu(\lambda)} \mu_j y_j^\lambda (\xi_j + x_j^\lambda)} \phi(\xi + 2x').$$

(9)

It must be observed that these formulas depend on the choice of the (ordered) basis of $V$ that diagonalizes $\Phi^\lambda$. However different choices of the basis lead to equivalent representations.
For a function $f$ on $G_\Phi$, we define

$$\pi_{\lambda,\eta}(f) = \int f(z, t)\pi_{\lambda,\eta}(z, t)^{-1} \, dz \, dt. \tag{10}$$

This definition has the effect that $\pi_{\lambda,\eta}(f * g) = \pi_{\lambda,\eta}(g)\pi_{\lambda,\eta}(f)$. The disadvantage of producing an inversion in the order of the two factors is compensated by a more natural formalism when dealing with vector-valued functions.

Observe that if $L$ is a left-invariant differential operator, then

$$\pi_{\lambda,\eta}(Lf) = d\pi_{\lambda,\eta}(L)\pi_{\lambda,\eta}(f).$$

**Definition 3.1.** Let $\nu = \max_{\lambda \in W^*} \nu(\lambda)$. We call $\Omega$ the Zariski-open set $\Omega \subseteq W^*$ such that $\nu(\lambda) = \nu$ for $\lambda \in \Omega$. For $\lambda \in \Omega$, we set

$$D(\lambda) = \prod_{j=1}^{\nu} |\mu_j|.$$

If $\nu = n$, then $D(\lambda) = |\det \Phi^\lambda|$.

**Proposition 3.2.** The Plancherel formula for $G_\Phi$ is

$$\|f\|_2^2 = \int_\Omega \int_{\mathbb{C}^{n-\nu}} \|\pi_{\lambda,\eta}(f)\|_\text{HS}^2 D(\lambda) \, d\eta \, d\lambda \tag{11}$$

for an appropriate normalization of the Lebesgue measure $d\lambda$ on $W^*$, and the inversion formula takes the form

$$f(z, t) = \int_\Omega \int_{\mathbb{C}^{n-\nu}} \text{tr}(\pi_{\lambda,\eta}(f)\pi_{\lambda,\eta}(z, t)) D(\lambda) \, d\eta \, d\lambda.$$

**Proof.** It follows from (9) that, for $\lambda \in \Omega$,

$$(\pi_{\lambda,\eta}(f)\phi)(\xi) = \int_{\mathbb{R}^{2\nu}} \mathcal{F}_{x', y', \lambda, \eta} f(x', y', 2\eta, \lambda) e^{2\iota \sum_{j=1}^{\nu} \mu_j (\xi_j - x_j)} \phi(\xi - 2x') \, dx' \, dy'$$

$$= \int_{\mathbb{R}^{2\nu}} K_{\lambda,\eta}(\xi, \xi') \phi(\xi') \, d\xi',$$

with

$$K_{\lambda,\eta}(\xi, \xi') = \mathcal{F}_{x', y', \lambda, \eta} f \left( \frac{\xi_1 - \xi'_1}{2}, \ldots, \frac{\xi_\nu - \xi'_\nu}{2}, -\mu_1 \frac{\xi_1 + \xi'_1}{2}, \ldots, -\mu_\nu \frac{\xi_\nu + \xi'_\nu}{2}, 2\eta, \lambda \right).$$

The conclusion follows from the fact that $\|\pi_{\lambda,\eta}(f)\|_\text{HS}^2 = \int |K_{\lambda,\eta}(\xi, \xi')|^2 d\xi \, d\xi'$ and from the Euclidean Plancherel formula.

When $\nu = n$, i.e. when there exists $\lambda \in W^*$ such that $\Phi^\lambda$ is non-degenerate, the Plancherel formula takes the simpler form

$$\|f\|_2^2 = \int_\Omega \|\pi_{\lambda}(f)\|_\text{HS}^2 |\det \Phi^\lambda| \, d\lambda.$$
Remark. It must be noticed that it is quite possible that all the $\Phi^\lambda$ are degenerate, even though there is no common radical that can be factored out to decompose $G_{\Phi}$ as the product of a nilpotent and an abelian group. An example is obtained by taking $V = \mathbb{C}^3$, $W = \mathbb{R}^2$, $\Phi = (\Phi_1, \Phi_2)$, with $\Phi_j(z, z') = z'^* A_j z$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We also observe that $G_{\Phi}$ is stratified (i.e. the vector fields $Z_v$ and $\bar{Z}_v$ generate the full complex Lie algebra) if and only if there is no $\lambda \neq 0$ such that $\Phi^\lambda = 0$. This remark will be recalled in Section 7.

3b. Hermite bases. In dealing with the representation $\pi_{\lambda, \eta}$ we privilege a particular orthonormal basis of $L^2(\mathbb{R}^{\nu(\lambda)})$ that depends on $\lambda$.

Denote by $h_j$ the $j$-th Hermite function on the real line:

$$h_j(t) = (2^j \sqrt{\pi j})^{-1/2} (-1)^j e^{t^2/2} \frac{d^j}{dt^j} e^{-t^2}.$$  \hfill (12)

Given a multi-index $m \in \mathbb{N}^{\nu(\lambda)}$, we set

$$h_m^\lambda(\xi) = \prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/4} h_{m_j}(\mu_j^{1/2} \xi_j).$$  \hfill (13)

As a further simplification in the notation, for $\xi \in \mathbb{R}^{\nu(\lambda)}$ we set

$$R_\lambda \xi = (|\mu_1|^{1/2} \xi_1, \ldots, |\mu_{\nu(\lambda)}|^{1/2} \xi_{\nu(\lambda)}).$$

Lemma 3.3. Let $\chi_{m,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t)$ be the matrix entry $\langle \pi_{\lambda, \eta}(x^\lambda, y^\lambda, t) h_m^\lambda, h_{m'}^\lambda \rangle$. There exist Schwartz functions $\psi_{m,m'}^\epsilon$ on $\mathbb{R}^{2\nu(\lambda)}$ depending only on $m, m'$ and on the signatures $\epsilon_j = \mu_j/|\mu_j|$ such that

$$\chi_{m,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t) = e^{i \langle \lambda(t) + 2\Re z^\epsilon \eta \rangle} \psi_{m,m'}^\epsilon \langle R_\lambda x^\lambda, R_\lambda y^\lambda \rangle.$$  

Proof. Write

$$h_m^\lambda(\xi) = \left( \prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/4} \right) e^{-\frac{1}{4} \sum_{j=1}^{\nu(\lambda)} |\mu_j| \xi_j^2} P_m(R_\lambda \xi),$$
with $P_m$ a real polynomial containing only monomials $\xi^\alpha$ with $\alpha \leq m$. Then

$$
\lambda_{m,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t)
$$

$$
eq e^{i(\lambda(t)+2\text{Re}\langle x',\eta \rangle)}\left(\prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/2}\right) \int_{\mathbb{R}^{\nu(\lambda)}} e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j y_j^\lambda (x_j^\lambda)} e^{-\frac{1}{2} \sum_{j=1}^{\nu(\lambda)} |\mu_j|((x_j^\lambda)^2)}
$$

$$
\times e^{-\frac{i}{2} \sum_{j=1}^{\nu(\lambda)} |\mu_j|((x_j^\lambda)^2)} |P_m(R_\lambda(x^\lambda))| |P_{m'}(R_\lambda(\xi))| d\xi
$$

$$
eq e^{i(\lambda(t)+2\text{Re}\langle x',\eta \rangle)}\left(\prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/2}\right) \int_{\mathbb{R}^{\nu(\lambda)}} e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j y_j^\lambda x_j^\lambda} e^{-\frac{1}{2} \sum_{j=1}^{\nu(\lambda)} |\mu_j|((x_j^\lambda)^2)}
$$

$$
\times e^{-\frac{i}{2} \sum_{j=1}^{\nu(\lambda)} |\mu_j|((x_j^\lambda)^2)} |P_m(R_\lambda(x^\lambda))| |P_{m'}(R_\lambda(\xi))| d\xi
$$

$$
eq e^{i(\lambda(t)+2\text{Re}\langle x',\eta \rangle)}\left(\prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/2}\right) \int_{\mathbb{R}^{\nu(\lambda)}} e^{-\sum_{j=1}^{\nu(\lambda)} |\mu_j|x_j^\lambda} \sum_{\alpha+\beta \leq m+m'} c_{m,m',\alpha,\beta}(R_\lambda x^\lambda)\alpha
$$

$$
\times \int_{\mathbb{R}^{\nu(\lambda)}} e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j y_j^\lambda x_j^\lambda} e^{-\sum_{j=1}^{\nu(\lambda)} |\mu_j|x_j^\lambda} (R_\lambda(\xi))^{\beta} d\xi
$$

$$
eq e^{i(\lambda(t)+2\text{Re}\langle x',\eta \rangle)}\left(\prod_{j=1}^{\nu(\lambda)} |\mu_j|^{1/2}\right) \sum_{\alpha+\beta \leq m+m'} c_{m,m',\alpha,\beta}(R_\lambda x^\lambda)\alpha
$$

$$
\times \prod_{j=1}^{\nu(\lambda)} (\text{sgn} \mu_j)^{\beta_j} \mathcal{F}(e^{-|\xi|^2} \xi^\beta)(2R_\lambda y_j).$$

The conclusion follows from the fact that the Fourier transform of a monomial times $e^{-|\xi|^2}$ equals $e^{-|\xi|^2/4}$ times a polynomial. □

**Remark.** As on the Heisenberg group, the functions $\psi_{m,m'}^{\lambda,\eta}$ can be expressed in terms of Laguerre functions (see e.g. [F]). However we shall not need their explicit expression, except for the case $m = m' = 0$. The proof of Lemma 3.3 shows that

$$
(14) \quad \lambda_{0,0}^{\lambda,\eta}(z, t) = e^{i(\lambda(t)+2\text{Re}\langle x',\eta \rangle)} e^{-\sum_{j=1}^{\nu(\lambda)} |\mu_j|x_j^\lambda}.
$$

**3c. Smoothly varying frames on $V$ and Schwartz functions on the group.**

Among the elements of $\Omega$ we select those $\lambda$ for which the number of distinct eigenvalues of $\Phi^\lambda$ is maximum. These elements form a subset $\Omega'$ which is Zariski-open, and therefore it carries the full Plancherel measure.

