APPENDIX
HÖRMANDER'S THEOREM

Our first aim is to prove Theorem 5.6 in Chapter I for sub-Laplacians (i.e. operators of the form L_1). The proof is based on a comparison between Sobolev and Lipschitz norms.

1. Lipschitz, Besov and Sobolev norms

Let $0 < \alpha < 1$. The *p-Lipschitz norm of order* α of a function $f \in L^p(\mathbb{R}^n)$ is defined as

$$\|f\|_{\Lambda^p_\alpha} = \|f\|_p + \sup_{h \neq 0} |h|^{-\alpha} \|\tau_h f - f\|_p \ , \tag{1.1}$$

where we have set $\tau_h f(x) = f(x-h)$. The space $\Lambda^p_\alpha(\mathbb{R}^n)$ is defined as the subspace of those functions in $L^p(\mathbb{R}^n)$ for which $\|f\|_{\Lambda^p_\alpha} < \infty$.

Notice that, for every $a > 0$, one obtains an equivalent norm on $\Lambda^p_\alpha(\mathbb{R}^n)$ by replacing $\sup_{h\neq 0}$ with $\sup_{0<|h|<a}$. This follows easily from the fact that, for $|h| \geq a$, $|h|^{-\alpha}\|\tau_h f - f\|_p \leq 2a^{-\alpha}\|f\|_p$.

Proposition 1.1. *The space Λ^∞_α consists of the Hölder-continuous functions of order α.*

Proof. It is quite clear that Hölder-continuous functions of order α belong to $\Lambda^\infty_\alpha(\mathbb{R}^n)$. Suppose therefore that $f \in \Lambda^\infty_\alpha(\mathbb{R}^n)$. If we prove that f is continuous, the rest follows easily.

Let φ be a continuous function supported on the unit ball and with $\int_{\mathbb{R}^n} \varphi = 1$. The functions $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ form an approximate identity for $\varepsilon \to 0$. Since $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$,

$$\begin{aligned} f * \varphi_\varepsilon(x) - f(x) &= \int_{\mathbb{R}^n} \big(f(x-y) - f(x)\big)\varphi_\varepsilon(y)\,dy \\ &= \int_{\mathbb{R}^n} \big(\tau_y f(x) - f(x)\big)\varphi_\varepsilon(y)\,dy \ , \end{aligned}$$

so that

$$\|f * \varphi_\varepsilon - f\|_\infty \leq \int_{\mathbb{R}^n} \|\tau_y f - f\|_\infty |\varphi_\varepsilon(y)|\,dy \ .$$

From the inequality

$$\|\tau_y f - f\|_\infty \leq \|f\|_{\Lambda^\infty_\alpha} |y|^\alpha \ ,$$

and the fact that φ_ε is supported on the ball of radius ε, we obtain that

$$\|f * \varphi_\varepsilon - f\|_\infty \le \|f\|_{\Lambda^\infty_\alpha}\varepsilon^\alpha \int_{\mathbb{R}^n} |\varphi_\varepsilon(y)|\,dy = \|f\|_{\Lambda^\infty_\alpha}\varepsilon^\alpha \int_{\mathbb{R}^n} |\varphi(y)|\,dy\ .$$

This gives the uniform convergence of $f * \varphi_\varepsilon$ to f. Since the φ_ε are continuous, the same is true for f. □

Remarks. We motivate the restriction $0 < \alpha < 1$. If the quantity in (1.1) is finite for some $\alpha > 1$, then $f = 0$. The exclusion of $\alpha = 1$ is less simple to explain. An indication that some problems arise with (1.1) is in Proposition 1.2 below, whose statement is false[1] for $\alpha = 1$.

A more general class of norms are the (p, q)-*Besov norms of order* α. For $1 \le q < \infty$, one sets[2]

$$\|f\|_{\Lambda^{p,q}_\alpha} = \|f\|_p + \left(\int_{|h|<1} \big(|h|^{-\alpha}\|\tau_h f - f\|_p\big)^q \frac{dh}{|h|^n}\right)^{\frac{1}{q}}\ , \tag{1.3}$$

and the Besov space $\Lambda^{p,q}_\alpha(\mathbb{R}^n)$ is defined as the space of L^p-functions for which (1.3) is finite. For $q = \infty$, one sets $\Lambda^{p,\infty}_\alpha(\mathbb{R}^n) = \Lambda^p_\alpha(\mathbb{R}^n)$.

We shall be mainly interested in the case $p = 2$, with $q = 2$ or $q = \infty$.

Proposition 1.2. *For $0 < \alpha < 1$, $\Lambda^{2,2}_\alpha(\mathbb{R}^n)$ coincides with the Sobolev space $H^\alpha(\mathbb{R}^n)$ and the two norms are equivalent.*

Proof. Observe that, denoting by

$$\hat f(\xi) = \int_{\mathbb{R}}^n f(x)e^{-ix\cdot\xi}\,dx$$

the Fourier transform of f, then $\widehat{\tau h f}(\xi) = e^{-ih\cdot\xi}\hat f(\xi)$. Hence, for $f \in L^2(\mathbb{R}^n)$ and using Plancherel's formula,

$$\begin{aligned}\|f\|^2_{\Lambda^{2,2}_\alpha} &\sim \|f\|_2^2 + \int_{\mathbb{R}^n} \|\tau_h f - f\|_2^2 \frac{dh}{|h|^{n+2\alpha}}\\ &\sim \int_{\mathbb{R}^n} |\hat f(\xi)|^2\,d\xi + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat f(\xi)|^2 \frac{|e^{-ih\cdot\xi} - 1|^2}{|h|^{n+2\alpha}}\,d\xi\,dh\ .\end{aligned} \tag{1.4}$$

Interchanging the order of integration, we are led to consider first the integral

$$I(\xi) = \int_{\mathbb{R}^n} \frac{|e^{-ih\cdot\xi} - 1|^2}{|h|^{n+2\alpha}}\,dh\ .$$

[1] For $\alpha = 1$ it is more appropriate to replace the "first-order difference" $\tau_h f - f$ with the "second-order difference" $\tau_{2h} f - 2\tau_h f + f$. With this modification, Proposition 1.2 becomes true also for $\alpha = 1$, and in fact it can be extended to all $\alpha < 2$. For larger values of α, one must use higher-order differences.

[2] As for Lipschitz norms, the restriction $|h| < 1$ in the domain of integration can be replaced by $|h| < a$ for any a positive or even infinite.

Using the inequalities $|e^{-ih\cdot\xi}-1|\le|h||\xi|$ for $|h|$ small, and $|e^{-ih\cdot\xi}-1|\le 2$ for $|h|$ large, we see that the integral is convergent for $0<\alpha<1$.

Obviously, $I(0)=0$. If $\xi\neq 0$, decompose ξ in polar coordinates as $\xi=|\xi|\omega$, with ω in the unit sphere. Changing variable, $|\xi|h=h'$, we find that

$$I(\xi)=|\xi|^{2\alpha}\int_{\mathbb{R}^n}\frac{|e^{-ih'\cdot\omega}-1|^2}{|h'|^{n+2\alpha}}\,dh'=|\xi|^{2\alpha}I(\omega)\ .$$

If A is an orthogonal transformation of $\mathbb{R}^n$, the change of variable $h'=Ah''$ shows that $I(\omega)=I(A\omega)$. Hence $I(\omega)$ is constant, and its constant value is non-zero. Putting this in (1.4), we find that

$$\|f\|^2_{\Lambda^{2,2}_\alpha}\sim\int_{\mathbb{R}^n}|\hat f(\xi)|^2(1+|\xi|^{2\alpha})\,d\xi\sim\|f\|_{(\alpha)}\ . \qquad \square$$

Lemma 1.3. *We have the following continuous inclusions*[3]

$$\Lambda^{2,2}_\alpha(\mathbb{R}^n)\subset\Lambda^{2,\infty}_\alpha(\mathbb{R}^n)\subset\Lambda^{2,2}_\beta(\mathbb{R}^n)\ ,$$

if $0<\beta<\alpha<1$.

Proof. Take $f\in\Lambda^{2,2}_\alpha(\mathbb{R}^n)$. From the inequality $|e^{it}-1|\le C_\alpha|t|^\alpha$, we have

$$\begin{aligned}\|\tau_hf-f\|_2^2&\sim\int_{\mathbb{R}^n}|\hat f(\xi)|^2|e^{-ih\cdot\xi}-1|^2\,d\xi\\&\lesssim\int_{\mathbb{R}^n}|\hat f(\xi)|^2|h|^{2\alpha}|\xi|^{2\alpha}\,d\xi\\&\le|h|^{2\alpha}\|f\|^2_{(\alpha)}\ .\end{aligned}$$

This gives the first inclusion by Proposition 1.2.

Take now $f\in\Lambda^{2,\infty}_\alpha(\mathbb{R}^n)$. Then

$$\|\tau_hf-f\|_2\le\|f\|_{\Lambda^{2,\infty}_\alpha}|h|^\alpha\ .$$

Therefore,

$$\begin{aligned}\int_{|h|<1}|h|^{-2\beta}\|\tau_hf-f\|_2^2\,\frac{dh}{|h|^n}&\le\|f\|^2_{\Lambda^{2,\infty}_\alpha}\int_{|h|<1}|h|^{2(\alpha-\beta)-n}\,dh\\&\lesssim\|f\|^2_{\Lambda^{2,\infty}_\alpha}\ .\end{aligned}$$

Adding the L^2-norm, one gets to the conclusion. $\square$

Corollary 1.4. *If* $s<\alpha<1$, $\Lambda^2_\alpha(\mathbb{R}^n)\subset H^s(\mathbb{R}^n)$, *and the inclusion is continuous.*

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[3]More generally, one has the following continuous inclusions:

$$\Lambda^{p,q_1}_\alpha(\mathbb{R}^n)\subset\Lambda^{p,q_2}_\alpha(\mathbb{R}^n)\ ,$$

if $q_1<q_2$, and

$$\Lambda^{p,\infty}_\alpha(\mathbb{R}^n)\subset\Lambda^{p,1}_\beta(\mathbb{R}^n)\ ,$$

if $\beta<\alpha$. These are the relevant inclusions with p fixed. See E.M. Stein, *Singular integrals and differentiability properties of functions*, Chap. V, Sect. 5.

2. Lipschitz norms defined in terms of flows

Lipschitz and Besov norms are defined in terms of the translations τ_h. One can interpret translations in $\mathbb{R}^n$ as generated by the flow of n very special vector fields.

If $h = (h_1, \dots, h_n) = h_1 e_1 + \cdots + h_n e_n$, then $\tau_h = \tau_{h_1 e_1} \circ \cdots \circ \tau_{h_n e_n}$; setting $X_j = -\partial_{x_j}$,

$$\tau_{h_j e_j} = \exp(h_j X_j) \ ,$$

and

$$\tau_h = \exp(h_1 X_1) \cdots \exp(h_n X_n) \ .$$

We shall discuss how more general vector fields can be used to introduce adapted Lipschitz norms, and to compare these among themselves and with the standard Lipschitz norms presented above. We shall restrict ourselves to $p = 2$.

Let Ω be an open set, X a (smooth) vector field on Ω, and let K, K' be two compact subsets of Ω with $K \subset \overset{\circ}{K'}$. We fix $a = a(X, K, K') > 0$ small enough so that

(i) the flow $\varphi_{X,t}(x)$ is defined on K for $|t| \le a$;
(ii) for $|t| \le a$, $\varphi_{X,t}$ maps K diffeomorphically into K'.

For $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset K$, we define the norm

$$\|f\|_{X,\alpha} = \|f\|_2 + \sup_{|t| \le a} |t|^{-\alpha} \| \exp(tX) f - f\|_2 \ . \tag{2.1}$$

Lemma 2.1. *Given K, the norm* (2.1) *does not depend, up to equivalence, on the choice of a, as long as (i) and (ii) are satisfied, for some K', by both a and a'.*

Proof. For the purpose of this proof, let us specify the value of a in (2.1) and write $\|f\|_{X,\alpha,a}$. Obviously, if $a < b$, $\|f\|_{X,\alpha,a} \le \|f\|_{X,\alpha,b}$.

On the other hand, let K_b be a compact subset of Ω such that (ii) is satisfied for $|t| < b$. For $a < |t| \le b$, we use the trivial majorization $\| \exp(tX) f - f\|_2 \le \| \exp(tX) f\|_2 + \|f\|_2$. Noticing that $\operatorname{supp}\big(\exp(tX) f\big) = \varphi_{X,t}^{-1}(\operatorname{supp} f) \subset K_b$,

$$\| \exp(tX) f\|_2^2 = \int_{K_b} \big|f\big(\varphi_{X,t}(x)\big)\big|^2 \, dx = \int_K |f(x)|^2 J_{\varphi_{X,-t}}(x) \, dx \ .$$

By compactness, $J_{\varphi_{X,-t}}$ is bounded on K uniformly for $|t| \le b$. Therefore,

$$\| \exp(tX) f\|_2 \le C \|f\|_2 \ . \tag{2.2}$$

This easily gives the inequality $\|f\|_{X,\alpha,b} \le C' \|f\|_{X,\alpha,a}$. $\square$

We shall not mention the choice of a in (2.2) unless it will be necessary.