Fix $\lambda_0 \in \Omega'$, and let $\mu_1, \ldots, \mu_\ell$ be the distinct eigenvalues of $\Phi^{\lambda_0}$, with multiplicities $m_1, \ldots, m_\ell$ respectively. By the implicit function theorem, there is a connected neighborhood $U$ of $\lambda_0$ in $\Omega'$ on which one can define real-analytic functions $\mu_i(\lambda)$ for $1 \leq i \leq \ell$, such that $\mu_i(\lambda_0) = \mu_i$ and $\mu_i(\lambda)$ is an eigenvalue of $\Phi^\lambda$ with multiplicity $m_i$. Also, $\mu_i(\lambda) \neq 0$ for $\lambda \in U$, except for at most one $i$ (in case $\nu < n$), for which $\mu_i(\lambda)$ is identically 0 on $U$. 
For each \( i \) and each \( \lambda \in U \), we can also find an orthonormal basis of the \( \mu_i(\lambda) \)-eigenspace of \( \Phi^\lambda \), in such a way that the \( k \)-th basis element depends analytically on \( \lambda \) for every \( k \).

At this point, we relabel the eigenvalues, allowing repetitions according to their multiplicity, and ordering them in such a way that \( \mu_{\nu+1}(\lambda) = \cdots = \mu_n(\lambda) = 0 \).

Hence, for each \( \lambda \in U \) we have an orthonormal basis \( \{ v_1^\lambda, \ldots, v_n^\lambda \} \) of \( V \), such that \( v_j^\lambda \) depends analytically on \( \lambda \) and

\[
\Phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk} \mu_j(\lambda).
\]

The corresponding coordinate functions \( z_j^\lambda = x_j^\lambda + iy_j^\lambda \) are then real-analytic in \( \lambda \) for \( \lambda \in U \).

Define the representations \( \pi_{\lambda,\eta} \) for \((\lambda, \eta) \in U \times \mathbb{C}^{n-\nu}\) according to this choice of the coordinates. If \( m, m' \in \mathbb{N}^\nu \), we set

\[
\hat{f}(\lambda, \eta; m, m') = \langle \pi_{\lambda,\eta}(f)h_m^\lambda, h_{m'}^\lambda \rangle = \int f(x^\lambda, y^\lambda, t)\chi_{\nu,m'}^\lambda(x^\lambda, y^\lambda, t) \, dx^\lambda \, dy^\lambda \, dt.
\]

**Lemma 3.4.** Let \( \phi(\lambda, \eta) \) be a \( C^\infty \) function with compact support in \( U \times \mathbb{C}^{n-\nu} \), and let \( m, m' \in \mathbb{N}^\nu \). There is a function \( f \in \mathcal{S}(G^\Phi) \) such that

(i) \( \pi_{\lambda,\eta}(f) = 0 \) for \( \lambda \not\in U \);

(ii) \( \hat{f}(\lambda, \eta; m, m') = \phi(\lambda, \eta) \) for \((\lambda, \eta) \in U \times \mathbb{C}^{n-\nu} \);

(iii) \( \hat{f}(\lambda, \eta; p, p') = 0 \) for \((p, p') \neq (m, m')\) and \((\lambda, \eta) \in U \times \mathbb{C}^{n-\nu} \).

**Proof.** Define

\[
f(z, t) = \int_{U \times \mathbb{C}^{n-\nu}} \phi(\lambda, \eta)\chi_{\nu,m'}^\lambda(x^\lambda, y^\lambda, t) D(\lambda) \, d\lambda \, d\eta,
\]

where \( x^\lambda, y^\lambda \) are the real coordinates of \( z \in V \) in the basis \( \{ v_1^\lambda \} \). As \( U \) is connected and contained in \( \Omega' \), the signatures \( \epsilon_j \) of the eigenvalues \( \mu_j(\lambda) \) are constant on \( U \).

Therefore,

\[
f(z, t) = \int_{U \times \mathbb{C}^{n-\nu}} \phi(\lambda, \eta)e^{i(\lambda(t)+2\text{Re}(z^\nu, \eta))} \psi_{\nu,m'}^\epsilon(R_\lambda x^\lambda, R_\lambda y^\lambda) D(\lambda) \, d\lambda \, d\eta,
\]

with \( \psi_{m',m}^\epsilon \) as in Lemma 3.3 and \( \epsilon \) fixed.

The fact that \( f \) is a Schwartz function can be easily deduced from the smoothness of the functions \( x_j^\lambda, y_j^\lambda, \mu_j(\lambda) \) and the fact that the \( \mu_j(\lambda) \) are bounded away from zero on the support of \( \phi \).

Taking Fourier transform in \( t \), we find that

\[
\int f(z, t)e^{-i\lambda(t)} \, dt = 0
\]

identically for \( \lambda \not\in U \), which implies that \( \pi_{\lambda,\eta}(f) = 0 \) for \( \lambda \not\in U \).
From the definition of $\chi_{\lambda,\eta}^{m,m'}$, we have that

\[
f(z, t) = \int_{U \times \mathbb{C}^{n-\nu}} \phi(\lambda, \eta) \langle \pi_{\lambda,\eta}(x^\lambda, y^\lambda, t) h_{m'}^\lambda, h_m^\lambda \rangle D(\lambda) \, d\lambda \, d\eta,
\]

where $A_{m,m'}^{\lambda,\eta} h_{m'}^\lambda = \phi(\lambda, \eta) h_m^\lambda$ and $A_{m,m'}^{\lambda,\eta} h_p^\lambda = 0$ if $p \neq m$.

By uniqueness of the Fourier transform, it follows that $\pi_{\lambda,\eta}(f) = A_{m,m'}^{\lambda,\eta}$ for $(\lambda, \eta) \in U \times \mathbb{C}^{n-\nu}$.

Hence

\[
\hat{f}(\lambda, \eta; p, p') = \langle A_{m,m'}^{\lambda,\eta} h_p^\lambda, h_{p'}^\lambda \rangle,
\]

and the conclusion follows.

3d. Fourier transform of vector-valued functions. Let $f$ be a function on $G_\Phi$ taking values in a finite-dimensional complex space $E$. Following (10), we define

\[
\pi_{\lambda,\eta}(f) = \int_{G_\Phi} \pi_{\lambda,\eta}(z, t)^{-1} \otimes f(z, t) \, dz \, dt \in \text{End} \left( L^2(\mathbb{R}^{\nu(\lambda)}) \right) \otimes E.
\]

Let $K$ be a function on $G_\Phi$ with values in $\text{Hom}(E, F)$, with $E$ and $F$ finite-dimensional spaces. Then the convolution operator

\[
f \mapsto f * K(z, t) = \int_{G_\Phi} K((w, u)^{-1}(z, t)) f(w, u) \, dw \, du
\]

maps $E$-valued functions into $F$-valued functions and it is left-invariant. We have

\[
\pi_{\lambda,\eta}(f * K) = \pi_{\lambda,\eta}(K) \pi_{\lambda,\eta}(f),
\]

if we understand that the composition of $T \otimes A \in \text{End} \left( L^2(\mathbb{R}^{\nu(\lambda)}) \right) \otimes \text{Hom}(E, F)$ with $U \otimes v \in \text{End} \left( L^2(\mathbb{R}^{\nu(\lambda)}) \right) \otimes E$ is $TU \otimes Av \in \text{End} \left( L^2(\mathbb{R}^{\nu(\lambda)}) \right) \otimes F$.

Let now $(\cdot, \cdot)$ be a Hermitian inner product on $E$ and let

\[
\langle f, g \rangle = \int_{G_\Phi} \langle f(z, t), g(z, t) \rangle \, dz \, dt.
\]

be the induced inner product on $L^2(G_\Phi) \otimes E$.

Introducing an orthonormal basis on $E$, one can easily express this pairing in terms of the Fourier transform of $f$ and $g$, using the polarized form of the Plancherel formula. In order to obtain a coordinate-free formula, consider the inner product $\langle\cdot, \cdot\rangle$ on $\text{HS}(L^2(\mathbb{R}^\nu)) \otimes E$ such that

\[
\langle T \otimes v, U \otimes w \rangle = \text{tr} (TU^*(v, w)),
\]

where $T, U$ are Hilbert-Schmidt operators on $L^2(\mathbb{R}^\nu)$, $v, w \in E$. We then have

\[
\langle f, g \rangle = \int_\Omega \int_{\mathbb{C}^{n-\nu}} \langle \pi_{\lambda,\eta}(f), \pi_{\lambda,\eta}(g) \rangle D(\lambda) \, d\eta \, d\lambda.
\]
We shall use this formula to define vector-valued distributions on $G\Phi$. In doing so, we adopt the convention that the pairing $\langle u, f \rangle$ between a distribution $u$ and a test function $f$ is linear in $u$ and anti-linear in $f$.

3e. The Fourier transform of $\Box_b^{(q)}$. We shall be primarily concerned with the situation where $E = F = \Lambda_q = \Lambda^{(0,q)}_V$, with the inner product naturally inherited from the inner product on $V$. If $\phi$ is a Schwartz $(0,q)$-form on $G\Phi$, then $\pi_{\lambda,\eta}(\phi) \in \text{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes \Lambda_q$.

We want to describe the image of $\Box_b^{(q)}$ under $\pi_{\lambda,\eta}$. Observe that $d\pi_{\lambda,\eta}(\Box_b^{(q)}) \in \text{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes \text{End}(\Lambda_q)$.

**Proposition 3.5.** Let $\{v_1^\lambda, \ldots, v_n^\lambda\}$ be an orthonormal basis of $V$ that diagonalizes $\Phi^\lambda$, and let $(z_1^\lambda, \ldots, z_n^\lambda)$ be the corresponding coordinates on $V$. For a strictly increasing multi-index $L$ with $|L| = q$, denote by $\omega_L^\lambda$ the elementary form $d\bar{z}^\lambda_L$. Then, for $\phi = \sum_{|L|=q} \phi_L \otimes \omega_L^\lambda \in S(\mathbb{R}^{\nu(\lambda)}) \otimes \Lambda_q$, we have

$$d\pi_{\lambda,\eta}(\Box_b^{(q)}) \left( \sum_{|L|=q} \phi_L \otimes \omega_L^\lambda \right) = \sum_{|L|=q} (H_{\lambda,\eta} + \alpha_L^\lambda) \phi_L \otimes \omega_L^\lambda,$$

where

$$H_{\lambda,\eta} = -\sum_{j=1}^{\nu(\lambda)} \left( \partial_{\xi_j}^2 - \mu_j^2 \xi_j^2 \right) + |\eta|^2,$$

and

$$\alpha_L^\lambda = \sum_{k \in L} \mu_k - \sum_{k \not\in L} \mu_k. \tag{18}$$

In particular, $d\pi_{\lambda,\eta}(\Box_b^{(q)})$ acts diagonally with respect to the basis $\{h_m^\lambda \otimes \omega_L^\lambda\}$ of $L^2(\mathbb{R}^{\nu(\lambda)}) \otimes \Lambda_q$. Precisely,

$$d\pi_{\lambda,\eta}(\Box_b^{(q)})(h_m^\lambda \otimes \omega_L^\lambda) = \left( \sum_{j=1}^{\nu(\lambda)} |\mu_j| (1 + 2m_j) + |\eta|^2 + \alpha_L^\lambda \right) h_m^\lambda \otimes \omega_L^\lambda. \tag{19}$$

**Proof.** For the given orthonormal basis we write $Z_j^\lambda, Z_k^\lambda$ be as in (2). From (7) we have

$$d\pi_{\lambda,\eta}(\Box_b^{(q)})(Z_j^\lambda, Z_k^\lambda) = 2\delta_{jk} \mu_k.$$

Notice that $d\pi_{\lambda,\eta}(\mathcal{L}) = \sum_{j=1}^{\nu(\lambda)} \partial_{\xi_j}^2 - \mu_j^2 \xi_j^2 - |\eta|^2 = -\mathcal{H}$. The result now follows from Proposition 2.1 and from the fact that the Hermite function $h_j(t)$ on the real line is an eigenfunction of the Hermite operator $-(d/dt)^2 + t^2$ with eigenvalue $2j + 1$. □
The next result will be needed in Section 6. When \( f \in \mathcal{S}(G_{\Phi}) \otimes E \) we still denote by \( \hat{f} \) the \( E \)-valued function

\[
\hat{f}(\lambda, \eta; m, m') = \int f(x^\lambda, y^\lambda, t) \chi_{m',m}^{\lambda,\eta}(x^\lambda, y^\lambda, t) \, dx^\lambda \, dy^\lambda \, dt.
\]

With an abuse of notation, we write

\[
\hat{f}(\lambda, \eta; m, m') = \langle \pi_{\lambda,\eta}(f) h^\lambda_m, h^\lambda_{m'} \rangle,
\]
keeping in mind that the inner product on the right-hand side is vector-valued.

We also denote by \( |\cdot| \) the norm on \( E \).

**Lemma 3.6.** For each positive integer \( N \), there exist a Sobolev norm \( \| \cdot \|_{N'} \) and a constant \( c_N > 0 \) such that for all \( f \in \mathcal{S}(G_{\Phi}) \otimes E \) we have

\[
|\hat{f}(\lambda, \eta; m, m')| \leq c_N \|f\|_{N'} (1 + |\eta|^2)^N (1 + |\lambda|)^N (1 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|)^N.
\]

**Proof.** Consider the operator \( \mathcal{L} \otimes I \) acting on \( \mathcal{S}(G_{\Phi}) \otimes E \), where \( I \) denotes the identity map on \( E \). Then

\[
\big( (\mathcal{L} \otimes I)f \big) \hat{}(\lambda, \eta; m, m) = \langle \pi_{\lambda,\eta}(\mathcal{L} \otimes I)f) h^\lambda_m, h^\lambda_{m'} \rangle = \langle \pi_{\lambda,\eta}(\mathcal{L} \otimes I)\pi_{\lambda,\eta}(f) h^\lambda_m, h^\lambda_{m'} \rangle = \langle \pi_{\lambda,\eta}(f) h^\lambda_m, d\pi_{\lambda,\eta}(\mathcal{L} \otimes I)h^\lambda_{m'} \rangle.
\]
(The fact that \( d\pi_{\lambda,\eta}(\mathcal{L} \otimes I) \) is self-adjoint on \( L^2(\mathbb{R}^\nu) \otimes E \) follows from the polarized form of the Plancherel formula, see (17).)

Then,

\[
\big( (\mathcal{L} \otimes I)f \big) \hat{}(\lambda, \eta; m, m) = (|\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|) \langle \pi_{\lambda,\eta}(f) h^\lambda_m, h^\lambda_{m'} \rangle = (|\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|) \hat{f}(\lambda, \eta; m, m).
\]

The conclusion follows easily, once we observe that, from (15) and Lemma 3.3,

\[
(1 + |\eta|^2)(1 + |\lambda|^2) \hat{f}(\lambda, \eta; m, m) = \int f(x^\lambda, y^\lambda, t) \mathcal{P}_{t,z''}^{\lambda,\eta} \chi_{m',m}^{\lambda,\eta}(x^\lambda, y^\lambda, t) \, dx^\lambda \, dy^\lambda \, dt,
\]
for a constant coefficient differential operator \( \mathcal{P}_{t,z''} \) in \( t \) and \( z'' \). \( \square \)
4. NON-SOLVABILITY OF $\square^{(q)}_b$

In this section we prove the negative result in Theorem 1. In fact we prove the stronger statement that, under the given assumption, the operator $\square^{(q)}_b$ is not even locally solvable.\footnote{A differential operator $P$ is said to be locally solvable at a point $x_0$ if there exists an open neighborhood $U$ of $x_0$ such that for any test function $f$ with support contained in $U$ there exists a distribution $u$ such that $Pu = f$ on $U$. For translation invariant operators, local solvability does not depend on $x_0$.}

We will use the following necessary criterion for local solvability, which is the vector-valued extension of the corresponding version for scalar operators, due to Corwin and Rothschild [CoRt].

**Lemma 4.1.** Let $\mathcal{M}$ be a homogeneous left-invariant differential operator from $\mathcal{S}(G_\Phi) \otimes E$ to $\mathcal{S}(G_\Phi) \otimes F$, and let $\mathcal{M}^* : \mathcal{S}(G_\Phi) \otimes F' \to \mathcal{S}(G_\Phi) \otimes E'$ be the adjoint operator. Suppose that there exists a non-trivial $\phi \in \mathcal{S}(G_\Phi) \otimes F'$ such that $\mathcal{M}^* \phi = 0$. Then $\mathcal{M}$ is not locally solvable.

**Proof.** We argue by contradiction. By Hörmander’s condition [Hö], $\mathcal{M}$ is locally solvable at a point $(z_0, t_0) \in G_\Phi$ if and only if there exist a neighborhood $U$ of $(z_0, t_0)$, $k \in \mathbb{N}$, and a constant $c > 0$ such that

$$\|g\|_{-k} \leq c\|\mathcal{M}^* g\|_k$$

for all $g \in C^\infty_0(U) \otimes F'$, where $\| \cdot \|_r$ denotes the Sobolev norm.

Suppose that $\mathcal{M}$ is locally solvable. Using the homogeneity of $\mathcal{M}$, the proof of Lemma 1 in [CoRt] goes through without changes to the case of vector-valued functions to imply that there exists $k \in \mathbb{N}$ such that the following holds. For each $\psi \in C^\infty_0(G_\Phi) \otimes F$ there exists $\{f_m\} \subseteq C^\infty_0(G_\Phi) \otimes E$ such that: (i) $\text{supp } f_m \subseteq \{(z, t) \mid |z| \leq m + 1\}$; (ii) $\mathcal{M} f_m = \psi$ on $\{(z, t) \mid |z| \leq m\}$; (iii) $|\mathcal{M} f_m(z, t)| \leq m^k$.

Given $\phi$ as in the statement, let $\psi \in C^\infty_0(G_\Phi) \otimes F$. Then

$$\left| \int_{|z,t| \leq m+1} \langle \phi(z,t), \psi(z,t) \rangle \, dzdt \right|$$

$$= \left| \int_{|z,t| \leq m+1} \langle \phi(z,t), \psi(z,t) - \mathcal{M} f_m(z,t) \rangle \, dzdt \right|$$

$$= \left| \int_{m \leq |z,t| \leq m+1} \langle \phi(z,t), \psi(z,t) - \mathcal{M} f_m(z,t) \rangle \, dzdt \right|$$

$$\leq c_\phi \int_{m \leq |z,t| \leq m+1} |\phi(z,t)| m^k \, dzdt,$$

which tends to 0 as $m \to +\infty$. Then $\phi = 0$, a contradiction. Hence, $\mathcal{M}$ is not locally solvable. $\square$

We state for future reference a lemma whose proof is essentially contained in the last part of Section 3.
Lemma 4.2. Given the partial differential equation \( (H_{\lambda, \eta} + \alpha^\lambda_L) f = 0 \), the following conditions are equivalent:

(i) there exists a non-trivial solution \( f \in S(\mathbb{R}^{\nu(\lambda)}) \);

(ii) \( \eta = 0 \) and the multi-index \( L \) is such that \( \mu_k \leq 0 \) for \( k \in L \) and \( \mu_k \geq 0 \) for \( k \notin L \).