Notice that, with $\Omega = \mathbb{R}^n$ and $X_j = -\partial_{x_j}$, the 2-Lipschitz norm $\|f\|_{\Lambda^2_\alpha}$ is equivalent to $\sum_{j=1}^n \|f\|_{X_j,\alpha}$.

The first comparison we make is between the Lipschitz norms associated with a vector field X and those associated with a modified vector field $\tilde X = \eta X$, for some $\eta \in \mathcal{D}(\Omega)$.

We expect that the trajectories $\tilde\gamma_x(t)$ of the flow generated by $\tilde X$ are contained in the trajectories $\gamma_x(t)$ of the flow generated by X, only travelling at a different speed. To be more precise, we expect that there is a real-valued function $\tau(x,t)$ such that

$$\tilde{\gamma}_x(t) = \gamma_x\big(\tau(x,t)\big) \ , \tag{2.3}$$

and $\tau(x,0) = 0$. Differentiating in t, we obtain the equation

$$\tilde{X}_{\tilde{\gamma}_x(t)} = \partial_t \tau(x,t) X_{\tilde{\gamma}_x(t)} \ .$$

If $X_{\tilde{\gamma}_x(t)} \neq 0$, this means

$$\eta \circ \gamma_x\big(\tau(x,t)\big) = \partial_t \tau(x,t) \ , \tag{2.4}$$

and surely (2.4) implies (2.3) in general. For $x \in \Omega$ fixed, the function τ satisfies (2.4) if and only if $\tau_x(t) = \tau(x,t)$ is a solution of the one-dimensional Cauchy problem

$$\begin{cases} \frac{d}{dt}\tau_x = \eta \circ \gamma_x(\tau_x) \\ \\ \tau_x(0) = 0 \ . \end{cases} \tag{2.5}$$

By Theorem 1.1 (iii) in Chapter I, $\gamma_x(t)$ is smooth in both x and t. It follows from (iv) in the same theorem that (2.5) has a unique solution which is smooth in both x and t.

It will be convenient soon to set the notation

$$\varphi_X(x,t) = \varphi_{X,t}(x) = \gamma_x(t) \ .$$

Proposition 2.2. *Given K and $\eta \in \mathcal{D}(\Omega)$, there is a constant $C = C(K,\eta)$ such that, for every $\alpha \in (0,1)$ and $f \in L^2(\Omega)$ supported on K,*

$$\|f\|_{\eta X,\alpha} \leq C \|f\|_{X,\alpha} \ .$$

Proof. If s,t are small enough and $|s| \leq |t|$,

$$\begin{aligned} \|\exp(t\eta X)f - f\|_2 &\leq \|\exp(t\eta X)f - \exp(sX)f\|_2 + \|\exp(sX)f - f\|_2 \\ &\leq \|\exp(t\eta X)f - \exp(sX)f\|_2 + |t|^\alpha |f|_{X,\alpha} \ . \end{aligned}$$

Hence

$$\|\exp(t\eta X)f - f\|_2 \leq |t|^\alpha |f|_{X,\alpha} + \inf_{|s|\leq|t|} \|\exp(t\eta X)f - \exp(sX)f\|_2 \ .$$

We have reduced matters to proving that

$$\inf_{|s|\leq|t|} \|\exp(t\eta X)f - \exp(sX)f\|_2 \lesssim |t|^\alpha \|f\|_{X,\alpha} \ . \tag{2.6}$$

We have

$$\inf_{|s|\le|t|}\|\exp(t\eta X)f-\exp(sX)f\|_2^2$$

$$=\inf_{|s|\le|t|}\int_\Omega\big|f\circ\varphi_X\big(x,\tau(x,t)\big)-f\circ\varphi_X(x,s)\big|^2\,dx \tag{2.7}$$

$$\le\frac{1}{2|t|}\int_{|s|\le|t|}\int_\Omega\big|f\circ\varphi_X\big(x,\tau(x,t)\big)-f\circ\varphi_X(x,s)\big|^2\,dx\,ds\ .$$

We fix a compact subset K' of Ω containing K in its interior, and we impose that $|s|,|t|\le\delta$, with δ small enough so that $K\subset\big\{\varphi_X\big(x,\tau(x,t)\big):x\in K',|t|\le\delta\big\}$ and $K\subset\big\{\varphi_X(x,s):x\in K',|s|\le\delta\big\}$

For each $t\in[-\delta,\delta]$, we want to make a change of variables $(x,s)\mapsto(y,u)$ in such a way that the integrand becomes $\big|f\circ\varphi_X(y,u)-f(y)\big|^2$. We must then set $y=\varphi_X(x,s)$. Observing that the identity $\varphi_{X,u+s}=\varphi_{X,u}\circ\varphi_{X,s}$ can be written as

$$\varphi_X(x,u+s)=\varphi_X\big(\varphi_X(x,s),u\big)=\varphi_X(y,u)\ ,$$

we must also set $u=\tau(x,t)-s$. The change of variables is then

$$(y,u)=\Phi_t(x,s)=\big(\varphi_X(x,s),\tau(x,t)-s\big)\ ,$$

which is a smooth function in all variables x,s,t.

The Jacobian matrix $J_{x,s}\Phi_t$ in the variables x,s is

$$J_{x,s}\Phi_t(x,s)=\begin{pmatrix}J_x\varphi_X(x,s) & X_{\varphi_X(x,s)}\\ \nabla_x\tau(x,t) & -1\end{pmatrix}\ .$$

At $t=0$, $\tau(x,0)=0$ for every x, therefore $\nabla_x\tau(x,0)=0$. At $s=0$, $\varphi_X(x,0)=x$, so that $J_x\varphi_X(x,0)=I$. Therefore

$$\det J_{x,s}\Phi_0(x,0)=-1\ .$$

It follows that for $|s|,|t|$ smaller than some $\delta'>0$, the determinant of $J_{x,s}\Phi_t(x,s)$ is bounded from below in absolute value, uniformly in $x\in K'$. Hence, for each point $x\in K'$ there is a neighborhood $U_x\times(-\delta_x,\delta_x)$, with $\partial_x<\delta'$, on which every Φ_t with $|t|<\delta_x$ is invertible. By compactness, it is possible to cover K' with a finite number $U_{x_1},\dots,U_{x_m}$ of such neighborhoods.

If $\{\psi_j\}_{j=1}^m$ is a smooth partition of unity on K' subordinated to the U_{x_j}, we set $f_j=f\psi_j$. We then choose a final $\bar\delta$, smaller than δ_{x_j} for every j, such that $\operatorname{supp} f_j\circ\varphi_{X,s}\subset U_{x_j}$ for $|s|<\bar\delta$.

Notice that, since $\tau(x,0)=0$, there is a constant A such that $|\tau(x,t)|\le A|t|$ for $|t|\le\delta'$ and $x\in K'$. Therefore $|u|\le(A+1)|t|$. Putting everything together, we have

$$\frac{1}{2|t|}\int_{|s|\le|t|}\int_\Omega\big|f_j\circ\varphi_X\big(x,\tau(x,t)\big)-f_j\circ\varphi_X(x,s)\big|^2\,dx\,ds$$

$$=\frac{1}{2|t|}\int_{|u|\le(A+1)|t|}\int_\Omega\big|f_j\circ\varphi_X(y,u)-f_j(y)\big|^2\Big|\det J_{x,s}\Phi_t|_{\Phi_t^{-1}(y,u)}\Big|^{-1}\,dy\,du$$

$$\lesssim\sup_{|u|\le(A+1)|t|}\|\exp(uX)f_j-f_j\|_2^2$$

$$\lesssim|t|^\alpha\|f_j\|_{X,\alpha}\ .$$

Summing over j, we obtain the same inequality for f. This and (2.7) imply (2.6). □

We show next that, on compact sets, the norm (2.1) is controlled by the Λ^2_α-norm. This will follow from the following general lemma, that shall be used again later on.

Lemma 2.3. *Let $\varphi(x,t)$ be defined on $K\times[0,\delta]\subset\Omega\times\mathbb{R}$, with values in Ω, such that*

(i) *φ is C^∞ in x, and all its derivatives are continuous in t;*
(ii) *$\varphi(x,t)-x=O(t^\mu)$ as $t\to 0$ uniformly in x, for some $\mu\in\mathbb{R}^+$.*

Then there is a constant $C=C(K,\varphi)$ such that, for $\alpha\in(0,1)$, t small enough and $f\in L^2(\Omega)$ supported on K,

$$\int_\Omega |f\circ\varphi(x,t)-f(x)|^2\,dx\le Ct^{2\mu\alpha}\|f\|^2_{\Lambda^2_\alpha}\ .$$

Proof. In analogy with the previous proof, we first notice that, taking t small and $h\in\mathbb{R}^n$ with $|h|\le t^\mu$,

$$\int_\Omega |f\circ\varphi(x,t)-f(x)|^2\,dx\lesssim\int_\Omega |f\circ\varphi(x,t)-f(x-h)|^2\,dx+\|\tau_hf-f\|_2^2\ ,$$

and therefore

$$\int_\Omega |f\circ\varphi(x,t)-f(x)|^2\,dx$$
$$\lesssim t^{-\mu n}\int_{|h|<t^\mu}\int_\Omega |f\circ\varphi(x,t)-f(x-h)|^2\,dx+t^{2\mu\alpha}\|f\|^2_{\Lambda^2_\alpha}\ .$$

For t fixed, we set $x-h=y$ and $\varphi(x,t)=y+u$. If $\Phi_t(x,h)=\big(x-h,\varphi(x,t)-x+h\big)$, we have

$$J_{x,h}\Phi_t(x,h)=\begin{pmatrix} I & -I\\ J_x\varphi(x,t)-I & I\end{pmatrix}\ .$$

Since $\varphi(x,0)=x$, $J_x\varphi(x,0)=I$ and therefore $\det J_{x,h}\Phi_0(x,h)=1$. The proof can be concluded as before, once we observe that

$$|u|\le|\varphi(x,t)-x|+|h|\lesssim t^\mu\ . \qquad \square$$

Proposition 2.4. *Given a smooth vector field X on Ω and K compact in Ω, there is a constant $C=C(X,K)$ such that, for every $f\in L^2(\Omega)$ supported on K,*

$$\|f\|_{X,\alpha}\le C\|f\|_{\Lambda^2_\alpha}\ .$$

Proof. Apply Lemma 2.3 with $\mu=1$ and $\varphi=\varphi_X$. □

3. Variations on the Baker-Campbell-Hausdorff formula

At this stage we introduce an algebraic formalism. We work with two "abstract" non-commuting indeterminates x and y. We are allowed to construct formal power series in x and y, keeping track of the order of factors in each monomial. We set $[x, y] = xy - yx$. Formal power series can be added amd multiplied among themeselves, and multiplied by scalars[4]. We denote as $\mathbb{R}[\![x, y]\!]$, or briefly as A, the algebra of formal power series generated by x and y. For any $a \in A$, the exponential series

$$e^a = \sum_{n\geq 0} \frac{1}{n!} a^n$$

is well defined and gives an invertible element of A, with $(e^a)^{-1} = e^{-a}$. The Baker-Campbell-Hausdorff formula says that, for $a, b \in A$,

$$e^a e^b = e^{s(a,b)} \ , \tag{3.1}$$

where s is itself a formal power series, containing only commutators

$$\begin{aligned} s(a,b) &= \sum_{j,k\geq 1} z_{j,k}(a,b) \\ &= a + b + \frac{1}{2}[a,b] + \frac{1}{12}\big[[a,b],b\big] - \frac{1}{12}\big[[a,b],a\big] + \cdots \ , \end{aligned} \tag{3.2}$$

where each $z_{j,k}(a, b)$ denotes a *fixed* linear combination of iterated commutators of a and b, each containing a j times and b k times. When $(a, b) = (x, y)$, we shall usually drop the arguments (x, y), and write $z_{j,k}$ as well as w_j, r_N etc. to denote expression involving commutators of x and y in *fixed* finite or infinite combinations.

A first instance of identity derived from (3.2) is the following conjugation formula:

$$e^x e^y e^{-x} = e^{y+[x,y]+\sum_{n\geq 2} b_n} \ , \tag{3.3}$$

where each b_n is a linear combination of commutators of x and y of order n (see (2.6) in Chapter I).

We shall perform some manipulations of this formula aimed to obtain expressions of $e^{[x,y]}$ and e^{x+y} as products of exponentials, where each exponent contains either x or y alone, or, alternatively, commutators of x and y of a sufficiently high order.

We first reduce the problem concerning the sum to a problem concerning commutators[5].

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[4]The scalar field can be $\mathbb{R}$ or $\mathbb{C}$, but consider real scalars here.