Recall that, given \( \lambda \in W^* \), we denote by \( n^+(\lambda) \) the number of positive eigenvalues of the form \( \Phi^\lambda \), and by \( n^-(\lambda) \) the number of negative eigenvalues.

**Definition 4.3.** We define \( \Omega_q \) to be the cone
\[
\Omega_q = \{ \lambda : n^+(\lambda) = q, n^-(\lambda) = n - q \}
\]}

Therefore, Theorem 1 can be restated by saying that \( \Box_b^{(q)} \) is (locally) solvable if and only if \( \Omega_q \) is empty (or equivalently if and only if \( \Omega_{n-q} = -\Omega_q \) is empty).

**Theorem 4.4.** Assume that \( \Omega_q \) is non-empty. Then there is a non-trivial \( \omega \in S(G_{\Phi}) \otimes \Lambda_q \) such that \( \Box_b^{(q)} \omega = 0 \).

**Proof.** Under the given assumptions, \( \Omega_{n-q} = \Omega_{n-q} \cap \Omega' \) is non-empty. As \( \nu = n \), there is no \( \eta \) in the parameters for the generic irreducible representations of \( G_{\Phi} \).

Let \( \lambda_0 \in U \subset \Omega_{n-q} \) be as in Section 3. Let \( z_j^\lambda = x_j^\lambda + iy_j^\lambda \) be the coordinates adapted to a corresponding smoothly varying frame on \( V \), and let \( \omega^\lambda_L = d\bar{z}^\lambda_L \), as in Section 3. Then \( \omega^\lambda_L \) varies smoothly with \( \lambda \).

Let \( \bar{L} \) be the multi-index of length \( q \) formed by those \( k \) for which \( \mu_k(\lambda) < 0 \) on \( U \). Slightly modifying the construction in the proof of Lemma 3.4 we take a \( C^\infty \)-function \( \phi(\lambda) \) with compact support in \( U \) and set
\[
\omega(z,t) = \int_U \phi(\lambda)\chi^\lambda_{0,0}(z,t)D(\lambda)\omega^\lambda_L d\lambda .
\]

It follows easily from (14) that \( \omega \in S(G_{\Phi}) \otimes \Lambda_q \). As in the proof of Lemma 3.4, it is easily shown that the only irreducible unitary representations of \( G_{\Phi} \) for which \( \pi_\lambda(\omega) \neq 0 \) are those with \( \lambda \) in the support of \( \phi \). For these \( \lambda \) we have
\[
\pi_\lambda(\omega) = \phi(\lambda)A^\lambda_{0,0} \otimes \omega^\lambda_L ,
\]
where \( A_{0,0} \) is the orthogonal projection onto the one-dimensional subspace of \( L^2(\mathbb{R}^n) \) spanned by \( h^\lambda_0 \).

It follows from Proposition 3.5 that, for \( \lambda \) in the support of \( \phi \),
\[
\pi_\lambda(\Box_b^{(q)} \omega) = \left( \sum_{j=1}^n |\mu_j| + \alpha^\lambda_L \right) \phi(\lambda)A^\lambda_{0,0} \otimes \omega^\lambda_L = 0 ,
\]
because
\[
\alpha^\lambda_L = \sum_{k \in L} \mu_k(\lambda) - \sum_{k \notin L} \mu_k(\lambda) = - \sum_{k=1}^n |\mu_k(\lambda)| .
\]
By uniqueness of the Fourier transform, $\Box_b^{(q)} \omega = 0$. □

5. The orthogonal projection on the null space of $\Box_b^{(q)}$

Assume that $\Omega_q$ is non-empty. It follows from Theorem 4.4 that the null space of $\Box_b^{(q)}$ is non-trivial in the space of Schwartz $(0, q)$-forms. We shall determine the null space in $L^2(G_{\Phi}) \otimes \Lambda_q$ and obtain an expression for the corresponding orthogonal projector involving a kind of Laplace transform.

Let $\{U_j\}$ be a locally finite open covering of $\Omega_{n-q}'$ such that each $U_j$ is relatively compact in $\Omega_{n-q}'$ and for each $\lambda \in U_j$ there is an orthonormal coordinate system $(z_1^\lambda, \ldots, z_n^\lambda)$ on $V$ that varies smoothly with $\lambda$ and diagonalizes $\Phi^\lambda$ as $\Phi^\lambda(z, z) = \sum_{k=1}^n \mu_k |z_k^\lambda|^2$. Let $\bar{L}$ the multi-index of length $q$ containing those $k$ for which $\mu_k < 0$.

Let also $\{\rho_j\}$ be a smooth partition of unity on $\Omega_{n-q}'$ subordinated to the given covering.

**Lemma 5.1.** Let $\omega \in L^2(G_{\Phi}) \otimes \Lambda_q$. The following are equivalent:

(i) $\omega$ is in the null space of $\Box_b^{(q)}$.

(ii) $\pi_\lambda(\omega) = 0$ a.e. outside of $\Omega_{n-q}'$ and, a.e. on each $U_j$, $\pi_\lambda(\omega) = T_j^\lambda \otimes \omega^\lambda_L$, where $T_j^\lambda$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, with range in the linear span of $h_0^\lambda$.

**Proof.** A form $\omega$ in $L^2(G_{\Phi}) \otimes \Lambda_q$ is in the null space of $\Box_b^{(q)}$ if and only if, for every $\tau \in S(G_{\Phi}) \otimes \Lambda_q$,

$$\langle \Box_b^{(q)} \omega, \tau \rangle = \langle \omega, \Box_b^{(q)} \tau \rangle$$

$$= \int_{\Omega} \langle \pi_\lambda(\omega), d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda) \, d\lambda$$

$$= 0.$$  

Assume that $\omega$ satisfies (ii). Then

$$\langle \Box_b^{(q)} \omega, \tau \rangle = \int_{\Omega_{n-q}} \langle \pi_\lambda(\omega), d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda) \, d\lambda$$

$$= \sum_j \int_{U_j} \rho_j(\lambda) \langle T_j^\lambda \otimes \omega_L^\lambda, d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda) \, d\lambda$$

$$= \sum_j \int_{U_j} \rho_j(\lambda) \sum_{m \in \mathbb{N}^n} \langle T_j^\lambda h_m^\lambda \otimes \omega_L^\lambda, d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) h_m^\lambda \otimes \omega_L^\lambda \rangle D(\lambda) \, d\lambda$$

$$= \sum_j \int_{U_j} \rho_j(\lambda) \sum_{m \in \mathbb{N}^n} \langle d\pi_\lambda(\Box_b^{(q)}) (T_j^\lambda h_m^\lambda \otimes \omega_L^\lambda), \pi_\lambda(\tau) h_m^\lambda \otimes \omega_L^\lambda \rangle D(\lambda) \, d\lambda$$

$$= 0.$$
by Proposition 3.5, because $T^\lambda h_m^\lambda \otimes \omega_L^\lambda$ is a scalar multiple of $h_0^\lambda \otimes \omega_L^\lambda$ and $\alpha_L^\lambda = -\sum_{k=1}^n |\mu_k|$. Hence (ii) implies (i).

Assume now that (i) holds, i.e. that (20) is satisfied for every Schwartz form $\tau$.

Take $\lambda_0 \in \Omega'$ and let $U$ be a neighborhood of $\lambda_0$ allowing a smoothly varying frame with coordinates $(z_1^\lambda, \ldots, z_n^\lambda)$ of $V$ for $\lambda \in U$. Let $\phi$ be a smooth function with compact support in $U$, $m, m' \in \mathbb{N}^n$ and $L$ a multi-index of length $q$. We set

$$
\tau(z,t) = \int_U \phi(\lambda) \chi_{m',m}(x^\lambda,y^\lambda,t) \omega_L^\lambda D(\lambda) \, d\lambda .
$$

As in the proof of Lemma 3.4, we find that, for $\lambda \in U$, $\pi_\lambda(\tau) = \phi(\lambda)A_{m,m'}^\lambda \otimes \omega_L^\lambda$ for $\lambda \in U$, where $A_{m,m'}^\lambda h_p^\lambda = \delta_{m,p} h_{m'}^\lambda$, and 0 otherwise.

Therefore

$$
\pi_\lambda(\square^q_0 \tau) = \left( \sum_{j=1}^n |\mu_j|(1 + 2m'_j) + \alpha_L^\lambda \right) \phi(\lambda)A_{m,m'}^\lambda \otimes \omega_L^\lambda
$$

for $\lambda \in U$ and 0 otherwise.

Since (20) holds,

$$
\int_U \left( \sum_{j=1}^n |\mu_j|(1 + 2m'_j) + \alpha_L^\lambda \right) \phi(\lambda) \langle \pi_\lambda(\omega), A_{m,m'}^\lambda \otimes \omega_L^\lambda \rangle D(\lambda) \, d\lambda = 0
$$

for every $\phi$. So, either $\sum_{j=1}^n |\mu_j|(1 + 2m'_j) + \alpha_L^\lambda = 0$, or $\langle \pi_\lambda(\omega), A_{m,m'}^\lambda \otimes \omega_L^\lambda \rangle = 0$ for a.e. $\lambda \in U$.

The first condition is satisfied if and only if $m' = 0$, $U \subset \Omega'_{n-q}$ and $L = \bar{L}$. This concludes the proof. □

In order to describe the projection operator, observe that, by translation invariance, it must have the form

$$
\omega \mapsto \omega \ast C_q ,
$$

where $C_q$ is a distribution taking values in $\text{End} (\Lambda_q)$. It is important at this point to make the following remark.

As we have already observed, each point in $\Omega'_{n-q}$ has a neighborhood $U$ on which we can define a smooth function $\lambda \mapsto \omega_L^\lambda$ with values in $\Lambda_q$ and where the multi-index $\bar{L}$ consists of the indices $j$ such that $\mu_j < 0$.

In general, this function cannot be extended to all of $\Omega'_{n-q}$. If two neighborhoods $U$ and $U'$ intersect, then the two corresponding choices of $\omega_L^\lambda$ differ by a scalar factor of absolute value 1.

This implies however that, at each $\lambda \in U \cap U'$, the two corresponding orthogonal projections of $\Lambda_q$ onto the linear span of $\omega_L^\lambda$ coincide. This orthogonal projection, that we call $P_{\lambda}$, is hence well defined and smooth on all of $\Omega'_{n-q}$.

\[3\]This is possible if $\Omega'_{n-q}$ is simply connected. One way to overcome the topological problems is to lift to the universal covering of $\Omega'_{n-q}$, as in [MR]. We have chosen to avoid this by introducing partitions of unity when strictly necessary.
In fact \( P_\lambda \) is well defined and smooth on all of \( \Omega_{n-q} \). In order to see this, we must regard the elements of \( \Lambda_q \) as multi-linear functionals on \( V \otimes \mathbb{R} \mathbb{C} \). The action of \( P_\lambda \) on a \((0,q)\)-form is then the composition of the form itself with the projection, in each component, onto the linear span of the \((0,1)\)-eigenvectors of \( \Phi_\lambda \) with negative eigenvalues. This operation is well defined and smooth on all of \( \Omega_{n-q} \).

**Theorem 5.2.** The orthogonal projection of \( L^2(R^n) \otimes \Lambda_q \) onto the null space of \( \Box^q_b \) maps a form \( \omega \) into \( \omega \ast C_q \), where \( C_q \in S'(\mathbb{R}^n) \otimes \text{End} (\Lambda_q) \) is given by

\[
C_q(z, t) = \int_{\Omega_{n-q}} e^{i\lambda(t)} e^{-|\Phi_\lambda|(z,z)} P_\lambda^z D(\lambda) \, d\lambda ,
\]

where \(|\Phi_\lambda|(z, z) = \sum_{k=1}^n |\mu_k||z^\lambda_k|^2\).

The formula for \( C_q \) must be interpreted in the sense of distributions. To be precise, if \( \psi \in S(G_\Phi) \otimes \text{End} (\Lambda_q) \), we have

\[
\langle C_q, \psi \rangle = \int_{\Omega_{n-q}} e^{i\lambda(t)} e^{-|\Phi_\lambda|(z,z)} \text{tr} \left( P_\lambda^z \psi(z, t)^* \right) D(\lambda) \, d\lambda \, dz \, dt
\]

\[
= \int_V \int_{\Omega_{n-q}} e^{-|\Phi_\lambda|(z,z)} \text{tr} \left( P_\lambda^z \mathcal{F}_t \psi(z, \lambda)^* \right) D(\lambda) \, d\lambda \, dz
\]

\[
= \sum_j \int_V \int_{U_j} e^{-|\Phi_\lambda|(z,z)} \langle \omega^\lambda_L, \mathcal{F}_t \psi(z, \lambda) \omega^\lambda_L \rangle D(\lambda) \, d\lambda \, dz .
\]

**Proof.** By Lemma 5.1, the Fourier transform of \( C_q \) is given by \( \pi_\lambda(C_q) = 0 \) for \( \lambda \in \Omega \setminus \Omega_{n-q} \) and \( \pi_\lambda(C_q) = A_{0,0}^\lambda \otimes P_\lambda^z \) for \( \lambda \in \Omega_{n-q} \). Therefore, if \( \psi \in S(G_\Phi) \otimes \text{End} (\Lambda_q) \),

\[
\langle C_q, \psi \rangle = \int_{\Omega_{n-q}} \langle A_{0,0}^\lambda \otimes P_\lambda^z, \pi_\lambda(\psi) \rangle D(\lambda) \, d\lambda
\]

\[
= \int_{\Omega_{n-q}} \text{tr} \left( P_\lambda^z \hat{\psi}(\lambda, 0; 0, 0)^* \right) D(\lambda) \, d\lambda .
\]

By (14),

\[
\hat{\psi}(\lambda, 0; 0, 0) = \int_{G_\Phi} \psi(z, t) \chi_{0,0}^\lambda(z, t) \, dz \, dt
\]

\[
= \int_V \mathcal{F}_t \psi(z, \lambda) e^{-|\Phi_\lambda|(z,z)} \, dz ,
\]

and this gives the proof. \( \square \)

The formula for \( C_q \) generalizes the classical Gindikin formula for the Cauchy-Szegö kernel on the \( \mathring{\text{Silov}} \) boundary of a Siegel domain of type II (see [G] Theorem 5.3 and [KS]). As it was mentioned in the Introduction, \( S \) is the \( \mathring{\text{Silov}} \) boundary of such a
domain if and only if $\Omega_n$ is non-empty. If this is the case, let $\Gamma \subset W$ be the conic hull of \{\Phi(z, z) : z \in V\}, and let
\[ D = \{(z, w) : \text{Im} w - \Phi(z, z) \in \Gamma\} \]
be the corresponding Siegel domain. Then $S$ is the Silov boundary of $D$. Since $\Omega_n$ is the dual open cone of $\Gamma$, according to Gindikin’s formula,
\[ C_0(z, t) = \int_{\Omega_n} e^{i\lambda(t)} e^{-\Phi^{\lambda}(z, z)} D(\lambda) d\lambda \]
is the (scalar-valued) convolution kernel of the orthogonal projection of $L^2(G_\Phi)$ onto the Hardy space consisting of boundary values of holomorphic $H^2$-functions on $D$ (see [OV]).

6. **Fundamental solution for $\Box^{(q)}_b$**

In this section we prove the positive part in Theorem 1 by constructing a tempered fundamental solution $K = K_q$ for $\Box^{(q)}_b$ when $\Omega_q$ is empty. Minor modifications to the formula will give a relative fundamental solution when $\Omega_q$ is non-empty.

The definition of fundamental solution requires the introduction of some more formalism.

Let $\phi \in S(G_\Phi) \otimes \text{Hom}(E, \Lambda_q)$, where $E$ is a finite-dimensional space. Because of the canonical identification of $\text{Hom}(E, \Lambda_q)$ with $E' \otimes \Lambda_q$, we can write
\[ \phi(z, t) = \sum_j \omega_j(z, t) \otimes \psi_j, \]
where the sum is finite, $\psi_j \in E'$ and $\omega_j \in S(G_\Phi) \otimes \Lambda_q$. We then set
\[ \Box^{(q)}_b \phi = \sum_j (\Box^{(q)}_b \omega_j) \otimes \psi_j. \]
(21)

This is consistent with the original definition of $\Box^{(q)}_b$ on forms, because of the identification $\Lambda_q \cong \text{Hom}(\mathbb{C}, \Lambda_q)$. If $E$ has an inner product, the action of $\Box^{(q)}_b$ can be extended to distributions in $S'(G_\Phi) \otimes \text{Hom}(E, \Lambda_q)$.

We then say that $K \in S'(G_\Phi) \otimes \text{End}(\Lambda_q)$ is a fundamental solution of $\Box^{(q)}_b$ if $\Box^{(q)}_b K = \delta_0 \otimes I$, i.e. if
\[ \langle K, \Box^{(q)}_b \phi \rangle = \text{tr} \phi(0), \]
for $\phi \in S(G_\Phi) \otimes \text{End}(\Lambda_q)$.

The existence of a fundamental solution implies that $\Box^{(q)}_b$ is solvable, because for $\omega \in S(G_\Phi) \otimes \Lambda_q$ we have
\[ \Box^{(q)}_b (\omega * K) = \omega * (\delta_0 \otimes I) = \omega. \]
In order to construct such a fundamental solution, we distinguish between the case $\nu = n$ and $\nu < n$. In the former case we must assume explicitly that $\Omega_q$ is empty. This assumption is automatically verified in the latter case.

6a. Case $\nu = n$. For $\lambda \in \Omega$ we define $\mathcal{K}_q^\lambda \in \text{End} L^2(\mathbb{R}^n) \otimes \text{End} (\Lambda_q)$ as follows. Keeping the notation in Proposition 3.5, let $\omega^\lambda_L$ denote the elementary form $d\bar{z}^\lambda L$. Then, for $\sum_{|L| = q} \psi_L \otimes \omega^\lambda_L \in L^2(\mathbb{R}^n) \otimes \Lambda_q$, we set

$$\mathcal{K}_q^\lambda \left( \sum_{|L| = q} \psi_L \otimes \omega^\lambda_L \right) = \sum_m \sum_{|L| = q} \frac{\langle \psi_L, h^\lambda_m \rangle}{\alpha^\lambda_L + \sum_{j=1}^n (1 + 2m_j)|\mu_j|} h^\lambda_m \otimes \omega^\lambda_L.$$  

Furthermore, for $\phi \in \mathcal{S}(G_\Phi) \otimes \text{End} (\Lambda_q)$, we define $K_q$ by setting

$$\langle K_q, \phi \rangle = \int_{\Omega} \langle \mathcal{K}_q^\lambda, \pi_\lambda(\phi) \rangle | \det \Phi^\lambda | d\lambda,$$

where the pairing $\langle \cdot, \cdot \rangle$ is defined in (17).