[5]A simpler equivalent statement of Lemma 3.1 would consist in the following identity:

$$e^{x+y} = e^x e^y \prod_{j=2}^{\infty} e^{w_j} \ ,$$

with the w_j as in (i). For our purposes it is preferable to focus on the properties of the remainders r_N such that$e^{r_N} = \prod_{j=N+1}^{\infty} e^{w_j}$.

Lemma 3.1. *There are two sequences of elements w_j and r_N of A such that*

(i) *each w_j is a (finite) linear combination of commutators of x and y of order j;*

(ii) *$r_N = \sum_{k=N+1}^{\infty} u_k^{(N)}$, where each $u_k^{(N)}$ is a linear combination of commutators of x and y of order k;*

(iii) *for every N,*

$$e^{x+y} = e^x e^y e^{w_2} \cdots e^{w_N} e^{r_N} \ . \tag{3.4}$$

Proof. Calling $z_n = \sum_{j+k=n} z_{j,k}$, we have

$$e^x e^y = e^{\sum_{n\geq 1} z_n} \ ,$$

and each z_n is a linear combination of commutators of x and y of order n.

For $N = 1$, multiply both sides sides by e^{-x-y} on the left. By the same formula,

$$\begin{aligned} e^{-x-y} e^x e^y &= e^{-x-y} e^{x+y+\sum_{n\geq 2} z_n} \\ &= e^{\sum_{n\geq 2} z'_n} \ , \end{aligned}$$

where each z'_n is again a linear combination of commutators of x and y of order n. Therefore

$$e^{x+y} = e^x e^y e^{-\sum_{n\geq 2} z'_n} \ .$$

This proves the case $N = 1$.

Inductively, suppose that (3.4) holds for a given N, with r_N as in (ii). Then there are elements $u_k^{(N+1)}$ as in (ii) and such that

$$\begin{aligned} e^{-u_{N+1}^{(N)}} e^{r_N} &= e^{-u_{N+1}^{(N)}} e^{\sum_{k\geq N+1} u_k^{(N)}} \\ &= e^{\sum_{k\geq N+2} u_k^{(N+1)}} \ . \end{aligned}$$

This concludes the proof. □

For exponentials of commutators, we have the following initial result. We change the notation slightly, writing x_1, x_2 in place of x, y.

Lemma 3.2. *Let c denote a commutator of x_1 and x_2 of order $p \geq 2$. Then*

$$e^c = e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_q}} e^{\sum_{n\geq p+1} v_n^{(p)}} \ , \tag{3.5}$$

where $i_1, \ldots, i_q \in \{1, 2\}$, $q = 3 \cdot 2^{p-1} - 2$, and each $v_n^{(p)}$ a linear combination of commutators of x_1 and x_2 of order n.

Proof. For $p = 2$, we can assume that $c = [x, y]$. From (3.3) we obtain that

$$e^{x_1} e^{x_2} e^{-x_1} e^{-x_2} = e^{x_2 + [x_1,x_2] + \sum_{n\geq 3} b_n} e^{-x_2} = e^{[x_1,x_2] + \sum_{n\geq 3} b'_n} \ ,$$

where the b'_n have the same properties as the b_n. Therefore,

$$e^{-[x_1,x_2]} e^{x_1} e^{x_2} e^{-x_1} e^{-x_2} = e^{\sum_{n\geq 3} b''_n} \ ,$$

which easily gives (3.5) with $q = 4$.

Inductively, assume that (3.5) holds up to some p. Here we explicitely write $v_n^{(p)}(x_1, x_2)$ to emphasize that each $v_n^{(p)}$ is a fixed expression in x_1, x_2. Take $c = [c', x_2]$, with c' a commutator of order $p - 1$. We apply (3.5) with $p = 2$ and x_1 replaced by c' to obtain

$$e^c = e^{c'} e^{x_2} e^{-c'} e^{-x_2} e^{\sum_{n\geq 3} v_n^{(1)}(c', x_2)} .$$

Since $v_n^{(1)}(c', x_2)$ is a commutator of c' and x_2 of order n and c' is itself a commutator of order $p-1$ in x_1, x_2, it follows from Lemma 2.1 in Chapter I, that $v_n^{(1)}(c', x_2)$ is a linear combination of commutators of x_1, x_2 of order at least $p + n - 1$. Hence

$$e^c = e^{c'} e^{x_2} e^{-c'} e^{-x_2} e^{\sum_{k\geq p+2} d_k(x_1, x_2)} .$$

We then insert the expansion (3.5) of $e^{c'}$ and $e^{-c'}$. If each expansion contains q factors equal to $e^{\pm x_1}$ or $e^{\pm x_2}$, we obtain altogether $2q + 2$ factors of this kind, plus two factors $e^{\pm \sum_{n\geq p+1} v_n^{(p)}}$ placed among them.

We can then shift these extra terms to the far right using the conjugation formula

$$e^{-x_1} e^{a_{p+1} + \sum_{n\geq p+2} a_n(x_1,x_2)} e^{x_1} = e^{a_{p+1} + \sum_{n\geq p+2} a'_n(x_1,x_2)} ,$$

derived from (3.3), and its analogue with x_2 in place of x_1 (notice that the leading term in the series remains unchanged). After doing so, we obtain that

$$\begin{aligned} e^c &= e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_{2q+2}}} \\ &\quad \times e^{v_{p+1}^{(p)} + \sum_{n\geq p+1} \alpha_n(x_1,x_2)} e^{-v_{p+1}^{(p)} + \sum_{n\geq p+2} \beta_n(x_1,x_2)} e^{\sum_{k\geq p+2} d_k(x_1,x_2)} \\ &= e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_{2q+2}}} e^{\sum_{k\geq p+2} d'_k(x_1,x_2)} . \end{aligned}$$

This gives (3.5). □

Corollary 3.3. *Let c denote a commutator of x_1 and x_2 of order p, and let $N \geq p + 1$. Then there is $q = q(p, N)$ such that*

$$e^c = e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_q}} e^{\sum_{n\geq N} v_n} , \tag{3.6}$$

where $i_1, \ldots, i_q \in \{1, 2\}$, and each $v_n = v_n^{(p,N)}$ a linear combination of commutators of x_1 and x_2 of order n.

Proof. We use induction on N. The case $N = p + 1$ is Lemma 3.2. Suppose (3.6) holds for N. We use the usual argument to extract the leading term v_N from the last exponent:

$$e^{-v_N} e^{\sum_{n\geq N} v_n} = e^{\sum_{n\geq N+1} v'_n} ,$$

so that

$$e^c = e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_q}} e^{v_N} e^{\sum_{n\geq N+1} v'_n} .$$

We write $v_N = \sum_{j=1}^r \lambda_j c_j$, where the c_j are commutators of x_1 and x_2 of order N, and the coefficients λ_j do not depend on x_1 and x_2. Making repeated use of (3.4) with $N = 1$, we obtain that

$$e^{v_N} = e^{\lambda_1 c_1} \cdots e^{\lambda_r c_r} e^{\sum_{\ell\geq 2} u_\ell} ,$$

where each u_ℓ is a linear combination of commutators of $c_1,\dots,c_r$ of order ℓ, hence a linear combination of commutators of x_1,x_2 of order greater than or equal to $2N$. Hence,

$$e^c = e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_q}} e^{\lambda_1 c_1} \cdots e^{\lambda_r c_r} e^{\sum_{n\ge N+1} v''_n} .$$

We next expand each factor $e^{\lambda_j c_j}$ according to Lemma 3.2 to derive the conclusion. □

In the same way we prove the following improvement of Lemma 3.1.

Corollary 3.4. *For every $N\ge 2$ there are an integer $q=q(N)$ and a sequence $\{\omega_n^{(N)}\}_{n\ge N}$ of linear combinations of commutators of x_1,x_2 of order n such that*

$$e^{x_1+x_2} = e^{\pm x_{i_1}} e^{\pm x_{i_2}} \cdots e^{\pm x_{i_q}} e^{\sum_{n\ge N} \omega_n^{(N)}} ,$$

with $i_1,\dots,i_q\in\{1,2\}$.

4. Operation on vector fields and Lipschitz norms

If X,Y are two commuting vector fields on Ω, it is quite easy to establish the inequality

$$\|f\|_{X+Y,\alpha} \le C(K)\big(\|f\|_{X,\alpha} + \|f\|_{Y,\alpha}\big)$$

when f is supported on a compact subset K. It is sufficient to observe that, by (2.2),

$$\begin{aligned}\big\|\exp\big(t(X+Y)\big)f-f\big\|_2 &= \|\exp(tX)\exp(tY)f-f\|_2\\ &\le \big\|\exp(tX)\big(\exp(tY)f-f\big)\big\|_2 + \|\exp(tX)f-f\|_2\\ &\le C(K)\|\exp(tY)f-f\|_2 + \|\exp(tX)f-f\|_2\\ &\le C(K)|t|^\alpha\big(\|f\|_{X,\alpha}+\|f\|_{Y,\alpha}\big) .\end{aligned}$$

The situation is much more complicated if X and Y do not commute. In this case we are also interested in norms like $\|f\|_{[X,Y],\alpha}$ and similar with higher-order commutators.

We shall use the formulas obtained in the previous section. When applied to exponentials of vector fields, these formulas do not make sense as such, because they contain infinite series that do not converge in general.

At each stage, we must replace infinite sums by truncations of sufficiently high order and introduce remainder terms, in complete analogy with the use of Taylor expansions with non-analytic functions. The starting point is the formulation of the Baker-Campbell-Hausdorff formula given in Theorem 2.2 in Chapter I. We must also introduce a parameter t tending to zero to which compare the remainders. An indeterminate x is then replaced by tX, with X a given vector field. Notice that a commutator of order $p+1$ is then replaced by the same commutator of the involved vector fields, multiplied by t^{p+1}.

We then obtain the following formulations of Corollaries 3.3 and 3.4.

Corollary 4.1. *Let X_1, X_2 be smooth vector fields on Ω.*

(i) *If W is a commutator of X_1, X_2 of order p, and $N \geq p+1$, let q, $i_1, \dots, i_q \in \{0,1\}$ be as in Corollary 3.3. For any compact $K \subset \Omega$ there is $\delta > 0$ such that, for $|t| < \delta$ and $f \in \mathcal{D}(\Omega)$ supported on K,*

$$\begin{aligned} \exp(t^p W) f(x) = \exp(\pm t X_{i_1}) \exp(\pm t X_{i_2}) \cdots \exp(\pm t X_{i_q}) f(x) \\ + O(t^N) \ , \end{aligned} \tag{4.1}$$

uniformly in x.

(ii) *Given $N \geq 2$, let q, $i_1, \dots, i_q \in \{0,1\}$ be as in Corollary 3.4. For any compact $K \subset \Omega$ there is $\delta > 0$ such that, for $|t| < \delta$ and $f \in \mathcal{D}(\Omega)$ supported on K,*

$$\begin{aligned} \exp\big(t(X_1 + X_2)\big) f(x) = \exp(\pm t X_{i_1}) \exp(\pm t X_{i_2}) \cdots \exp(\pm t X_{i_q}) f(x) \\ + O(t^N) \ . \end{aligned} \tag{4.2}$$

Before applying these formulas, we must add some considerations on the transformations of Ω induced by the corresponding compositions of flows.

Let us consider (4.1) on a compact set $K' \supset K$ in Ω. Making use of (2.2), it can be rewritten as

$$\exp(\mp t X_{i_q}) \cdots \exp(\mp t X_{i_2}) \exp(\mp t X_{i_!}) \exp(t^p W) f(x) = f(x) + O(t^N) \ . \tag{4.3}$$

Composition of the flows in each factor gives rise to a function $\varphi(x,t)$, smooth on $K' \times (-\delta, \delta)$ and with values in Ω, such that

$$\exp(\mp t X_{i_q}) \cdots \exp(\mp t X_{i_2}) \exp(\mp t X_{i_!}) \exp(t^p W) f(x) = f\big(\varphi(x,t)\big) \ . \tag{4.4}$$

Take $f \in \mathcal{D}(\Omega)$, supported on K' and equal to a coordinate function x_j on K. Plugging it into (4.3), we see that the j-th component φ_j of φ satisfies

$$\varphi_j(x,t) = x_j + O(t^N) \ . \tag{4.5}$$

This shows that φ satisfies the hypotheses of Lemma 2.3.

Theorem 4.2. *Let X_1, X_2 be smooth vector fields on Ω. Given $\sigma \in (0,1)$, we have the following estimates, for every $\alpha < 1$ and $f \in L^2(\Omega)$ supported on K:*

(i) *if W is a commutator of X_1, X_2 of order p,*

$$\|f\|_{W,\frac{\alpha}{p}} \leq C_{p,\sigma}\big(\|f\|_{X_1,\alpha} + \|f\|_{X_2,\alpha}\big) + C'\|f\|_{\Lambda^2_\sigma} \ ; \tag{4.6}$$

(ii)

$$\|f\|_{X_1+X_2,\alpha} \leq C_\sigma\big(\|f\|_{X_1,\alpha} + \|f\|_{X_2,\alpha}\big) + C'\|f\|_{\Lambda^2_\sigma} \ . \tag{4.7}$$

The constants $C_{p,\sigma}$ in (4.6) and C_σ in (4.7) do not depend on α or on the specific vector fields X_1, X_2.