**Theorem 6.1.** Let $\Omega_q$ be empty and $\nu = n$. Then $K_q$ is a well defined tempered distribution on $G_\Phi$, that is, $K_q \in \mathcal{S}'(G_\Phi) \otimes \text{End} (\Lambda_q)$. Moreover, $K_q$ is a global, homogeneous, fundamental solution for $\Box^q_b$.

In the proof we will need the following result.

**Lemma 6.2.** There is $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ there exists a constant $c_{N,n} \geq 0$ such that for each multi-index $L$ we have

$$\sum_m \frac{1}{(\alpha^\lambda_L + \sum_{j=1}^n (1 + 2m_j)|\mu_j|)^N} \leq c_{n,N} \left( 1 + |\lambda|^n \right) | \det \Phi^\lambda |.$$

Assuming the validity of the lemma we prove Theorem 6.1.

**Proof of Theorem 6.1.** We begin by showing that $K = K_q \in \mathcal{S}'(G_\Phi) \otimes \text{End} (\Lambda_q)$.

For fixed $\lambda \in \Omega$, consider the orthonormal basis $(d\bar{z}^\lambda K)^* \otimes d\bar{z}^\lambda L$ of $\Lambda_q^* \otimes \Lambda_q \cong \text{End} (\Lambda_q)$, where we have set $\nu^*(w) = \langle u, v \rangle$ for $u, v$ in any inner product space. If $\phi \in \mathcal{S}(G_\Phi) \otimes \text{End} (\Lambda_q)$, we write

$$\phi = \sum_{K,L} \phi_{KL}(d\bar{z}^\lambda K)^* \otimes d\bar{z}^\lambda L,$$

where the $\phi_{KL}$ are scalar-valued functions.

By (17) and Lemma 3.6 we have

$$| \langle K, \phi \rangle | = \int_{\Omega} \sum_{K,L} \frac{\phi_{KL}(\lambda; m, m)}{(\alpha^\lambda_K + \sum_{j=1}^n (1 + 2m_j)|\mu_j|)^N} | \det \Phi^\lambda | d\lambda \leq c \| \phi \|_{N'} \int_{\Omega} \sum_{L} S(L, \lambda) | \det \Phi^\lambda | (1 + |\lambda|)^N d\lambda.$$
where $S(L, \lambda)$ denotes the left hand side in (24).

From Lemma 6.2 it follows that for $N$ large enough,

$$|\langle K, g \rangle| \leq c \|\phi\|_{N'},$$

which shows that $K \in S'(G_{\Phi}) \otimes \text{End} (\Lambda_q)$.

We now show that $K$ is a fundamental solution for $\Box^{(q)} b$. For $\phi \in S(G_{\Phi}) \otimes \text{End} (\Lambda_q)$, we have

$$\langle \Box^{(q)} b K, \phi \rangle = \langle K, \Box^{(q)} b \phi \rangle = \int_{\Omega} \langle \pi_\lambda(K), \pi_\lambda(\Box^{(q)} b \phi) \rangle \det \Phi |\lambda| \, d\lambda = \text{tr} \phi(0).$$

This proves the proposition. \qed

**Proof of Lemma 6.2.** We wish to estimate the left hand side of (24).

We split the sum for $m \in \mathbb{N}^n$ as

$$\sum_{E \subseteq \{1, \ldots, n\}} \left( \sum_{\substack{m_j = 0 \text{n.o. } j \notin E \atop m_j \geq 1 \text{n.o. } j \in E}} 1 \right) \frac{\alpha^\lambda_L + \sum_{j=1}^n (1 + 2m_j) |\mu_j| \langle h_m, \pi_\lambda(\phi_{LL}) h_m^\lambda \rangle}{(\alpha^\lambda_L + \sum_{j=1}^n (1 + 2m_j) |\mu_j|)^N}$$

and we write $|E|$ to denote the cardinality of $E$.

For each $L$ fixed, we may relabel coordinates in order to have $\alpha^\lambda_L = \sum_{j=1}^p |\mu_j| - \sum_{j=p+1}^n |\mu_j|$. Then,

$$\alpha^\lambda_L + \sum_{j=1}^n (1 + 2m_j) |\mu_j| = \sum_{j=1}^p |\mu_j| + 2 \sum_{j=1}^n m_j |\mu_j|.$$

Notice that $p \geq 1$ since $\nu = n$ and $\Omega_q$ is empty.
Let $E = \{j_1, \ldots, j_k\}$. If $|E| = k \geq 2$,

$$
\sum_{\substack{m_j \geq 1 \\
j \in E}} \frac{1}{(1 + 2m_j)|\mu_j|} \left(1 + \sum_{j=1}^{n} (1 + 2m_j)|\mu_j|\right)^N
$$

$$
\leq \sum_{\substack{m_j \geq 1 \\
j = 1, \ldots, k}} \frac{1}{(1 + \sum_{i=1}^{k} m_i|\mu_i|) \left(1 + \sum_{i=1}^{k} m_i|\mu_i|\right)^N}
$$

$$
\leq \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{1}{(\sum_{i=1}^{k} x_i|\mu_{ji}|) \left(1 + \sum_{i=1}^{k} x_i|\mu_{ji}|\right)^N} \prod_{j=1}^{k} \frac{1}{|\mu_{ji}|}
$$

where the last inequality follows from estimates for the eigenvalues of a Hermitean form (see e.g. [MR] Lemma 4.2).

Next, if $|E| = 1$, the corresponding sum is bounded by a constant times

$$
\sum_{m \geq 1} \frac{1}{m|\mu_{j0}|(1 + m|\mu_{j0}|)^N} \leq \frac{1}{|\mu_{j0}|} + \int_1^{+\infty} \frac{1}{x|\mu_{j0}|(1 + x|\mu_{j0}|)^N} dx
$$

$$
\leq c \frac{1}{|\mu_{j0}|},
$$

and the claimed estimate follows as before.

Finally, if $|E| = 0$, the corresponding term equals $1/(\sum_{j=1}^{p} |\mu_j|)$ for which we easily obtain the estimate. This proves the lemma. \(\Box\)

If $\Omega_q$ is non-empty, define $K^\lambda$ by (22) if $\lambda \notin \Omega_q$, and by the same formula with the sum in $L$ extended only to $L \neq \bar{L}$ if $\lambda \in \Omega_q$, where $\bar{L}$ is the multi-index introduced in the proof of Theorem 4.4. Then define $K_{rel}$ according to (23).

**Corollary 6.3.** If $\Omega_q$ is non-empty, $K_{rel}$ is a relative fundamental solution of $\Box^{(q)}_b$, i.e. $\Box^{(q)}_b K_{rel} = \delta_0 \otimes I - C_q$.

We now treat the case in which the form $\Phi^\lambda$ is degenerate for all $\lambda$, that is the maximum rank $\nu$ of $\Phi^\lambda$ is strictly less than $n$. We split this case into two subcases: when $\nu < n - 1$ and when $\nu = n - 1$. The former case is technically similar to the case $\nu = n$. Instead, the latter case requires a more involved definition of the fundamental solution. The difference between these two cases somehow resembles the difference in the formulas for the fundamental solution of the classical Laplacian in $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n > 2$. 
6b. Case $\nu < n - 1$. We now assume that the form $\Phi^\lambda$ is degenerate for all $\lambda$ and that the maximum rank $\nu$ of $\Phi^\lambda$ is strictly less than $n - 1$. As before, we denote by $\Omega'$ the Zariski-open cone of the $\lambda \in W^*$ for which rank $\Phi^\lambda = \nu$ and by $\Omega''$ the subcone of $\Omega$ where the number of distinct eigenvalues of $\Phi^\lambda$ is maximum.

For $\lambda \in \Omega'$, $\eta \neq 0$ and for $\sum_{|L| = q} \psi_L \otimes \omega^\lambda_L \in L^2(\mathbb{R}^\nu) \otimes \Lambda_q$, we set

$$K^\lambda_{q, \eta} \left( \sum_{|L| = q} \psi_L \otimes \omega^\lambda_L \right) = \sum_m \sum_{|L| = q} \frac{\langle \psi_L, h^\lambda_m \rangle}{|\eta|^2 + \alpha^\lambda_L + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|} h^\lambda_m \otimes \omega^\lambda_L.$$  

Furthermore, for $\phi \in \mathcal{S}(G_\Phi) \otimes \text{End} (\Lambda_q)$, we define $K_q$ by setting

$$\langle K_q, \phi \rangle = \int_{\Omega'} \int_{\mathbb{C}^{n-k}} \langle \mathcal{K}^\lambda_{q, \eta}, \pi_\lambda(\phi) \rangle d\eta D(\lambda) d\lambda.$$  

Essentially the same proof of Theorem 6.1 proves the following.

**Theorem 6.4.** Let $\nu < n - 1$ and $K_q$ be defined by (26). Then $K_q \in \mathcal{S}'(G_\Phi) \otimes \text{End} (\Lambda_q)$ and it is a global, homogeneous, fundamental solution for $\Box^{(q)}_n$.

6c. Case $\nu = n - 1$. As before, let $\Omega'$ be the subcone of $\Omega$ where the number of distinct eigenvalues of $\Phi^\lambda$ is maximum. We must treat with special care the values of $\lambda$ for which there exists at least a multi-index $L$ such that $\alpha^\lambda_L + \sum_{j=1}^\nu |\mu_j| = 0$. (The existence of such $\lambda$ was excluded in the case $\nu = n$, because of the assumption $\Omega_q = \emptyset$, while in the case $\nu < n - 1$ such $\lambda$ do not cause any inconvenience since the function $1/|\eta|^2$ is locally integrable in $\mathbb{C}^k$ when $k > 1$.) Let $\Gamma$ be the subcone of $\Omega'$ consisting of such $\lambda$.