Proof. We prove only (4.6). Take N such that $N\sigma \geq 1$, and let φ the function in (4.4); for $|t|$ small,

$$\begin{aligned}
&\exp(t^pW)f(x) - f(x) \\
&\qquad = \exp(\pm tX_{i_1})\cdots\exp(\pm tX_{i_{q-1}})\exp(\pm tX_{i_q})f\big(\varphi(x,t)\big) - f(x) \\
&\qquad = \exp(\pm tX_{i_1})\cdots\exp(\pm tX_{i_{q-1}})\exp(\pm tX_{i_q})\Big(f\big(\varphi(x,t)\big) - f(x)\Big) \\
&\qquad\quad + \exp(\pm tX_{i_1})\cdots\exp(\pm tX_{i_{q-1}})\big(\exp(\pm tX_{i_q})f(x) - f(x)\big) \\
&\qquad\quad + \cdots \\
&\qquad\quad + \exp(\pm tX_{i_1})f(x) - f(x)\ .
\end{aligned}$$

Choosing $|t| < \delta$ small enough and using (2.2), we obtain that

$$\|\exp(t^pW)f - f\|_2 \leq 2q\big(\|f\|_{X_1,\alpha} + \|f\|_{X_2,\alpha}\big)|t|^\alpha + C\|f\|_{\Lambda^2_\sigma}|t|^{N\sigma}\ ,$$

$q = q(p,N)$ being as in Corollary 3.3. Since $|t|$ is small, $|t|^{N\sigma} \leq |t|^\alpha$. This shows that, for $s > 0$ and small enough,

$$\|\exp(sW)f - f\|_2 \leq 2q\big(\|f\|_{X_1,\alpha} + \|f\|_{X_2,\alpha}\big)s^{\frac{\alpha}{p}} + C\|f\|_{\Lambda^2_\sigma}s^{\frac{\alpha}{p}}\ .$$

For $s < 0$, it is sufficient to notice that $-W$ is also a commutator of X_1, X_2 of order p (just interchange X_1 and X_2 in the innermost Lie bracket). □

We can proceed now to the proof the main result about Lipschitz norms, which will be preceded by a lemma.

Lemma 4.3. *Let $0 < \beta < \alpha < 1$. Given $\varepsilon > 0$, there is a constant C_ε such that*

$$\|f\|_{\Lambda^2_\beta} \leq C_\varepsilon\|f\|_2 + \varepsilon\|f\|_{\Lambda^2_\alpha}\ .$$

Proof. Let a be the constant appearing in (2.1). If $0 < \delta < a$,

$$\begin{aligned}
\|f\|_{\Lambda^2_\beta} &\leq \|f\|_2 + \sup_{\delta\leq|h|<a}|h|^{-\beta}\|\tau_hf - f\|_2 + \sup_{|h|<\delta}|h|^{-\beta}\|\tau_hf - f\|_2 \\
&\leq (1+\delta^{-\beta})\|f\|_2 + \delta^{\alpha-\beta}\sup_{|h|<\delta}|h|^{-\alpha}\|\tau_hf - f\|_2 \\
&\leq (1+\delta^{-\beta})\|f\|_2 + \delta^{\alpha-\beta}\|f\|_{\Lambda^2_\alpha}\ .
\end{aligned}$$

It is then sufficient to take $\delta = \varepsilon^{\frac{1}{\alpha-\beta}}$. □

Theorem 4.4. *Let K be a compact subset of Ω, and suppose that the smooth vector fields $X_1, \dots, X_k$, together with their iterated commutators X^I with $|I| \leq m$, span $\mathbb{R}^n$ at each point of K. Then, for every $\alpha \in (0,1)$, there is a constant C_α such that, for every $f \in L^2(\Omega)$ with support contained in K,*

$$\|f\|_{\Lambda^2_{\frac{\alpha}{m}}} \leq C_\alpha\sum_{j=1}^k\|f\|_{X_j,\alpha}\ . \tag{4.8}$$

Proof. Given $x \in K$, there are multi-indices $I_1, \dots, I_n$, with $|I_k| \le m$ for each k, such that the vectors $\{X_x^{I_k}\}$ form a basis of $\mathbb{R}^n$. By continuity, the same holds for the vectors $\{X_{x'}^{I_k}\}$ for each x' in a neighborhood U_x of x. By the inverse function theorem, there are smooth functions $\eta_{j,k}$ on U_x such that

$$\partial_{x_j} = \sum_{k=1}^{n} \eta_{j,k}(x) X^{I_k} \ ,$$

on U_x, for $j = 1, \dots, n$.

Covering K with a finite number of such neighborhoods, and with the aid of a subordinated partition of unity, we then conclude that there are smooth functions $\eta_{j,I}$ on a neighborhood U of K such that

$$\partial_{x_j} = \sum_{|I| \le m} \eta_{j,I}(x) X^I \ ,$$

on U, for $j = 1, \dots, n$. By restricting Ω if necessary, we can as well assume that $U = \Omega$.

Denote by E_j the coordinate vector field ∂_{x_j}. Applying (4.7) and Proposition 2.2, we obtain that

$$\begin{aligned} \|f\|_{E_j, \frac{\alpha}{m}} &\le C_\sigma \sum_{|I| \le m} \|f\|_{\eta_{j,I} X^I, \frac{\alpha}{m}} + C' \|f\|_{\Lambda^2_\sigma} \\ &\le C_{X_1, \dots, X_k, \sigma} \sum_{|I| \le m} \|f\|_{X^I, \frac{\alpha}{m}} + C' \|f\|_{\Lambda^2_\sigma} \ , \end{aligned}$$

for every $\sigma > 0$. Since $|I| \le m$, we also have, by (4.6)

$$\|f\|_{X^I, \frac{\alpha}{m}} \le C \|f\|_{X^I, \frac{\alpha}{|I|}} \le C_{m,\sigma} \big(\|f\|_{X_1, \alpha} + \|f\|_{X_2, \alpha} \big) + C' \|f\|_{\Lambda^2_\sigma} \ .$$

Since

$$\|f\|_{\Lambda^2_{\frac{\alpha}{m}}} \sim \sum_{j=1}^{n} \|f\|_{E_j, \frac{\alpha}{m}} \ ,$$

we conclude that

$$\|f\|_{\Lambda^2_{\frac{\alpha}{m}}} \le C_{X_1, \dots, X_k, \sigma} \big(\|f\|_{X_1, \alpha} + \|f\|_{X_2, \alpha} \big) + C' \|f\|_{\Lambda^2_\sigma} \ .$$

We finally fix $\sigma < \frac{\alpha}{m}$ and apply Lemma 4.3, with $\beta = \sigma$, α replaced by $\frac{\alpha}{m}$ and $\varepsilon = \frac{1}{2C'}$, to majorize the last term. $\square$

Theorem 5.6 in Chapter I is now an immediate consequence of this result, once we observe that, by the mean value theorem,

$$\|f\|_{X_j, \alpha} \le C \big(\|f\|_2 + \|X_j f\|_2 \big) \ ,$$

for any $\alpha < 1$. Given $s < \frac{1}{m}$, choose α such that $s < \frac{\alpha}{m} < \frac{1}{m}$, and apply Corollary 1.4.

5. Hypoellipticity of sub-Laplacians: preliminaries

We adopt the following notation. If D is a differential operator with smooth coefficients in Ω, denote by D^* the *formal adjoint* of D, i.e. the differential operator such that

$$\int_\Omega D^* f(x)\overline{g(x)}\,dx = \int_\Omega f(x)\overline{Dg(x)}\,dx \ ,$$

for $f, g \in \mathcal{D}(\Omega)$.

Suppose that $\mathcal{X} = \{X_1, \dots X_k\}$ is a generating system of vector fields on Ω. Let

$$L = -\sum_{j=1}^k X_j^2 \ .$$

Introduce the norm

$$|\!|\!| f |\!|\!|_{\mathcal{X}} = \Big(\|f\|_2^2 + \sum_{j=1}^k \|X_j f\|_2^2 \Big)^{\frac{1}{2}}$$

on $\mathcal{D}(\Omega)$. For Ω' open in Ω, we consider also the dual norm

$$|\!|\!| u |\!|\!|'_{\mathcal{X},\Omega'} = \sup_{f \in \mathcal{D}(\Omega') : |\!|\!| f |\!|\!|_{\mathcal{X}} \le 1} \big| \langle u, f \rangle \big|$$

on $\mathcal{D}'(\Omega')$. Clearly,

$$|\!|\!| f |\!|\!|'_{\mathcal{X},\Omega'} \le \|f\|_2 \le |\!|\!| f |\!|\!|_{\mathcal{X}} \ , \tag{5.1}$$

whenever applicable.

The reason for introducing these norms is the following.

First of all, notice that, by definition of hypoellipticity, L is hypoelliptic on Ω if and only if it is hypoelliptic on every relatively compact subdomain. Let Ω' be one such subdomain.

If

$$X_j = \sum_{\ell=1}^n a_{j,\ell}(x)\partial_{x_\ell} \ ,$$

then, for functions $f, g \in \mathcal{D}(\Omega')$,

$$\begin{aligned}
\int_{\Omega'} X_j f(x)\overline{g(x)}\,dx &= \sum_{\ell=1}^n \int_{\Omega'} a_{j,\ell}(x)\partial_{x_\ell} f(x)\overline{g(x)}\,dx \\
&= -\sum_{\ell=1}^n \int_{\Omega'} f(x)\overline{\partial_{x_\ell}\big(a_{j,\ell}(x)g(x)\big)}\,dx \\
&= -\int_{\Omega'} f(x)\overline{\big(X_j + b_j(x)\big)g(x)}\,dx \ ,
\end{aligned}$$

with $b_j = \sum_{\ell=1}^n \partial_{x_\ell} a_{j,\ell}$. In other words, $X_j^* = -X_j - b_j$. Notice that the functions $a_{j,\ell}$ and b_j are real valued.

Next, denote by $\langle\ ,\ \rangle$ the inner product in $L^2(\Omega')$. For $f \in \mathcal{D}(\Omega')$,

$$\begin{aligned}\langle Lf, f\rangle &= \sum_{j=1}^{k} \langle X_j f, (X_j + b_j) f\rangle \\ &= \sum_{j=1}^{k} \|X_j f\|_2^2 + \sum_{j=1}^{k} \langle X_j f, b_j f\rangle\ .\end{aligned}$$

Hence,

$$\Re\mathrm{e}\langle Lf, f\rangle = \sum_{j=1}^{k} \|X_j f\|_2^2 + \sum_{j=1}^{k} \Re\mathrm{e}\langle X_j f, b_j f\rangle\ .$$

Now,

$$\begin{aligned}\Re\mathrm{e}\langle X_j f, b_j f\rangle &= \frac{1}{2}\big(\langle X_j f, b_j f\rangle + \langle b_j f, X_j f\rangle\big) \\ &= \frac{1}{2}\big(\langle X_j f, b_j f\rangle - \langle (X_j + b_j) b_j f, f\rangle\big) \\ &= -\frac{1}{2}\langle (X_j b_j + b_j^2) f, f\rangle \\ &= -\frac{1}{2}\int_{\Omega'} (b_j^2 + X_j b_j)|f|^2\, dx\ .\end{aligned}$$

It follows that

$$\begin{aligned}\|\!|f|\!\|_{\mathcal{X}}^2 &= \|f\|_2^2 + \Re\mathrm{e}\langle Lf, f\rangle - \sum_{j=1}^{k} \Re\mathrm{e}\langle X_j f, b_j f\rangle \\ &= \frac{1}{2}\int_{\Omega'} (2 + b_j^2 + X_j b_j)|f|^2\, dx + \Re\mathrm{e}\langle Lf, f\rangle \\ &\le C\|f\|_2^2 + \|\!|Lf|\!\|'_{\mathcal{X},\Omega'}\|\!|f|\!\|_{\mathcal{X}}\ .\end{aligned}$$

Using the inequality $\|f\|_2^2 \le \|f\|_2 \|\!|f|\!\|_{\mathcal{X}}$ we can simplify by a factor $\|\!|f|\!\|_{\mathcal{X}}$, obtaining that

$$\|\!|f|\!\|_{\mathcal{X}} \le C\|f\|_2 + \|\!|Lf|\!\|'_{\mathcal{X},\Omega'}\ .$$

Observe that the hypotheses of Theorem 4.4 are satisfied on all of Ω' with the same $m = m(\Omega')$ and the same constants. By Theorem 5.6 in Chapter I, it follows that, for $f \in \mathcal{D}(\Omega')$ and $s < \frac{1}{m}$,

$$\|f\|_{(s)} \le C_s\big(\|f\|_2 + \|\!|Lf|\!\|'_{\mathcal{X},\Omega'}\big)\ . \tag{5.2}$$

The hardest part of the proof will be the extension of (5.2) to general L^2-functions on Ω' with compact support. This will be done in Section 7. In Section 6 we introduce calculus and estimates with Bessel potentials.

Once this is done, we shall prove a “bootstrapping” argument which extends this implication to Sobolev norms of arbitrary orders. This will be done in Section 8, and it will easily lead us to the conclusion of the proof.

6. Bessel potentials and pseudo-differential operators

For $\gamma \in \mathbb{R}$, let K_γ be the distribution defined as the inverse Fourier transform of $(1+|\xi|^2)^\gamma$, i.e. such that

$$\langle K_\gamma, \varphi\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^\gamma \hat\varphi(\xi)\,d\xi\ ,$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The convolution operator

$$\varphi \longmapsto K_\gamma * \varphi = \mathcal{F}^{-1}\big((1+|\xi|^2)^\gamma\hat\varphi(\xi)\big) \tag{6.1}$$

maps $\mathcal{S}(\mathbb{R}^n)$ continuously onto itself. By duality, it also maps $\mathcal{S}'(\mathbb{R}^n)$ onto itself continuously.

Notice that $K_\gamma * K_{\gamma'} * \varphi = K_{\gamma+\gamma'}\varphi$ and that, for $j \in \mathbb{N}$,

$$K_j * \varphi = (1-\Delta)^j\varphi\ .$$

For this reason the operator (6.1) is denoted by $(1-\Delta)^\gamma$ for arbitrary[6] γ. The identity

$$\big((1-\Delta)^\gamma\varphi\big)\hat{}\,(\xi) = (1+|\xi|^2)^\gamma\hat\varphi(\xi)$$

extends from integer to real values of γ.

The operator $(1-\Delta)^{-\frac{s}{2}}$ is called the *Bessel potential of order s*.

The following facts follow directly from the definition of Sobolev spaces and the Plancherel formula.

Lemma 6.1. *The Sobolev space $H^s(\mathbb{R}^n)$ consists of those $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1-\Delta)^{\frac{s}{2}}f \in L^2(\mathbb{R}^n)$. In other words,*

$$H^s(\mathbb{R}^n) = (1-\Delta)^{-\frac{s}{2}}\big(L^2(\mathbb{R}^n)\big)\ .$$

If $f \in L^2(\mathbb{R}^n)$ and $0 \le |\alpha| \le s$,

$$\big\|(1-\Delta)^{-\frac{s}{2}}\partial^\alpha f\big\|_2 \le \|f\|_2\ . \tag{6.2}$$

In particular, $(1-\Delta)^{-\frac{s}{2}}$ is bounded on $L^2(\mathbb{R}^n)$ for $s \ge 0$.

The properties of Bessel potentials that we shall need can be better understood in the more general context of *pseudo-differential operators* ("ψdo" in short). More precisely, we will consider "localized" versions $M_\chi(1-\Delta)^{-\frac{s}{2}}$ of the Bessel potential, where M_χ is the multiplication operator by a function $\chi \in \mathcal{D}(\Omega)$. Localized Bessel potentials of order s are examples of pseudo-differential operators of order s. In order to understand the definition of a ψdo, consider first a differential operator $D = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$ of order $\le m$ with coefficients $a_\alpha \in \mathcal{S}(\mathbb{R}^n)$. Passing to the Fourier transform, we have, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\widehat{Df}(\xi) &= \sum_{|\alpha|\le m} i^{|\alpha|}\widehat{a_\alpha} * (\xi^\alpha\hat f)(\xi)\\ &= \int_{\mathbb{R}^n}\sum_{|\alpha|\le m} i^{|\alpha|}\widehat{a_\alpha}(\xi-\eta)\eta^\alpha\hat f(\eta)\,d\eta\ .\end{aligned}$$

---

[6] All we said is true also for complex exponents, but we shall not need them here.

Denote, as usual, by $\mathcal{F}$ the Fourier transform, and set

$$k(\xi,\eta)=\sum_{|\alpha|\le m} i^{|\alpha|}\widehat{a_\alpha}(\xi)\eta^\alpha\ .$$

We have shown that D is the conjugation $D=\mathcal{F}^{-1}T_k\mathcal{F}$, where T_k is the integral operator with kernel $k(\xi-\eta,\eta)$. It is easy to verify that k satisfies the estimates

$$\left|\partial_\eta^\beta k(\xi,\eta)\right|\le C_{\beta,N}(1+|\xi|)^{-N}(1+|\eta|)^{m-|\beta|}\ , \tag{6.3}$$

for every multi-index β and every $N>0$.

Definition. *We call special ψdo of order $m\in\mathbb{R}$ an operator if $D=\mathcal{F}^{-1}T_k\mathcal{F}$, where*

$$T_kf(\xi)=\int_{\mathbb{R}^n} k(\xi-\eta,\eta)f(\eta)\,d\eta\ , \tag{6.4}$$

and k is a smooth function on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying (6.3) *for every β and N.*

This definition is *ad hoc*, with stronger hypotheses than in the general theory. One usually defines a ψdo as an operator of ther form

$$Df(x)=(2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\eta)\hat f(\eta)e^{i\eta\cdot x}\,d\eta\ ,$$

where the function a (called the *symbol* of D) satisfies the condition

$$\left|\partial_x^\alpha\partial_\eta^\beta a(x,\eta)\right|\le C_{\alpha,\beta}(1+|\eta|)^{m-|\beta|}\ ,$$

for every pair of multi-indices α,β. The kernel k appearing in our definition is

$$k(\xi,\eta)=\int_{\mathbb{R}^n} a(x,\eta)e^{-ix\cdot\xi}\,dx\ .$$

The following remarks are rather obvious.

(1) Differential operators with coefficients in $\mathcal{S}(\mathbb{R}^n)$ are special ψdo of the same order.
(2) For $\chi\in\mathcal{S}(\mathbb{R}^n)$ and $\gamma\in\mathbb{R}$, $M_\chi(1-\Delta)^\gamma$ is a special ψdo of order 2γ.
(3) As a particular case of the previous two, pointwise multiplication by a Schwartz function is a special ψdo of order 0.
(4) If k satisfies (6.3) for some m, then $\partial_\eta^\beta k$ satisfies (6.3) with m replaced by $m-|\beta|$.

Lemma 6.2. *Let D be a special ψdo of order m. Then D maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ continuously, and its adjoint $D^*:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$, defined by the condition*[7]

$$\langle D^*f,g\rangle=\langle f,Dg\rangle$$

[7]We use here the sesquilinear pairing $\langle\Phi,f\rangle=\overline{\langle f,\Phi\rangle}=\Phi(\bar f)$ between a distribution Φ and a function f.

for $f, g \in \mathcal{S}(\mathbb{R}^n)$ is also a special ψdo of order m.

Proof. Let $D = \mathcal{F}^{-1}T_k\mathcal{F}$, with k as in (6.4), Because the Fourier transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, it is sufficient to prove that T_k is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. But, for $N > n$,

$$\begin{aligned} |T_k f(\xi)| &\leq C_N \int_{\mathbb{R}^n} (1+|\xi-\eta|)^{-N}(1+|\eta|)^m |f(\eta)|\, d\eta \\ &\leq C_N \sup_{\eta\in\mathbb{R}^n} (1+|\eta|)^m |f(\eta)| \int_{\mathbb{R}^n} (1+|\tau|)^{-N}\, d\tau \\ &\leq C'_N \sup_{\eta\in\mathbb{R}^n} (1+|\eta|)^m |f(\eta)| \ , \end{aligned}$$

where the right-hand side is one of the norms defining the Schwartz topology. Therefore, T_k is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. Since the inclusion of $L^\infty(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ is also continuous, we have proved the first statement.

Using the extended Parseval identity $\langle \Phi, f\rangle = (2\pi)^{-n}\langle \hat{\Phi}, \hat{f}\rangle$ and setting

$$k^*(\xi,\eta) = \overline{k(-\xi,\eta+\xi)} \ ,$$

we have

$$\begin{aligned} \langle D'f, g\rangle &= (2\pi)^{-n}\langle T_k\hat{f}, \hat{g}\rangle \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(\xi-\eta,\eta)\hat{f}(\eta)\, d\eta\, \overline{\hat{g}(\xi)}\, d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{f}(\eta) \int_{\mathbb{R}^n} k(\xi-\eta,\eta)\overline{\hat{g}(\xi)}\, d\xi\, d\eta \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{f}(\eta) \int_{\mathbb{R}^n} \overline{k^*(\eta-\xi,\xi)\hat{g}(\xi)}\, d\xi\, d\eta \\ &= (2\pi)^{-n}\langle \hat{f}, T_{k'}\mathcal{F}^{-1}g\rangle \\ &= \langle f, \mathcal{F}^{-1}T_{k^*}\mathcal{F}g\rangle \ . \end{aligned}$$

Then for every N,

$$|\partial_\eta^\beta k^*(\xi,\eta)| \leq C_N(1+|\xi|)^{-N}(1+|\eta+\xi|)^{m-|\beta|} \ .$$

If $|\xi| \leq \frac{|\eta|}{2}$, then $|\eta+\xi| \sim |\eta|$, and $(1+|\eta+\xi|)^m \sim (1+|\eta|)^m$ for m both positive and negative.

If $|\xi| > \frac{|\eta|}{2}$ and $m-|\beta| \geq 0$,

$$\begin{aligned} (1+|\xi|)^{-N}(1+|\eta+\xi|)^{m-|\beta|} &\leq (1+|\xi|)^{-N}(1+|\eta|+|\xi|)^{m-|\beta|} \\ &\leq 3^{m-|\beta|}(1+|\xi|)^{-N+m-|\beta|} \\ &\leq 3^{m-|\beta|}(1+|\xi|)^{-N+m-|\beta|}(1+|\eta|)^{m-|\beta|} \ . \end{aligned}$$

If $|\xi| > \frac{|\eta|}{2}$ and $m-|\beta| < 0$,

$$\begin{aligned} (1+|\xi|)^{-N}(1+|\eta+\xi|)^{m-|\beta|} &\leq (1+|\xi|)^{-N} \\ &\leq (1+|\xi|)^{-N-m+|\beta|}\Big(1+\frac{|\eta|}{2}\Big)^{m-|\beta|} \\ &\leq 2^{-m+|\beta|}(1+|\xi|)^{-N+m-|\beta|}(1+|\eta|)^{m-|\beta|} \ . \end{aligned}$$

Since N is arbitrary, k^* satisfies (6.3). $\square$

Corollary 6.3. *If D is a special ψdo of order m, then $D(1-\Delta)^\gamma$ and $(1-\Delta)^\gamma D$ are special ψdo of order $m+2\gamma$.*

Proof. If $D=\mathcal{F}^{-1}T_k\mathcal{F}$, then $D(1-\Delta)^\gamma=\mathcal{F}^{-1}T_{k'}\mathcal{F}$, with $k'(\xi,\eta)=k(\xi,\eta)(1+|\eta|^2)^\gamma$, and the verification of (6.3) with m replaced by $m+2\gamma$ is fairly simple.

For the other operator, it is sufficient to consider its adjoint, equal to $D^*(1-\Delta)^\gamma$, and we are therefore in the previous situation. □

The composition of two ψdo's $D_1=\mathcal{F}^{-1}T_{k_1}\mathcal{F}$ and $D_2=\mathcal{F}^{-1}T_{k_2}\mathcal{F}$ can be formally defined as $D_1D_2=\mathcal{F}^{-1}T_{k_1}T_{k_2}\mathcal{F}$. It is not clear however that this formal definition corresponds to the actual composition of the individual factors when applied to Schwartz functions. In order to settle this point, we show that special ψdo's can be continuously extended to Sobolev spaces of any order.