Moreover, let

$$\mathcal{E}_\lambda = \{ L : |L| = q, \alpha^\lambda_L + \sum_{j=1}^\nu |\mu_j| = 0 \},$$

and

$$\mathcal{D}_\lambda = \{ (L, m) : L \in \mathcal{E}_\lambda, m = 0 \in \mathbb{N}^\nu \}.$$  

Let $\{U_k\}$ be an open covering of $\Omega'$ such that on each $U_k$ a smoothly varying frame can be chosen according to Section 3c. In particular, on each $U_k$ we have well defined functions $\mu_j = \mu_j(\lambda)$ parametrizing the eigenvalues of $\Phi^\lambda$. We order them in such a way that $\mu_{\nu+1}(\lambda) = \cdots = \mu_n(\lambda) = 0$. Let $\{\rho_k\}$ be a smooth partition of unity subordinated to this covering.

In the present situation, we need to modify the definition of the fundamental solution of $\Box^{(q)}_n$ as follows. Let $\mathcal{U} = \{ (\lambda, \eta) : \lambda \in \Gamma, |\eta| < 1 \}$.

We set $K_q = K' + K''$ where, for $\phi \in \mathcal{S}(G_\Phi) \otimes \text{End} (\Lambda_q)$ we define

$$\langle K', \phi \rangle = \sum_k \int_{(\Omega \times \mathbb{C}^{n-k}) \setminus \mathcal{U}} \rho_k(\lambda) \langle \mathcal{K}^\lambda_{q, \eta}, \pi_\lambda(\phi) \rangle dy D(\lambda) d\lambda,$$
with $K_{q}^{\lambda,\eta}$ defined by (25), and

$$
\langle K'', \phi \rangle = \sum_{k} \int_{\mathcal{U}} \rho_k(\lambda) \sum_{(L,m) \notin \mathcal{D}_\lambda} \frac{\phiLL(\lambda, \eta; m, m)}{\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|} \, d\eta \, D(\lambda) \, d\lambda
\tag{28}
$$

$$
+ \sum_{k} \int_{\mathcal{U}} \rho_k(\lambda) \sum_{\lambda \in \mathcal{E}_\lambda} \frac{\phiLL(\lambda, \eta; 0, 0) - \phiLL(\lambda, 0; 0)}{|\eta|^2} \, d\eta \, D(\lambda) \, d\lambda,
$$

where $\hat{f}(\lambda, \eta; m, m)$ is given by (15).

**Theorem 6.5.** Let $\nu = n - 1$ and let $K_q = K' + K''$ be defined as above. Then $K_q \in \mathcal{S}'(G_{\phi}) \otimes \text{End}(\Lambda_q)$. Moreover, $K_q$ is a fundamental solution for $\Box^{(q)}_b$.

**Proof.** Notice that for $(\lambda, m) \in (\Omega \setminus \Gamma) \times \mathbb{N}^\nu$ or $\lambda \in \Gamma$ and $(L, m) \notin \mathcal{D}_\lambda$ we have

$$
\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j| \geq \sum_{j=1}^p |\mu_j|,
$$

for some integer $p$, $1 \leq p \leq \nu$. Then, combining Lemma 3.6 with an argument analogous to that given in the proof of Lemma 6.2, we obtain that

$$
|\langle K', \phi \rangle| \leq c \sum_{|L|=q} \int_{\mathcal{U}} \int_{\mathbb{C}^{n-\nu}} |\phiLL(\lambda, \eta; m, m)| \, d\eta \, (1 + |\lambda|^\nu) \, d\lambda
\leq c\|\phi\|_{\mathcal{N}^{\nu}}.
$$

The fact that $|\langle K'', \phi \rangle| \leq c\|\phi\|_{\mathcal{N}^{\nu}}$ follows from standard arguments. This shows that $K \in \mathcal{S}'(G_{\phi}) \otimes \text{End}(\Lambda_q)$.

Finally, we prove that $K$ is a fundamental solution of $\Box^{(q)}_b$. Let $\phi \in \mathcal{S}(G_{\phi}) \otimes \text{End}(\Lambda_q)$. By Proposition 3.5, arguing as in the proof of Lemma 3.6, we have that

$$
(\Box^{(q)}_b \phi)^* (\lambda, \eta; m, m) = \langle \pi_{\lambda,\eta}(\Box^{(q)}_b \phi) h^\lambda_m, h^\lambda_m \rangle
= \langle d\pi\lambda,\eta(\Box^{(q)}_b \phi) \pi_{\lambda,\eta}(\phi) h^\lambda_m, h^\lambda_m \rangle
= \langle \pi_{\lambda,\eta}(\phi) h^\lambda_m, d\pi_{\lambda,\eta}(\Box^{(q)}_b \phi) h^\lambda_m \rangle.
$$

Hence,

$$
(\Box^{(q)}_b \phi)^{\ast}_{LL}(\lambda, \eta; m, m) = (\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|) \langle \pi_{\lambda,\eta}(\phiLL) h^\lambda_m, h^\lambda_m \rangle
= (\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^\nu (1 + 2m_j)|\mu_j|) \phiLL(\lambda, \eta; m, m).
$$

From this, it also follows that, for $L \in \mathcal{E}_\lambda$, $(\Box^{(q)}_b \phi)^{\ast}_{LL}(\lambda, \eta; 0, 0) = |\eta|^2 \phiLL(\lambda, \eta; 0, 0)$. 

Then, for $\phi \in S(G_\Phi) \otimes \text{End} (\Lambda_q)$ we have
\[
\langle \Box_b^{(q)} K_q, \phi \rangle = \langle K_q, \Box_b^{(q)} \phi \rangle = \sum_{|L|=q} \sum_{m \in \mathbb{N} \nu} \int_\Omega' \int_{\mathbb{C}^{n-q}} \phi_{LL}(\lambda, \eta; m, m) D(\lambda) \, d\lambda
\]
which is what we wished to prove.  

\section{Hypoellipticity of $\Box_b^{(q)}$}

We now turn to Theorem 2. We begin by noticing that if the operator $L$ is hypoelliptic then $\text{span}_R \{ \Phi(z,z) \} = W$. Indeed, if $\text{span}_R \{ \Phi(z,z) \}$ is a proper subspace of $W$, $L$ cannot be hypoelliptic since it is an operator on a proper subgroup of $G_\Phi$.

The fact that $\text{span}_R \{ \Phi(z,z) \} = W$ is equivalent to saying that the group $G_\Phi$ is stratified, and also to saying that there is no $\lambda \neq 0$ such that $\Phi^{\lambda} = 0$. If this is the case and if $\{ V_1, \ldots, V_{2n} \}$ is an enumeration of the vector fields $Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n$, we have
\[
\| V_j V_k f \|_{L^2} \leq c \| \mathcal{L} f \|_{L^2},
\]
for each $f \in S(G_\phi)$ and $j, k = 1, \ldots, 2n$.

We introduce non-isotropic Sobolev norms as follows. Let $k \in \mathbb{N}$ and let $B_k$ be the set of all products of the form $V_{i_1}, \ldots, V_{i_j}$, where $1 \leq i_j \leq 2n$ and $j \leq k$. For $f \in S(G_\phi)$ we set
\[
\| f \|_{(k)} = \sum_{P \in B_k} \| Pf \|_{L^2}.
\]
It is well known that, for functions with a fixed compact support, any ordinary Sobolev norm is controlled by a non-isotropic norm, see [FS].

Because of (29), for $k$ even
\[
\| f \|_{(k)} \approx \sum_{j=0}^{k/2} \| \mathcal{L}^j f \|_{L^2}.
\]
If we extend Sobolev norms to forms in $S(G_\phi) \otimes \Lambda_q$ in the obvious way, (30) remains valid replacing $\mathcal{L}$ by $\mathcal{L} \otimes I$, where $I$ is the identity on $\Lambda_q$.

\begin{theorem}
The following conditions are equivalent:
\begin{enumerate}
\item span$_R \{ \Phi(z,z) \} = W$ and there exists $C > 0$ such that for each $\phi \in S(G_\phi) \otimes \Lambda_q$
\[
\|(\mathcal{L} \otimes I) \phi\|_{L^2} \leq C \|\Box_b^{(q)} \phi\|_{L^2};
\]
\item $\Box_b^{(q)}$ is hypoelliptic;
\item there exists no non-zero $\lambda \in W^*$ such that $n^+(\lambda) \leq n - q$ and $n^- (\lambda) \leq q$;
\end{enumerate}
\end{theorem}
(iv) there exists \( \delta \in (0, 1) \) such that, for every \( \lambda \in \Omega \) and every multi-index \( L \) with \(|L| = q\), \(-\alpha_L^\lambda < (1 - \delta) \sum_{j=1}^{\nu(\lambda)} |\mu_j(\lambda)|\), where \( \alpha_L^\lambda \) is defined in (18);

(v) for every \( \lambda \neq 0 \), \( \Phi^\lambda \) has at least \( \max(q + 1, n - q + 1) \) eigenvalues with the same sign, or at least \( \min(q + 1, n - q + 1) \) pairs of eigenvalues with opposite signs.

Remark. Condition (v) above is the natural generalization of the \( Y(q) \) condition to quadratic manifolds of higher codimension mentioned in the Introduction.

Proof. We preliminary show that conditions (iii) and (v) are equivalent. The rest of the proof gives the implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i).

It is easy to see that conditions (iii) and (v) are both equivalent to the following condition: There exists no non-zero \( \lambda \in W^* \) such that

\[
\begin{align*}
\min(n^+(\lambda), n^-(\lambda)) &\leq \min(q, n - q) \\
\max(n^+(\lambda), n^-(\lambda)) &\leq \max(q, n - q).
\end{align*}
\]

Next, if (i) holds, formula (29) implies that

\[
\|\phi\|_{(2)} \leq c(\|\boxq_b \phi\|_{L^2} + \|\phi\|_{L^2}),
\]

for every \( \phi \) smooth, with support in a fixed compact set. Using the fact that \( \mathcal{L} \otimes I \) and \( \boxq_b \) commute, it follows by induction that

\[
\|\phi\|_{(k+2)} \leq c(\|\boxq_b \phi\|_{(k)} + \|\phi\|_{(k)}),
\]

for every \( k \) even. This implies that \( \boxq_b \) is hypoelliptic by standard arguments (see [T1] Ch. 2 Sect. 5). Thus (i) implies (ii).

In order to prove that (ii) implies (iii), we show that if (iii) does not hold, we can construct a non-smooth solution of the homogeneous equation \( \boxq_b u = 0 \).

Suppose then that there exists a \( \lambda_0 \neq 0 \) such that \( n^+(\lambda_0) \leq n - q \) and \( n^-(\lambda_0) \leq q \). Then there exists a multi-index \( L \) with \(|L| = \nu(\lambda_0)\) such that

\[
\alpha_L^{\lambda_0} + \sum_{j=1}^{\nu(\lambda_0)} |\mu_j(\lambda_0)| = 0.
\]

Because the eigenvalues are homogeneous functions of \( \lambda \), the same equality holds for all \( \lambda = s\lambda_0 \), with \( s > 0 \).

By Proposition 3.5,

\[
d\pi\lambda,0(\boxq_b)u_\lambda = 0,
\]

whenever \( \lambda = s\lambda_0 \), \( s > 0 \) and \( u_\lambda = h_0^\lambda \otimes \omega_L^\lambda \) (see Prop. 3.5 for notation). Notice that we can take the basis \( \{v^\lambda_1, \ldots, v^\lambda_n\} \) that diagonalizes \( \Phi^\lambda \) to be the same for all \( \lambda = s\lambda_0 \), with \( s > 0 \). Also, notice that \( \nu(\lambda) = \nu(\lambda_0) \), and we denote this value by \( \nu_0 \).
We define \( u \in \mathcal{S}'(G_\Phi) \otimes \Lambda_q \) as follows. For \( \phi \in \mathcal{S}(G_\Phi) \otimes \Lambda_q \) we set

\[
(32) \quad \langle u, \phi \rangle = \int_0^{+\infty} \langle u, \pi_{s\lambda_0,0}(\phi) \rangle \, ds
\]

\[
= \int_0^{+\infty} \langle h_0^{s\lambda_0}, \pi_{s\lambda_0,0}(\phi) h_0^{s\lambda_0} \rangle \, ds
\]

\[
= \int_0^{+\infty} \int_{\mathbb{C}^{10}} \mathcal{F}_{t,z'} \phi(z',0,s\lambda_0) e^{-s\sum_{i=1}^q |\mu_j(\lambda_0)||z||^2} \, dz' \, ds. \tag{32}
\]

We show that \( u \) is homogeneous of degree \(-2\) with respect to the dilations \( r \cdot (z,t) = (r z, r^2 t) \) on \( G_\Phi \). Making the change of variables \( z' \mapsto r^{-1} z' \) and \( s \mapsto r^2 s \), we have

\[
\langle u, \phi(r \cdot) \rangle = r^{-2(n+m)} \int_0^{+\infty} \int_{\mathbb{C}^{10}} \mathcal{F}_{t,z'} \phi(r z',0,s\lambda_0/r^2) e^{-s\sum_{i=1}^q |\mu_j(\lambda_0)||z||^2} \, dz' \, ds
\]

\[
= r^{-2(n+m)+2} \int_0^{+\infty} \int_{\mathbb{C}^{10}} \mathcal{F}_{t,z'} \phi(z',0,s\lambda_0) e^{-s\sum_{i=1}^q |\mu_j(\lambda_0)||z||^2} \, dz' \, ds
\]

\[
= r^{-Q+2} \langle u, \phi(r \cdot) \rangle,
\]

where \( Q = 2(n+m) \) denotes the homogeneous dimension of \( G_\Phi \).

As a distribution, \( u \) is homogeneous of degree \( \sigma \) if \( \langle u, \phi(r \cdot) \rangle = r^{-Q-\sigma} \langle u, \phi \rangle \), for all \( r > 0 \) and \( \phi \in \mathcal{S}(G_\Phi) \otimes \Lambda_q \). Thus, \( u \) is homogeneous of degree \(-2\) and non-trivial, hence it cannot coincide with a smooth function.

For \( \phi \in \mathcal{S}(G_\Phi) \otimes \Lambda_q \) we have

\[
\langle \Box_b^{(q)} u, \phi \rangle = \langle u, \Box_b^{(q)} \phi \rangle
\]

\[
= \int_0^{+\infty} \langle u_{s\lambda_0}, d\pi_{s\lambda_0,0}(\Box_b^{(q)}) \pi_{s\lambda_0,0}(\phi) \rangle \, ds
\]

\[
= 0,
\]

because of (31). Then \( u \) is a solution of the equation \( \Box_b^{(q)} u \equiv 0 \), henceforth implying that \( \Box_b^{(q)} \) is not hypoelliptic.

We prove that (iii) implies (iv). If (iii) holds, the quantity

\[
A(\lambda, L) = \frac{\alpha_k^2 + \sum_{j=1}^{q} |\mu_j(\lambda)|}{\sum_{j=1}^{q} |\mu_j(\lambda)|}
\]

is well defined for \( \lambda \in \Omega \) and \( |L| = q \), because the denominator does not vanish.

Proving condition (iv) is equivalent to proving that

\[
(33) \quad \inf \left\{ A(\lambda, L) : \lambda \in \Omega, |L| = q \right\} > 0.
\]

Suppose then that (33) does not hold. Let \( \{\lambda_k\} \subseteq \Omega \) and \( \{L_k\} \) be multi-indices such that \( A(\lambda_k, L_k) \to 0 \) as \( k \to +\infty \). Since \( A \) is homogeneous of degree 0 in \( \lambda \), we may assume that \( |\lambda_k| = 1 \) for all \( k \). By passing to a subsequence we may also assume that \( \lambda_k \to \lambda_0 \), with \( |\lambda_0| = 1 \).
By condition (iii), either \( n^+(\lambda_0) > n - q \) or \( n^-(\lambda_0) > q \). Assume for instance that \( n^+(\lambda_0) > n - q \), and let \( \delta > 0 \) be a strict lower bound for the positive eigenvalues of \( \Phi^{b\lambda} \). By Rouché’s theorem, \( \Phi^{b\lambda} \) has at least \( n - q + 1 \) eigenvalues larger than \( \delta \) for \( k \) large enough. Then for every \( L \) with \( |L| = q \),

\[
\alpha_L^\lambda + \sum_1^\nu |\mu_j(\lambda_k)| > 2\delta,
\]

for \( k \) large. Since \( \sum_1^\nu |\mu_j(\lambda_k)| \) remains bounded, we have a contradiction.

Finally, we show that (iv) implies (i). Condition (iv) implies that

\[
\sup_{\lambda \in \Omega, |L| = q} \frac{\sum_1^\nu |\mu_j(\lambda)|}{\alpha_L^\lambda + \sum_1^\nu |\mu_j(\lambda)|} \leq \frac{1}{\delta},
\]

which in turn is equivalent to

\[
\sup_{\lambda \in \Omega, |L| = q} \frac{|\eta|^2 + \sum_1^\nu |\mu_j(\lambda)|(1 + 2m_j)}{\alpha_L^\lambda + |\eta|^2 + \sum_1^\nu |\mu_j(\lambda)|(1 + 2m_j)} \leq \frac{1}{\delta}.
\]

Observe that, by Proposition 3.5, for \( \lambda \in \Omega \), these quantities are precisely the eigenvalues of \( d\pi_{\lambda,\eta}(L \otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1} \). Therefore this is a bounded operator and

\[
\|d\pi_{\lambda,\eta}(L \otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1}\| \leq \frac{1}{\delta}
\]

for every \( \lambda \in \Omega \), where \( \| \cdot \| \) denotes the operator norm.

Therefore, for \( \phi \in S(G_\Phi) \otimes \Lambda_q \), by (11), we have

\[
\|L \otimes I\phi\|_{L^2}^2 = \int_\Omega \int_\mathbb{C}^{n-\nu} \|\pi_{\lambda,\eta}((L \otimes I)(\phi))\|_{HS}^2 D(\lambda) \, d\eta \, d\lambda
\]

\[
\leq \int_\Omega \int_\mathbb{C}^{n-\nu} \|d\pi_{\lambda,\eta}(L \otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1}\| \|\pi_{\lambda,\eta}(\Box_b^{(q)}\phi)\|_{HS}^2 D(\lambda) \, d\eta \, d\lambda
\]

\[
\leq \frac{1}{\delta} \|\Box_b^{(q)}\phi\|_{L^2}^2.
\]

This proves that (iv) implies (i) and finishes the proof. \( \square \)

This proves the theorem. \( \square \)

**Remark.** We have in fact proved that \( \Box_b^{(q)} \) is hypoelliptic if and only if the following Rockland condition is satisfied: For every \( (\lambda, \eta) \neq (0,0) \) \( d\pi_{\lambda,\eta}(\Box_b^{(q)}) \) is injective on \( S(G_\Phi) \otimes \Lambda_q \). An extension of Helffer-Nourrigat theorem [HeN] to systems of differential operators does not seem to appear in the literature.
References


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