Theorem 6.4. *A ψdo D of order m extends to a bounded operator from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for every $s\in\mathbb{R}$.*

Proof. We first prove that a special ψdo of order 0 extends to a bounded operator on $L^2(\mathbb{R}^n)$. By the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and the Plancherel formula, this is equivalent to proving that if $k(\xi,\eta)$ satisfies (6.3) with $m=0$, then T_k is bounded on $L^2(\mathbb{R}^n)$. If $N>n$,

$$|T_kf(\xi)|\le C_N\int_{\mathbb{R}^n}(1+|\xi-\eta|)^{-N}|f(\eta)|\,d\eta=C_N(1+|\cdot|)^{-N}*|f|(\xi)\ .$$

Therefore,

$$\begin{aligned}\|T_kf\|_2&\le C_N\|f\|_2\int_{\mathbb{R}^n}(1+|\xi|)^{-N}\,d\xi\\&\le C'_N\|f\|_2\ .\end{aligned}$$

For general m and s, we have to show that $D_0=(1-\Delta)^{\frac{s-m}{2}}D(1-\Delta)^{-\frac{s}{2}}$ is bounded on $L^2(\mathbb{R}^n)$. It follows from Corollary 6.3 that D_0 is a ψdo of order 0. □

Proposition 6.5. *Let D_1,D_2 be special ψdo of order m_1,m_2 respectively, then D_1D_2 is a special ψdo of order m_1+m_2.*

Proof. If k_1,k_2 are the kernels associated to D_1,D_2 respectively, we consider the composition $T_{k_1}T_{k_2}$. If $f\in\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}T_{k_1}T_{k_2}f(\xi)&=\int_{\mathbb{R}^n}k_1(\xi,\tau)T_{k_2}f(\tau)\,d\tau\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}k_1(\xi-\tau,\tau)k_2(\tau-\eta,\eta)f(\eta)\,d\eta\,d\tau\\&=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}k_1(\xi-\tau,\tau)k_2(\tau-\eta,\eta)\,d\tau\right)f(\eta)\,d\eta\\&=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}k_1(\xi-\eta-\tau,\tau+\eta)k_2(\tau,\eta)\,d\tau\right)f(\eta)\,d\eta\ .\end{aligned}$$

This shows that $T_{k_1}T_{k_2}=T_{\tilde{k}}$, with

$$\tilde{k}(\xi,\eta)=\int_{\mathbb{R}^n}k_1(\xi-\tau,\tau+\eta)k_2(\tau,\eta)\,d\tau\ . \tag{6.5}$$

We have to show that $\tilde{k}$ satisfies (6.3) for $m = m_1 + m_2$. By (6.3),

$$|\tilde{k}(\xi,\eta)| \le C_N(1+|\eta|)^{m_2}\int_{\mathbb{R}^n}(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau\ .$$

If $|\tau| < \frac{|\eta|}{2}$, then $1+|\tau+\eta| \sim 1+|\eta|$, so that

$$\begin{aligned}\int_{|\tau|<\frac{|\eta|}{2}}&(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau\\ &\lesssim (1+|\eta|)^{m_1}\int_{|\tau|<\frac{|\eta|}{2}}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N}\,d\tau\\ &\le (1+|\eta|)^{m_1}\int_{\mathbb{R}^n}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N}\,d\tau\ .\end{aligned}$$

If we split the last integral in two parts, according to whether $|\tau|$ is bigger or smaller than $|\xi-\tau|$, and make the change of variable $\tau' = \xi-\tau$ in one of the two, we see that

$$\int_{\mathbb{R}^n}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N}\,d\tau = 2\int_{|\tau|<|\xi-\tau|}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N}\,d\tau\ .$$

Since $|\xi| \le |\tau|+|\xi-\tau|$, if $|\tau|<|\xi-\tau|$ then $|\xi-\tau| > \frac{|\xi|}{2}$, and therefore

$$\begin{aligned}\int_{|\tau|<|\xi-\tau|}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N}\,d\tau &\le \Big(1+\frac{|\xi|}{2}\Big)^{-N}\int_{\mathbb{R}^n}(1+|\tau|)^{-N}\,d\tau\\ &\le C_N(1+|\xi|)^{-N}\ .\end{aligned}$$

We have so proved that, for any N,

$$\begin{aligned}\int_{|\tau|<\frac{|\eta|}{2}}&(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau\\ &\le C_N(1+|\xi|)^{-N}(1+|\eta|)^{m_1}\ .\end{aligned}\tag{6.6}$$

If $|\tau| > \frac{|\eta|}{2}$, we separate the case $m_1 \ge 0$ from the case $m_1 < 0$. If $m_1 \ge 0$, we use the fact that $|\tau+\eta| \le 3|\tau|$, so that

$$\begin{aligned}\int_{|\tau|>\frac{|\eta|}{2}}&(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau\\ &\lesssim \int_{|\tau|>\frac{|\eta|}{2}}(1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N+m_1}\,d\tau\\ &\lesssim \int_{\mathbb{R}^n}(1+|\xi-\tau|)^{-N+m_1}(1+|\tau|)^{-N+m_1}\,d\tau\\ &\lesssim (1+|\xi|)^{-N+m_1}\\ &\le (1+|\xi|)^{-N+m_1}(1+|\eta|)^{m_1}\ .\end{aligned}$$

If $m_1 < 0$, we use the fact that $(1+|\tau+\eta|)^{m_1} \le 1$, together with the fact that $(1+|\tau|)^{m_1} \le (1+|\eta|/2)^{m_1}$, to obtain the inequality

$$\begin{aligned}\int_{|\tau|>\frac{|\eta|}{2}} &(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau \\ &\lesssim (1+|\eta|)^{m_1}\int_{|\tau|>\frac{|\eta|}{2}} (1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N-m_1}\,d\tau \\ &\lesssim (1+|\eta|)^{m_1}\int_{|\tau|>\frac{|\eta|}{2}} (1+|\xi-\tau|)^{-N-m_1}(1+|\tau|)^{-N-m_1}\,d\tau \\ &\lesssim (1+|\xi|)^{-N-m_1}(1+|\eta|)^{m_1}\ .\end{aligned}$$

The two cases can be combined together to give the inequality

$$\begin{aligned}\int_{|\tau|>\frac{|\eta|}{2}} &(1+|\xi-\tau|)^{-N}(1+|\tau+\eta|)^{m_1}(1+|\tau|)^{-N}\,d\tau \\ &\le C_N(1+|\xi|)^{-N+|m_1|}(1+|\eta|)^{m_1}\ .\end{aligned} \tag{6.7}$$

Putting (6.6) and (6.7) together, we have that

$$|\tilde{k}(\xi,\eta)| \le C_N(1+|\xi|)^{-N+|m_1|}(1+|\eta|)^{m_1+m_2}\ ,$$

for any N.

Consider now a derivative of $\tilde{k}$ in η,

$$\partial_\eta^\beta \tilde{k}(\xi,\eta) = \sum_{\alpha\le\beta} c_{\beta,\alpha}\int_{\mathbb{R}^n} \partial_\eta^\alpha k_1(\xi-\tau,\tau+\eta)\partial_\eta^{\beta-\alpha}k_2(\tau,\eta)\,d\tau\ .$$

By remark (4) above, each term can be treated as before, with m_1 replaced by $m_1 - |\alpha|$ and m_2 replaced by $m_2 - |\beta| + |\alpha|$. □

Proposition 6.6. *Let D_1, D_2 be special ψdo of order m_1, m_2 respectively. Then $[D_1, D_2]$ is a special ψdo of order $m_1 + m_2 - 1$.*

Proof. Let k_1, k_2 be the kernels associated to D_1, D_2, and let $\tilde{k}, \tilde{k}'$ be such that $T_{\tilde{k}} = T_{k_1}T_{k_2}$, $T_{\tilde{k}'} = T_{k_2}T_{k_1}$. Then $\tilde{k}$ is given by (6.5), and

$$\begin{aligned}\tilde{k}'(\xi,\eta) &= \int_{\mathbb{R}^n} k_2(\xi-\tau,\tau+\eta)k_1(\tau,\eta)\,d\tau \\ &= \int_{\mathbb{R}^n} k_2(\xi-\tau,\tau+\eta)k_1(\tau,\eta)\,d\tau \\ &= \int_{\mathbb{R}^n} k_1(\xi-\tau,\eta)k_2(\tau,\xi-\tau+\eta)\,d\tau\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{k}(\xi,\eta) - \tilde{k}'(\xi,\eta) &= \int_{\mathbb{R}^n} k_1(\xi-\tau,\tau+\eta)k_2(\tau,\eta)\,d\tau \\ &\qquad - \int_{\mathbb{R}^n} k_1(\xi-\tau,\eta)k_2(\tau,\xi-\tau+\eta)\,d\tau \\ &= \int_{\mathbb{R}^n} \big(k_1(\xi-\tau,\tau+\eta) - k_1(\xi-\tau,\eta)\big)k_2(\tau,\eta)\,d\tau \\ &\qquad + \int_{\mathbb{R}^n} k_1(\xi-\tau,\eta)\big(k_2(\tau,\eta) - k_2(\tau,\xi-\tau+\eta)\big)\,d\tau \\ &= h_1(\xi,\eta) + h_2(\xi,\eta)\ .\end{aligned}$$

We prove that each of h_1 and h_2 satisfies (6.3) with $m = m_1 + m_2 - 1$. The proof requires some modifications to that of Proposition 6.5.

For h_1, we first integrate for $|\tau| < \frac{|\eta|}{2}$. Since $|t\tau + \eta| \sim |\eta|$ for $0 < t < 1$, by the mean value theorem

$$\begin{aligned}\big|k_1(\xi-\tau,\tau+\eta) - k_1(\xi-\tau,\eta)\big| &\le |\tau| \sup_{0<t<1} \big|\nabla_\eta k_1(\xi-\tau, t\tau+\eta)\big| \\ &\le C_N |\tau| (1+|\xi-\tau|)^{-N}(1+|\eta|)^{m_1-1} \ .\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{|\tau|<\frac{|\eta|}{2}} \big|k_1(\xi-\tau,\tau+\eta) - k_1(\xi-\tau,\eta)\big|\big|k_2(\tau,\eta)\big|\,d\tau \\ &\qquad \le C_N (1+|\eta|)^{m_1+m_2-1} \int_{|\tau|<\frac{|\eta|}{2}} (1+|\xi-\tau|)^{-N}(1+|\tau|)^{-N+1}\,d\tau \\ &\qquad \le C_N (1+|\xi|)^{-N+1}(1+|\eta|)^{m_1+m_2-1} \ .\end{aligned}$$

Passing to the integral for $|\tau| > \frac{|\eta|}{2}$, we write

$$\begin{aligned}&\int_{|\tau|>\frac{|\eta|}{2}} \big|k_1(\xi-\tau,\tau+\eta) - k_1(\xi-\tau,\eta)\big|\big|k_2(\tau,\eta)\big|\,d\tau \\ &\qquad \le \int_{|\tau|>\frac{|\eta|}{2}} |k_1(\xi-\tau,\tau+\eta)||k_2(\tau,\eta)|\,d\tau \\ &\qquad\quad + \int_{|\tau|>\frac{|\eta|}{2}} |k_1(\xi-\tau,\eta)||k_2(\tau,\eta)|\,d\tau \ .,\end{aligned}$$

and estimate the two terms separately, distinguishing between the cases $m_1 \ge 0$ and $m_1 < 0$. The proof goes as in Proposition 6.5[8]. The derivatives of h_1 are estimated in a similar way.

For h_2, the integral must be split according to whether $|\xi - \tau| < \frac{|\eta|}{2}$ or $|\xi - \tau| > \frac{|\eta|}{2}$, and there is no substantial difference. □

Bessel potentials can be used to approximate L^2-functions by $H^s(\mathbb{R}^n)$-functions with $s > 0$.

For $\delta > 0$, the operator $(1 - \delta^2\Delta)^\gamma$ is naturally defined by

$$\big((1-\delta^2\Delta)^\gamma \varphi\big)\hat{}\,(\xi) = (1+\delta^2|\xi|^2)^\gamma \hat\varphi(\xi) \ .$$

Lemma 6.7. *Let $s > 0$. If $f \in L^2(\mathbb{R}^n)$, then $(1-\delta^2\Delta)^{-\frac{s}{2}} f \in H^s(\mathbb{R}^n)$ and*

$$\lim_{\delta\to 0} \big\|(1-\delta^2\Delta)^{-\frac{s}{2}} f - f\big\|_2 = 0 \ .$$

Proof. The first statement follows from the trivial estimate $(1+\delta^2|\xi|^2)^{-\frac{s}{2}} \le C_\delta (1+|\xi|^2)^{-\frac{s}{2}}$. By Plancherel's formula,

$$\big\|(1-\delta^2\Delta)^{-\frac{s}{2}} f - f\big\|_2^2 = (2\pi)^n \int_{\mathbb{R}^n} \Big|\frac{1}{(1+\delta^2|\xi|^2)^{\frac{s}{2}}} - 1\Big|^2 |\hat f(\xi)|^2\,d\xi \ .$$

[8] Notice that at this stage of the proof of Proposition 6.5, the factor $(1+|\eta|)^{m_1}$ was introduced by brute force. The same can be done here with the exponent $m_1 - 1$.

Since $\left|\frac{1}{(1+\delta^2|\xi|^2)^{\frac{s}{2}}}-1\right| \le 2$, $\delta \to 0$, the integral tends to 0 by dominated convergence. □

For $\delta < 1$, the constant C_δ appearing in the proof above is of the order of δ^{-s}. This means that, whereas the L^2-norms of $(1-\delta^2\Delta)^{-\frac{s}{2}}f$ remain bounded as $\delta \to 0$, the H^s-norms can blow up, at most like δ^{-s}. In the same way, one can see that intermediate H^r-norms (i.e. with $0 < r < s$) of $(1-\delta^2\Delta)^{-\frac{s}{2}}f$ can blow up at most like δ^{-r}. In particular, if $|\alpha| \le s$,

$$\left\|\partial^\alpha(1-\delta^2\Delta)^{-\frac{s}{2}}f\right\|_2 \le C\delta^{-|\alpha|}\ ,$$

for δ small.

The following statement is a generalization of this fact.

Proposition 6.8. *Let D be a special ψdo of order m, $0 \le m \le s$. For every $f \in L^2(\mathbb{R}^n)$ and every $\delta < 1$,*

$$\left\|D(1-\delta^2\Delta)^{-\frac{s}{2}}f\right\|_2 \le C\delta^{-m}\|f\|_2\ , \quad \left\|(1-\delta^2\Delta)^{-\frac{s}{2}}Df\right\|_2 \le C\delta^{-m}\|f\|_2\ . \tag{6.8}$$

Proof. Since $\left((1-\delta^2\Delta)^{-\frac{s}{2}}D\right)^* = D^*(1-\delta^2\Delta)^{-\frac{s}{2}}$, it is sufficient to prove the first estimate. If $D = \mathcal{F}^{-1}T_k\mathcal{F}$, then $(1-\delta^2\Delta)^{-\frac{s}{2}} = \mathcal{F}^{-1}T_{k'}\mathcal{F}D\mathcal{F}^{-1}$, with

$$k'(\xi,\eta) = k(\xi,\eta)(1+\delta^2|\eta|^2)^{-\frac{s}{2}}\ .$$

Hence, for every N,

$$\begin{aligned}|k'(\xi,\eta)| &\le C_N\frac{(1+|\eta|)^m}{(1+|\xi|)^N(1+\delta|\eta|)^s}\\ &\le C_N\delta^{-m}\frac{1}{(1+|\xi|)^N(1+\delta|\eta|)^{s-m}}\\ &\le C_N\delta^{-m}\frac{1}{(1+|\xi|)^N}\ .\end{aligned}$$

Therefore,

$$|T_{k'}f(\xi)| \le C_N\delta^{-m}\int_{\mathbb{R}^n}\frac{1}{(1+|\xi-\eta|)^N}|f(\eta)|\,d\eta\ .$$

Fixing $N > n$, this implies that

$$\|T_{k'}f\|_2 \le C\delta^{-m}\|f\|_2\ . \qquad \square$$

Proposition 6.9. *Let D be a differential operator on $\mathbb{R}^n$ of order 1 and with coefficients in $\mathcal{D}(\mathbb{R}^n)$. For every $f \in L^2(\mathbb{R}^n)$ and every $\delta < 1$,*

$$\left\|\left[(1-\delta^2\Delta)^{-1}, D\right]f\right\|_2 \le C\|f\|_2\ . \tag{6.9}$$

Proof. Let $v = (1-\delta^2\Delta)^{-1}f$. Then

$$\begin{aligned}\left[(1-\delta^2\Delta)^{-1}, D\right]f &= (1-\delta^2\Delta)^{-1}Df - D(1-\delta^2\Delta)^{-1}f\\ &= (1-\delta^2\Delta)^{-1}\left(D(1-\delta^2\Delta)v - (1-\delta^2\Delta)Dv\right)\\ &= \delta^2(1-\delta^2\Delta)^{-1}[\Delta, D]v\ .\end{aligned}$$

As we have observed, $[\Delta, D]$ is a differential operator of order at most 2. Applying Proposition 6.8 twice, we obtain that

$$\left\|\left[(1-\delta^2\Delta)^{-1}, D\right]f\right\|_2 \le C\|v\|_2 = C\left\|(1-\delta^2\Delta)^{-1}f\right\|_2 \le C^2\|f\|_2\ . \qquad \square$$

7. Back to L

Recall that, for $f \in \mathcal{D}(\Omega')$ and $s < \frac{1}{m}$,

$$\|f\|_{(s)} \le C_s\big(\|f\|_2 + \|\!|Lf|\!\|'_{\mathcal{X},\Omega'}\big) \ .$$

This inequality was labeled (5.2) in Section 5. We first extend it to $f \in H^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \Omega'$.

Lemma 7.1. *Suppose that $f \in H^2(\mathbb{R}^n)$, with $\operatorname{supp} f \subset \Omega'$, and that $\|\!|Lf|\!\|'_{\mathcal{X},\Omega'} < \infty$. Then (5.2) holds for $s < \frac{1}{m}$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be supported on the unit ball and with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, and set $f_\varepsilon = \varphi_\varepsilon * f$.

If $\varepsilon < d(\operatorname{supp} f, \partial\Omega')$, $f_\varepsilon \in \mathcal{D}(\Omega') \subset H^2(\mathbb{R}^n)$. Moreover, for $|\alpha| \le 2$,

$$\lim_{\varepsilon\to 0} \|\partial^\alpha f - \partial^\alpha f_\varepsilon\|_2 = \lim_{\varepsilon\to 0} \|\partial^\alpha f - \varphi_\varepsilon * \partial^\alpha f\|_2 = 0 \ , \tag{7.1}$$

since the φ_ε form an approximate identity for $\varepsilon \to 0$. We have so proved that $f_\varepsilon \to f$ in $H^2(\mathbb{R}^n)$, hence in $H^s(\mathbb{R}^n)$, since $s < 2$.

From (7.1) we also obtain that

$$\lim_{\varepsilon\to 0} \|Lf - Lf_\varepsilon\|_2 = 0 \ ,$$

just because L is a second-order operator with bounded coefficients on Ω'. By (5.1),

$$\lim_{\varepsilon\to 0} \|\!|Lf - Lf_\varepsilon|\!\|'_{\mathcal{X},\Omega'} = 0 \ .$$

We can then apply (5.2) to the f_ε and pass to the limit. □

Lemma 7.2. *Let K be a compact subset of Ω'. If $f \in L^2(\mathbb{R}^n)$, with $\operatorname{supp} f \subseteq K$, and $\|\!|Lf|\!\|'_{\mathcal{X},\Omega'} < \infty$, then, for $s < \frac{1}{m}$, $f \in H^s(\mathbb{R}^n)$ and (5.2) holds with a constant $C_{s,K}$ also depending on K.*

Proof. We can assume that the vector fields X_j have compact support in Ω (if not multiply each of them by a function in $\mathcal{D}(\Omega)$ which is identically equal to 1 on a neighborhood of $\overline{\Omega'}$; since we will apply the X_j and L only to functions compactly supported in Ω', this modification will not affect our operations). This modification allows us to apply the results of the previous sections to the X_j and to L.

For $\delta > 0$ define

$$f_\delta = (1 - \delta^2\Delta)^{-1} f \ .$$

The discussion at the beginning of Section 7 shows that $f_\delta \in H^2(\mathbb{R}^n)$. If $\chi \in \mathcal{D}(\Omega')$ is identically equal to 1 on a neighborhood of K, then also $\chi f_\delta \in H^2(\mathbb{R}^n)$. Applying Lemma 7.1 to χf_δ, we have that

$$\|\chi f_\delta\|_{(s)} \le C_s\big(\|f_\delta\|_2 + \|\!|L\chi f_\delta|\!\|'_{\mathcal{X},\Omega'}\big) \ . \tag{7.2}$$

We prove that the norms $\|\!|L\chi f_\delta|\!\|'_{\mathcal{X},\Omega'}$ are uniformly bounded for δ small. Since $f = \chi f = \chi(1 - \delta^2\Delta) f_\delta$,

$$\begin{aligned} Lf &= L\chi(1 - \delta^2\Delta) f_\delta \\ &= (1 - \delta^2\Delta) L\chi f_\delta - \delta^2[L\chi, \Delta] f_\delta \ . \end{aligned}$$

Applying $(1-\delta^2\Delta)^{-1}$ to both sides, we obtain that

$$L\chi f_\delta = (1-\delta^2\Delta)^{-1}Lf + \delta^2(1-\delta^2\Delta)^{-1}[L\chi,\Delta]f_\delta\ ,$$

so that

$$\|\!|L\chi f_\delta\|\!|'_{\mathcal{X},\Omega'} \le \big\|\!\big|(1-\delta^2\Delta)^{-1}Lf\big\|\!\big|'_{\mathcal{X},\Omega'} + \delta^2\big\|\!\big|(1-\delta^2\Delta)^{-1}[L\chi,\Delta]f_\delta\big\|\!\big|'_{\mathcal{X},\Omega'}\ . \tag{7.3}$$

Now,

$$\begin{aligned}
\|\!|(1-\delta^2\Delta)^{-1}Lf\|\!|'_{\mathcal{X},\Omega'} &= \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big|\langle(1-\delta^2\Delta)^{-1}Lf,\varphi\rangle\big|\\
&= \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big|\langle Lf,(1-\delta^2\Delta)^{-1}\varphi\rangle\big|\\
&= \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big|\langle Lf,\chi(1-\delta^2\Delta)^{-1}\varphi\rangle\big|\\
&\le \|\!|Lf\|\!|'_{\mathcal{X},\Omega'} \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \|\!|\chi(1-\delta^2\Delta)^{-1}\varphi\|\!|_{\mathcal{X}}\ .
\end{aligned}$$

By (6.6),

$$\|\chi(1-\delta^2\Delta)^{-1}\varphi\|_2 \le \|\varphi\|_2\ .$$

Moreover, setting $D_j g = X_j\chi g$, we can apply Proposition 6.9 and obtain that, for δ small,

$$\begin{aligned}
\big\|X_j\chi(1-\delta^2\Delta)^{-1}\varphi)\big\|_2 &= \|D_j(1-\delta^2\Delta)^{-1}\varphi\|_2\\
&\le \|(1-\delta^2\Delta)^{-1}D_j\varphi\|_2 + \big\|[D_j,(1-\delta^2\Delta)^{-1}]\varphi\big\|_2\\
&\le C_K\big(\|D_j\varphi\|_2 + \|\varphi\|_2\big)\\
&\le C_K\big(\|(X_j\chi)\varphi\|_2 + \|\chi X_j\varphi\|_2 + \|\varphi\|_2\big)\\
&\le C_K\|\!|\varphi\|\!|_{\mathcal{X}}
\end{aligned}$$

(notice that the constant depends on the choice of χ, i.e. on K).

Therefore,

$$\|\!|\chi'(1-\delta^2\Delta)^{-1}\varphi\|\!|_{\mathcal{X}} \le C_K\|\!|\varphi\|\!|_{\mathcal{X}}\ ,$$

and hence

$$\|\!|(1-\delta^2\Delta)^{-1}Lf\|\!|'_{\mathcal{X},\Omega'} \le C_K\|\!|Lf\|\!|'_{\mathcal{X},\Omega'}\ . \tag{7.4}$$

Consider now the last term in (7.3),

$$\delta^2\big\|\!\big|(1-\delta^2\Delta)^{-1}[L\chi,\Delta]f_\delta\big\|\!\big|'_{\mathcal{X},\Omega'} \le \delta^2\sum_{j=1}^k \big\|\!\big|(1-\delta^2\Delta)^{-1}[X_j^2\chi,\Delta]f_\delta\big\|\!\big|'_{\mathcal{X},\Omega'}\ .$$

Each summand in the right-hand side can be estimated as follows:

$$\begin{aligned}
&\big\|\!\big|(1-\delta^2\Delta)^{-1}[X_j^2\chi,\Delta]f_\delta\big\|\!\big|'_{\mathcal{X},\Omega'}\\
&\quad= \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1}\big|\big\langle f_\delta,[\Delta,\chi(X_j^*)^2](1-\delta^2\Delta)^{-1}\varphi\big\rangle\big|\\
&\quad= \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1}\big|\big\langle f,(1-\delta^2\Delta)^{-1}[\Delta,\chi(X_j^*)^2](1-\delta^2\Delta)^{-1}\varphi\big\rangle\big|\\
&\quad\le \|f\|_2 \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1}\big\|(1-\delta^2\Delta)^{-1}[\Delta,\chi(X_j^*)^2](1-\delta^2\Delta)^{-1}\varphi\big\|_2\ .
\end{aligned} \tag{7.5}$$

We have that

$$\begin{aligned}[\Delta, \chi(X_j^*)^2] &= \Delta\chi(X_j^*)^2 - \chi(X_j^*)^2\Delta \\ &= [\Delta, \chi](X_j^*)^2 + \chi\big(\Delta(X_j^*)^2 - (X_j^*)^2\Delta\big) \\ &= [\Delta, \chi](X_j^*)^2 + \chi\big([\Delta, X_j^*]X_j^* + X_j^*[\Delta, X_j^*]\big) \\ &= [\Delta, \chi](X_j^*)^2 + \chi\big(2[\Delta, X_j^*]X_j^* + \big[X_j^*, [\Delta, X_j^*]\big]\big) \\ &= D_1 X_j^* + D_2 \ ,\end{aligned}$$

where D_1, D_2 are second-order operators with compact support in Ω'. Therefore

$$\begin{aligned}&\big\|(1-\delta^2\Delta)^{-1}[\Delta, \chi(X_j^*)^2](1-\delta^2\Delta)^{-1}\varphi\big\|_2 \\ &\qquad \le \big\|(1-\delta^2\Delta)^{-1}D_1X_j^*(1-\delta^2\Delta)^{-1}\varphi\big\|_2 \\ &\qquad\quad + \big\|(1-\delta^2\Delta)^{-1}D_2(1-\delta^2\Delta)^{-1}\varphi\|_2 \\ &\qquad \le \big\|(1-\delta^2\Delta)^{-1}D_1(1-\delta^2\Delta)^{-1}X_j^*\varphi\big\|_2 \\ &\qquad\quad + \big\|(1-\delta^2\Delta)^{-1}D_1[X_j^*, (1-\delta^2\Delta)^{-1}]\varphi\big\|_2 \\ &\qquad\quad + \big\|(1-\delta^2\Delta)^{-1}D_2(1-\delta^2\Delta)^{-1}\varphi\|_2 \ .\end{aligned}$$

Since X_j, D_1 and D_2 depend only on the choice of χ, i.e. on K, we have, by Proposition 6.8,

$$\begin{aligned}\big\|(1-\delta^2\Delta)^{-1}D_1(1-\delta^2\Delta)^{-1}X_j^*\varphi\big\|_2 &\le \big\|D_1(1-\delta^2\Delta)^{-1}X_j^*\varphi\big\|_2 \\ &\le C_K\delta^{-2}\|X_j^*\varphi\big\|_2 \\ &\le C_K\delta^{-2}\big(\|\varphi\|_2 + \|X_j\varphi\|_2\big) \\ &\le C_K\delta^{-2}\big(\|\varphi\|_2 + \|X_j\varphi\|_2\big) \\ &\le C_K\delta^{-2}|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .\end{aligned}$$

Similarly,

$$\|(1-\delta^2\Delta)^{-1}D_2(1-\delta^2\Delta)^{-1}\varphi\|_2 \le C_K\delta^{-2}\|\varphi\|_2 \le C_K\delta^{-2}|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .$$

Finally, by Propositions 6.8 and 6.9 together,

$$\begin{aligned}\big\|(1-\delta^2\Delta)^{-1}D_1[X_j^*, (1-\delta^2\Delta)^{-1}]\varphi\big\|_2 &\le C_K\delta^{-2}\|[X_j^*, (1-\delta^2\Delta)^{-1}]\varphi\big\|_2 \\ &\le C_K\delta^{-2}\|\varphi\|_2 \\ &\le C_K\delta^{-2}|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .\end{aligned}$$

Putting these various estimates into (7.5), we have

$$\big|\!\big|\!\big|(1-\delta^2\Delta)^{-1}[X_j^2, \Delta]f_\delta\big|\!\big|\!\big|'_{\mathcal{X},\Omega'} \le C_K\|f\|_2 \ . \tag{7.6}$$

Then (7.4) and (7.6) give that

$$|\!|\!|Lf_\delta|\!|\!|'_{\mathcal{X},\Omega'} \le C_K\big(|\!|\!|Lf|\!|\!|'_{\mathcal{X},\Omega'} + \|f\|_2\big) \ ,$$

and from (7.2) we have

$$\|\chi' f_\delta\|_{(s)} \le C_{s,K}\big(\|f\|_2 + |\!|\!|Lf|\!|\!|'_{\mathcal{X},\Omega'}\big) \ . \tag{7.7}$$

It follows that the norms $\|\chi' f_\delta\|_{(s)}$ are bounded for δ small. Let $\{\delta_j\}$ be a sequence tending to 0 such that $\chi' f_{\delta_j}$ have a weak limit in $H^s(\mathbb{R}^n)$. The norm of the weak limit is not larger than the right-hand side in (7.7). But, by Lemma 6.7, $\chi' f_{\delta_j}$ tend to f in $L^2(\mathbb{R}^n)$. Then the weak limit must be f. □

8. Hypoellipticity of L

Recall that, for $r \in \mathbb{R}$,

$$H^r(\mathbb{R}^n) = (1-\Delta)^{-\frac{r}{2}}\big(L^2(\mathbb{R}^n)\big)\ .$$

The following statement has the effect of "translating" (5.2) from the level of $H^0(= L^2)$-functions to the level of H^r-functions. The exponent r can be both positive or negative.

Lemma 8.1. *Let K be a compact subset of Ω', and let $\chi \in \mathcal{D}(\Omega')$ be identically equal to* 1 *on a neighborhood of K. If $f \in H^r(\mathbb{R}^n)$, with* $\operatorname{supp} f \subseteq K$, *and* $\|\!|\chi(1-\Delta)^{\frac{r}{2}} Lf\|\!|'_{\mathcal{X},\Omega'} < \infty$, *then, for $s < \frac{1}{m}$, $f \in H^{r+s}(\mathbb{R}^n)$ and*

$$\|f\|_{(r+s)} \le C_{s,K}\big(\|f\|_{(r)} + \|\!|\chi(1-\Delta)^{\frac{r}{2}} Lf\|\!|'_{\mathcal{X},\Omega'}\big)\ .$$

Proof. The statement follows immediately if we show that Lemma 7.2 can be applied to to $g = \chi(1-\Delta)^{\frac{r}{2}} f$, with K replaced by $K' = \operatorname{supp}\chi \subset \Omega'$.

By assumption, $g \in L^2(\mathbb{R}^n)$ and $\operatorname{supp} g \subseteq K'$. We have to show that

$$\|\!|Lg\|\!|'_{\mathcal{X},\Omega'} < \infty\ . \tag{8.1}$$

Assuming, as in the proof of Lemma 7.2, that the X_j and L are compactly supported in Ω, in our hypotheses, (8.1) is equivalent to

$$\|\!|\big[\chi(1-\Delta)^{\frac{r}{2}}, L\big] f\|\!|'_{\mathcal{X},\Omega'} < \infty\ .$$

As in the proof of Lemma 7.2,

$$\begin{aligned}
&\|\!|\big[\chi(1-\Delta)^{\frac{r}{2}}, L\big] f\|\!|'_{\mathcal{X},\Omega'} \\
&\qquad = \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big|\big\langle f, \big[L^*, (1-\Delta)^{\frac{r}{2}}\chi\big]\varphi\big\rangle\big| \\
&\qquad = \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big|\big\langle (1-\Delta)^{\frac{r}{2}} f, (1-\Delta)^{-\frac{r}{2}}\big[L^*, (1-\Delta)^{\frac{r}{2}}\chi\big]\varphi\big\rangle\big| \\
&\qquad \le \|f\|_{(r)} \sup_{\varphi\in\mathcal{D}(\Omega'):\|\!|\varphi\|\!|_X\le 1} \big\|(1-\Delta)^{-\frac{r}{2}}\big[L^*, (1-\Delta)^{\frac{r}{2}}\chi\big]\varphi\big\|_2\ .
\end{aligned}$$

Now,

$$\begin{aligned}
\big[L^*, (1-\Delta)^{\frac{r}{2}}\chi\big] &= \sum_{j=1}^k \big[(1-\Delta)^{\frac{r}{2}}\chi, (X_j^*)^2\big] \\
&= \sum_{j=1}^k \big[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\big]X_j^* + \sum_{j=1}^k X_j^*\big[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\big] \\
&= 2\sum_{j=1}^k \big[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\big]X_j^* + \sum_{j=1}^k \Big[X_j^*, \big[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\big]\Big]\ .
\end{aligned}$$

By Proposition 6.6, $\left[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\right]$ is a special ψdo of order r, and Corollary 6.3 implies that its composition with $(1-\Delta)^{-\frac{r}{2}}$ is of order 0. Therefore, by Theorem 6.4,

$$\left\|(1-\Delta)^{-\frac{r}{2}}\left[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\right]X_j^*\varphi\right\|_2 \le C\|X_j^*\varphi\|_2 \le C|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .$$

For the same reason,

$$\left\|(1-\Delta)^{-\frac{r}{2}}\left[X_j^*, \left[(1-\Delta)^{\frac{r}{2}}\chi, X_j^*\right]\right]\varphi\right\|_2 \le C\|\varphi\|_2 \ .$$

Therefore,

$$|\!|\!|\left[(1-\Delta)^{\frac{r}{2}}\chi, L\right]f|\!|\!|'_{\mathcal{X},\Omega'} \le C\|f\|_{(r)} \ ,$$

and we are finished. □

We can now prove the final theorem.

Theorem 8.2. *L is hypoelliptic in Ω.*

Proof. . Let $u \in \mathcal{D}'(\Omega)$ be such that Lu coincides with a C^∞-function on an open subset Ω'. We can assume that Ω' is relatively compact. Multiplying, if necessary, u by a function in $\mathcal{D}(\Omega)$ that is identically equal to 1 on a neighborhood of $\overline{\Omega'}$, we can also assume that u has compact support in Ω. We then extend u to all of $\mathbb{R}^n$, by imposing that it vanishes on $\mathbb{R}^n \setminus K$. Since any distribution with compact support has finite order, there is $r \in \mathbb{R}$ such that $u \in H^r(\mathbb{R}^n)$.

For a fixed ball B in Ω', we choose two functions $\chi, \chi' \in \mathcal{D}(\Omega')$ such that χ is identically equal to 1 on a neighborhood of $\bar{B}$, and χ' is identically equal to 1 on a neighborhood of $\operatorname{supp}\chi$. In particular $\chi\chi' = \chi$.

If we prove that $|\!|\!|\chi'(1-\Delta)^{\frac{r}{2}}L(\chi u)|\!|\!|'_{\mathcal{X},\Omega'} < \infty$, it follows from Lemma 8.1 that $\chi u \in H^{r+s}(\mathbb{R}^n)$ for $s < \frac{1}{m}$. We can the reapply Lemma 8.1 repeatedly, to conclude that $\chi u \in H^N(\mathbb{R}^n)$ for every N. By the Sobolev embedding theorem[9], $\chi u \in C^\infty(\mathbb{R}^n)$, i.e. $u \in C^\infty(B)$, and we are finished.

We consider separately each summand coming from application of Leibniz's formula

$$L(\chi u) = \chi L u + (L\chi)u + \sum_{j=1}^k (X_j\chi)X_j u \ .$$

For the first term, notice that $\chi L u \in \mathcal{D}(\mathbb{R}^n)$ by hypothesis. Therefore, applying $\chi'(1-\Delta)^{\frac{r}{2}}$ to it, we obtain a function in every Sobolev space, i.e. a C^∞-function, which also has compact support. Therefore

$$|\!|\!|\chi'(1-\Delta)^{\frac{r}{2}}(\chi L u)|\!|\!|'_{\mathcal{X},\Omega'} \le \left\|\chi'(1-\Delta)^{\frac{r}{2}}(\chi L u)\right\|_2 < \infty \ .$$

[9]Directly, we have

$$\int_{\mathbb{R}^n} |\widehat{\chi u}(\xi)|^2(1+|\xi|)^{2N}\,d\xi < \infty$$

for every N. Applying Schwartz's inequality,

$$\widehat{\partial^\alpha(\chi u)}(\xi) = (i\xi)^\alpha\widehat{\chi u}(\xi) \in L^1(\mathbb{R}^n) \ ,$$

for every multi-index α. Therefore $\partial^\alpha(\chi u)$ is continuous.

For the second term, we observe that $(L\chi)u \in H^r(\mathbb{R}^n)$, i.e. $(1-\Delta)^{\frac{r}{2}}\big((L\chi)u\big) \in L^2(\mathbb{R}^n)$, and we are done.

For each of the other terms, setting $\tilde{X}_j = (X_j\chi)X_j$, we must then prove that, for some constant $C > 0$ and every $\varphi \in \mathcal{D}(\Omega')$,

$$\big|\big\langle \chi'(1-\Delta)^{\frac{r}{2}}\tilde{X}_j u, \varphi\big\rangle\big| \leq C|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .$$

We have

$$\begin{aligned}\big\langle \chi'(1-\Delta)^{\frac{r}{2}}\tilde{X}_j u, \varphi\big\rangle &= \big\langle \tilde{X}_j\big(\chi'(1-\Delta)^{\frac{r}{2}}u\big), \varphi\big\rangle + \big\langle \big[\chi'(1-\Delta)^{\frac{r}{2}}, \tilde{X}_j\big]u, \varphi\big\rangle \\ &= \big\langle (1-\Delta)^{\frac{r}{2}}u, \chi'\tilde{X}_j^*\varphi\big\rangle + \big\langle \big[\chi'(1-\Delta)^{\frac{r}{2}}, \tilde{X}_j\big]u, \varphi\big\rangle\end{aligned}$$

Then

$$\big|\big\langle (1-\Delta)^{\frac{r}{2}}u, \chi'\tilde{X}_j^*\varphi\big\rangle\big| \leq \|u\|_{H^r}\|\tilde{X}_j^* u\|_2 \leq C\|u\|_{H^r}|\!|\!|\varphi|\!|\!|_{\mathcal{X}} \ .$$

Since $\big[\chi'(1-\Delta)^{\frac{r}{2}}, \tilde{X}_j\big]$ is a special ψdo of order r,

$$\big|\big\langle \big[\chi'(1-\Delta)^{\frac{r}{2}}, \tilde{X}_j\big]u, \varphi\big\rangle\big| \leq C\|u\|_{H^r}\|\varphi\|_2 \ .$$

Putting everything together, we obtain the desired estimate. □